Exercises IV

November 12th

E4.1 *Spontaneous Emission*

If an atom is not in its ground state, the experiment shows that it will change into the ground state emitting light. However, an eigenstate is stable and in the context of perturbation theory we may only explain emission stimulated by an external electromagnetic field, which is the principle of the laser.

In order to explain spontaneous emission we have to take some facts from quantum electrodynamics where the electromagnetic field itself will is quantised. We choose the Coulomb gauge $\mathrm{div}\vec{A} = 0$. In vacuum thus $\Phi = 0$ and $\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A}$ as well as $\vec{B} = \mathrm{rot}\vec{A}$. hold. For the vector potential we assume $\vec{A}(\vec{r},t) = \vec{A}_+(\vec{r},t) + \vec{A}_-(\vec{r},t)$ with

$$\vec{A}_+(\vec{r},t) = \frac{c\sqrt{\hbar}}{2\pi}\sum_{\lambda=1}^{2}\int \mathrm{d}^3k\,\frac{1}{\sqrt{\omega_{\vec{k}}}}\,\vec{e}(\vec{k},\lambda)\,g_\lambda(\vec{k})\,\mathrm{e}^{\mathrm{i}(\vec{k}\vec{r}-\omega_{\vec{k}}t)},\quad \omega_{\vec{k}} = c|\vec{k}|,$$

where $\vec{A}_- = \vec{A}_+^*$ in order to have a real $\vec{A}(\vec{r},t)$. $\vec{e}(\vec{k},\lambda)$ are the two polarisation vectors that are perpendicular to $\vec{k}$ in Coulomb gauge (why?). $g_\lambda(\vec{k})$ describes the distribution of wave vectors. The introduction of the normalisation $1/\sqrt{\omega_{\vec{k}}}$ will be apparent later.

(1) Wer interpret $\vec{A}_+(\vec{r},t)$ as wave packet. Let $g_\lambda(\vec{k})$ be sharply concentrated around $\vec{k}_0$. Show that the field energy is given by $H = \frac{1}{8\pi}\int \mathrm{d}^3r(|\vec{E}|^2 + |\vec{B}|^2) = \hbar\omega_{\vec{k}_0}$. However, a sharply concentrated wave packet is considered as a particle, this particle thus has the dispersion relation of a photon..

(2) Now, we interpret $g_\lambda^*(\vec{k})$ as annihilation operator $a_{\vec{k},\lambda}$ and $g_\lambda(\vec{k})$ as creation operator $a_{\vec{k},\lambda}^+$ of a photon with momentum $\vec{k}$ and polarisation λ. We require the operatos to satisfy the commutation relation of the harmonic oscillator, i.e. $[a_{\vec{k},\lambda}, a_{\vec{k}',\lambda'}^+] = \delta_{\vec{k}\vec{k}'}\delta_{\lambda\lambda'}$, $[a_{\vec{k},\lambda}, a_{\vec{k}',\lambda'}] = [a_{\vec{k},\lambda}^+, a_{\vec{k}',\lambda'}^+] = 0$. Show that the Hamiltonian of the field is then given by $H = \frac{\hbar}{2}\int \mathrm{d}^3k\sum_\lambda \omega_{\vec{k}}(a_{\vec{k},\lambda}^+ a_{\vec{k},\lambda} + a_{\vec{k},\lambda}a_{\vec{k},\lambda}^+)$. Why did the normalisation $1/\sqrt{\omega_{\vec{k}}}$ prove useful?

(3) Deduce the interaction of a particle with the electromagnetic field:

$$H' = -\frac{q}{mc}\vec{A}(\vec{r},t)\cdot\vec{p} + \frac{q^2}{2mc^2}|\vec{A}(\vec{r},t)|^2 + q\Phi(\vec{r},t).$$

Here, the identification $\vec{\jmath} = \frac{q}{m}\vec{p}$ is useful to be able to applay formulas from classical electrodynamics. The term quadratic in $\vec{A}$ contains each two creation and annihilation operators and thus describes processes like Compton scattering. Only the linear term contributes to the emission of photons.

(4) Let $V(t) = F\mathrm{e}^{-\mathrm{i}\omega t} + F^+\mathrm{e}^{\mathrm{i}\omega t}$ be a time-dependent perturbation. Deduce the following formula for the transition rate, i.e. the transition probability per time unit (cf. the derivation of Fermi's Golden Rule in the lecture):

$$w_{m\to n} = \frac{2\pi}{\hbar}\left(\delta(E_n - E_m - \hbar\omega)|\langle n|F|m\rangle|^2 + \delta(E_n - E_m + \hbar\omega)|\langle n|F^+|m\rangle|^2\right).$$

(5) Now, we consider the spontaneous emission of a photon with wave number $\vec{k}$ and polarisation λ by an atom doing a transition from the state $|m\rangle$ to $|n\rangle$. The

radiation field does a transition from its ground state $|0\rangle$ to a one-photon state $a^+_{\vec{k},\lambda}|0\rangle$. With the Golden Rule (or (4), respectively), show for the transition rate:

$$w_{m\to n,\vec{k},\lambda} = \frac{1}{2\pi|\vec{k}|c}\delta(E_m - E_n - \hbar\omega_{\vec{k}})\Big|\langle n|\vec{\jmath}(\vec{r})\cdot\vec{e}(\vec{k},\lambda)^*\,\mathrm{e}^{-\mathrm{i}\vec{k}\vec{r}}|m\rangle\Big|^2.$$

Deduce the power radiated into the solid angle $\mathrm{d}\Omega$:

$$\frac{\mathrm{d}P_{m\to n,\vec{k},\lambda}}{\mathrm{d}\Omega} = \frac{\omega_{\vec{k}}^2}{2\pi c^3}\;\big|\langle n|\vec{\jmath}(\vec{k})\cdot\vec{e}(\vec{k},\lambda)^*|m\rangle\big|^2,$$

where $\vec{\jmath}(\vec{k}) = \vec{\jmath}(\vec{r})\mathrm{e}^{-\mathrm{i}\vec{k}\vec{r}}$ and $\vec{k}$ in the matrix element has to satisfy $|\vec{k}| = \frac{E_m - E_n}{\hbar c}$.

(6) Show that $\vec{k}\vec{r} \ll 1$. Thus, the exponential series in $\vec{\jmath}(\vec{k})$ can be truncated (long wave approximation; Theoretical Physics II, A8.2).

(7) For the first term of this series, the electric dipole radiation, show that

$$\frac{\mathrm{d}P_{m\to n,\vec{k},\lambda}}{\mathrm{d}\Omega} = \frac{q^2\omega_{\vec{k}}^4}{2\pi c^3}\;\big|\vec{d}_{nm}\cdot\vec{e}(\vec{k},\lambda)^*\big|^2,\quad \text{where}\quad \vec{d}_{nm} = \langle n|\vec{r}|m\rangle.$$

The total power of radiation results in $P_{m\to n} = \frac{4q^2\omega_{\vec{k}}^4}{3c^3}|\vec{d}_{nm}|^2$.

(8) Discuss which transitions are possible. What is the polarisation of the light?

Homework IV

Return: November 19th

Note: In H4.1 and H4.2 we use the notation introduced in E4.1.

H4.1 *Lifetime for dipole transitions*

(1) For the probability $w_{m\to n}$ per time unit for a photon being emitted into the solid angle $\mathrm{d}\Omega$ ($\theta_{\vec{k},\lambda}$ is the angle between $\vec{e}(\vec{k},\lambda)^*$ and $\vec{d}_{nm}$):

$$\frac{\mathrm{d}w_{m\to n,\vec{k},\lambda}}{\mathrm{d}\Omega} = \frac{q^2\omega_{\vec{k}}^3}{2\pi c^3\hbar}|\vec{d}_{nm}|^2\cos^2\theta_{\vec{k},\lambda}.$$

(2) By adding up the polarisations and by integrating the angle, show for the total transition probability per time unit:

$$w_{m\to n} = \frac{4q^2\omega_{\vec{k}}^3}{3c^3\hbar}|\vec{d}_{nm}|^2.$$

(3) The lifetime τ of a state $|m\rangle$ depends on the probabilities via $\frac{1}{\tau} = \sum_n w_{m\to n}$, where the sum contains all allowed final states $|n\rangle$. Compute the lifetime of the $2p$ state of the hydrogen atom.

(15 points)

H4.2 *Higher multipole orders*

The term following the electric dipole radiation in the long wave expansion reads

$$\langle n| -\mathrm{i}\,(\vec{k}\cdot\vec{r})\vec{\jmath}(\vec{r})\cdot\vec{e}(\vec{k},\lambda)^*|m\rangle.$$

(1) Show that this matrix element can be decomposed as follows:

$$\underbrace{-\frac{\mathrm{i}q}{2m}\langle n|(\vec{k}\times\vec{e}(\vec{k},\lambda)^*)\cdot\vec{L}|m\rangle}_{\text{magnetic dipole transition}} \underbrace{-\frac{q}{2}\frac{E_m - E_n}{\hbar}\langle n|(\vec{k}\cdot\vec{r})(\vec{e}(\vec{k},\lambda)^*\cdot\vec{r})|m\rangle}_{\text{electric quadrupole transition}}.$$

(2) Discuss which transitions are possible and when they are. What is the suppression of these transitions compared to electric dipole transitions?

(15 points)