Exercises V

November 19th

E5.1 *The Born Approximation*

The Born approximation for (elastic) potential scattering in first order reads

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{2\pi m^2}{\hbar^4}\left|\int \mathrm{d}^3r \frac{1}{(2\pi)^{\frac{3}{2}}} V(\vec{r})\, \mathrm{e}^{\mathrm{i}\vec{r}\cdot(\vec{k}_i-\vec{k}_f)}\right|^2, \quad \text{where} \quad |\vec{k}_f| = |\vec{k}_i| = k. \qquad (*)$$

Briefly stated, the differential cross section is given by the Fourier transform of the potential, taken at the value of the momentum transfer, i.e.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{2\pi m^2}{\hbar^4}|(\mathcal{F}V)(\vec{q})|^2, \quad \vec{q} = \vec{k}_i - \vec{k}_f.$$

Anticipating the lecture, deduce $(*)$ from the Golden Rule with your tutor's help.

(1) Show for the momentum transfer that $|\vec{q}| = 2k\sin\frac{\vartheta}{2}$. ϑ is the scattering angle.

(2) Perform the angular integration in $(*)$ for a radially symmetric potential:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{4m^2}{\hbar^4}\frac{1}{|\vec{q}|^2}\left|\int_0^\infty \mathrm{d}r\, r\, V(r)\sin(|\vec{q}|r)\right|^2.$$

(3) Consider a Yukawa potential $V(r) = \frac{\kappa}{r}\,\mathrm{e}^{-\frac{r}{r_0}}$. Zeige mit (2), daß

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\kappa}{4E\sin^2\frac{\vartheta}{2} + \frac{\hbar^2}{2mr_0^2}}\right)^2.$$

(4) Which famous formula does one obtain in (3) taking the limit of the Coulomb potential? Does $\hbar$ still occur? Discuss the case of forward scattering.

(5) Show for a Gaussian potential $V(r) = V_0\exp(-\frac{r^2}{2r_0^2})$ that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{2\pi m^2 r_0^6 V_0^2}{\hbar^4}\,\mathrm{e}^{-4k^2r_0^2\sin^2\frac{\vartheta}{2}}.$$

How does this cross section behave for large scattering angles?

(6) Show for a spherically symmetric potential well with depth V_0 and radius r_0:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m^2V_0}{\pi\hbar^4}\frac{1}{|\vec{q}|^4}\left(r_0\cos(|\vec{q}|r_0) - \frac{\sin(|\vec{q}|r_0)}{|\vec{q}|}\right)^2.$$

How does this cross section behave for large scattering energies?

(7) Discuss $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ in (3), (5) and (6) for small energies ($kr_0 \ll 1$). Can one distinguish these potentials by scattering slow projectiles? What do they have in common?

(8) Let $V(\vec{r})$ be the Coulomb potential of a charge distribution ϱ, i.e.

$$V(\vec{r}) = Q\int \mathrm{d}^3r'\,\frac{\varrho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}.$$

Show

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{32\pi^3m^2Q^2}{\hbar^4}\frac{1}{|\vec{q}|^4}|(\mathcal{F}\varrho)(\vec{q})|^2.$$

$|(\mathcal{F}\varrho)(\vec{q})|^2$ is called *form factor*. What does one obtain for a pointlike charge?

Homework V

Return: November 26th

H5.1 *Pion–proton scattering*

(1) Compute $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ for the superposition of a Yukawa and a Coulomb potential:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\kappa}{4E\sin^2\frac{\vartheta}{2} + \frac{\hbar^2}{2mr_0^2}} + \frac{q_1 q_2}{4E\sin^2\frac{\vartheta}{2}} \right)^2 .$$

(2) What will happen if the detector is placed in forward direction ($\vartheta = 0$)?

(3) Compare the three contributions of the Coulomb potential, the Yukawa potential and their interference in $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ depending on E for $\sin\frac{\vartheta}{2} > 0.1$, i.e. away from the forward direction. Let $\kappa = 0.07\,\hbar c$, $r_0 = 1.4\,\mathrm{fm}$, $mc^2 = 140\,\mathrm{MeV}$, $q_1 = q_2 = e$.

(8 points)

H5.2 *Scattering of an electron by a hydrogen atom*

Let the atom be in the ground state. At first, consider elastic scattering.

(1) Compute the total charge density for the hydrogen atom to be

$$\varrho(\vec{r}) = e\delta^{(3)}(\vec{r}) - \frac{e}{\pi r_0^3}\,\mathrm{e}^{-\frac{2r}{r_0}} .$$

(2) Using E5.1.8, compute the differential cross section to be

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{4}{r_0^2}\frac{1}{|\vec{q}|^4}\left(1 - \frac{16}{(4+|\vec{q}|^2 r_0^2)^2}\right)^2 .$$

Now take the scattering to be inelatic, i.e. the atom can be excited to $|n\ell m\rangle$.

(3) Show by transferring the derivation of the formula for the differential cross section for elastic scattering to the inelastic case that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{2\pi m}{\hbar^2|\vec{k}_i|}\int \mathrm{d}k\, k^2 \delta\left(\frac{\hbar^2|\vec{k}\,|^2}{2m} + E_{n\ell} - \frac{\hbar^2|\vec{k}_i|^2}{2m} - E_{1s}\right)$$
$$\times \left| \iint \frac{1}{(2\pi)^{\frac{3}{2}}}\psi^*_{n\ell m}(\vec{r}\,')\psi_{1s}(\vec{r}\,')\left(-\frac{e^2}{r} + \frac{e^2}{|\vec{r}-\vec{r}\,'|}\right)\mathrm{e}^{\mathrm{i}\vec{r}\cdot(\vec{k}_i-\vec{k})}\mathrm{d}^3r\,\mathrm{d}^3r' \right|^2 .$$

Here, k_i denotes the wave number vector of the incoming particle.

(4) Why does the potential of the nucleus not contribute to the inelastic scattering?

(5) Show in (3) by applying the convolution theorem that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{32\pi^3 m^2 e^4}{\hbar^4}\frac{|\vec{k}_f|}{|\vec{k}_i|}\frac{1}{|\vec{q}\,|^4}|(\mathcal{F}(\psi^*_{n\ell m}\psi_{1s}))(\vec{q}\,)|^2 \quad \text{with} \quad |\vec{k}_f|^2 = \frac{2m}{\hbar^2}\left(E_{1s} - E_{n\ell} + \frac{\hbar^2|\vec{k}_i|^2}{2m}\right),$$

$$|\vec{q}\,|^2 = 2\left(|\vec{k}_i|^2 + \frac{(E_{1s}-E_{n\ell})m}{\hbar^2} - \frac{|\vec{k}_i|}{\hbar}\sqrt{m\left(2(E_{1s}-E_{n\ell}) + \frac{\hbar^2|\vec{k}_i|^2}{m}\right)}\cos\vartheta\right).$$

(6) Compute $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ explicitly for the case that the atom is excited to the $2s$ state.

(22 points)