Exercises VI

November 26th

E6.1 *Expansion of the planar wave in terms of Bessel functions*

The goal of this assignment is to prove the identity

$$e^{i\vec{x}\cdot\vec{y}} = 4\pi\sum_{\ell=0}^{\infty} i^\ell j_\ell(|\vec{x}||\vec{y}|)\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\frac{\vec{x}}{|\vec{x}|})Y_{\ell m}(\frac{\vec{y}}{|\vec{y}|}) = \sum_{\ell=0}^{\infty} i^\ell(2\ell+1)j_\ell(|\vec{x}||\vec{y}|)P_\ell(\frac{\vec{x}}{|\vec{x}|}\cdot\frac{\vec{y}}{|\vec{y}|})$$

describing the planar wave, i.e. the motion of a free particle, in spherical coordinates. Here, the spherical Bessel function $j_\ell(kr)$ is the solution (regular at the origin) of the free radial Schrödinger equation with angular momentum ℓ and energy $E = \frac{\hbar^2 k^2}{2m}$.

(1) Explain why the expansion $e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} c_{\ell m}(\frac{\vec{k}}{|\vec{k}|})j_\ell(|\vec{k}|r)Y_{\ell m}(\vartheta,\varphi)$ holds.

(2) Show $e^{ikr\cos\vartheta} = \sum_{\ell=0}^{\infty}\sqrt{\frac{2\ell+1}{4\pi}}A_\ell j_\ell(kr)P_\ell(\cos\vartheta)$ for the choice $\vec{k} = k\vec{e}_z$.

(3) The P_ℓ are orthogonal. Deduce $A_\ell j_\ell(kr) = \frac{1}{2}\sqrt{4\pi(2\ell+1)}\int_{-1}^{1} dz P_\ell(z)e^{ikrz}$.

(4) With Rodrigues' formula, show: $\int_{-1}^{1} dz P_\ell(z)e^{ikrz} = \frac{i^\ell 2^{\ell+1}\ell!}{(2\ell+1)!}(kr)^\ell + \mathcal{O}((kr)^{\ell+1})$.

(5) From the asymptotics of $j_\ell(z)$ for $z\to 0$, deduce that $A_\ell = i^\ell\sqrt{4\pi(2\ell+1)}$.

(6) Finally, using the addition theorem for $Y_{\ell m}$, show the general formula.

E6.2 *Partial wave decomposition*

In the lecture, the scattering amplitude was written in terms of Legendre polynomials:

$$f(\vartheta) = \sum_{\ell=0}^{\infty}(2\ell+1)f_\ell(k)P_\ell(\cos\vartheta), \quad f_\ell(k) = \frac{1}{2ik}(e^{2i\delta_\ell(k)}-1) = \frac{1}{k}e^{i\delta_\ell(k)}\sin\delta_\ell(k).$$

(1) Sketch the angular dependance of the differential cross section $\frac{d\sigma}{d\Omega}(\vartheta)$ for pure s and p wave scattering, respectively, and for the interference of the two terms.

(2) Sketch the *Argand amplitude* $A_\ell = kf_\ell$ in the complex plane.

(3) Write the total cross section $\sigma_{\rm tot}$ as a function of the partial total cross sections $\sigma_\ell = 4\pi(2\ell+1)\frac{1}{k^2}\sin^2\delta_\ell(k)$. Which function sets a bound on the total cross section? When is the value of this bound really taken?

(4) State a relation between $\sigma_{\rm tot}$ and the forward scattering amplitude.

If $\frac{u_\ell(r)}{r}$ solves the radial Schrödinger equation then for the scattering phase one has:

$$\sin\delta_\ell = -\frac{2mk}{\hbar^2}\int_0^\infty rV(r)u_\ell(r)j_\ell(kr)dr \quad \text{with} \quad k^2 = \frac{2mE}{\hbar^2}.$$

The proof of this formula uses the asymptotics of the solutions and applies the Wronskian (Theoretical Physics II, H4.3). One obtains the first Born approximation (equivalent to the ansatz in E5.1) by replacing the actual, but in general unknown solution $u_\ell(r)$ by the solution $rj_\ell(kr)$ of the free Schrödinger equation.

(5) Let $kR \ll 1$. Compute δ_0 and σ_0 for a potential well or barrier $V(r) = V_0\Theta(R-r)$.

(6) Calculate the *scattering length* $a = -\lim_{k\to 0} f_0(k)$ and write σ_0 in terms of a.

(7) Compare the scattering phases for a repulsive and an attractive potential. How does the wave function change compared to the free s wave? Sketch $r\psi(kr)$.

Homework VI

Return: December 3rd

H6.1 *Scattering on a hard sphere*

For a radially symmetric potential $V(r)$ vanishing for $r > a$, one can write the solution of the Schrödinger equation for $r > a$ as

$$\psi(\vec{r}) = \frac{1}{\sqrt{2\pi}^3}\sum_{\ell=0}^{\infty} \mathrm{i}^\ell (2\ell+1) A_\ell(r) P_\ell(\cos\vartheta),$$

where the radial wave functions for $r > a$ are given by

$$A_\ell(r) = \mathrm{e}^{\mathrm{i}\delta_\ell}\left(\cos\delta_\ell j_\ell(kr) - \sin\delta_\ell n_\ell(kr)\right),$$

such that one has $A_\ell(r) = j_\ell(kr)$ for the free case. Here, $j_\ell(z) = (-z)^\ell \left(\frac{1}{z}\partial_z\right)^\ell \frac{\sin z}{z}$ are called spherical Bessel and $n_\ell(z) = -(-z)^\ell \left(\frac{1}{z}\partial_z\right)^\ell \frac{\cos z}{z}$ are called spherical Neumann functions. Consider the scattering problem for an infinitely hard sphere of radius a.

(1) From the condition $A_\ell(a) = 0$, deduce a formula for $\tan\delta_\ell$.

(2) Give the total cross section σ_{tot} in the limits of low ($ka \ll 1$) and high energies ($ka \gg 1$), respectively. Make use of the asymptotic behaviour of j_ℓ and n_ℓ in order to show for $ka \ll 1$ that the sum is dominated by the s wave contribution. For $ka \gg 1$, break the sum at $\ell_{\max} \approx ka$. With $(2n+1)!! = 1\cdot 3\cdot 5\cdot\ldots\cdot(2n+1)$, the asymptotic behaviour is:

$$\lim_{z\to 0}: \quad j_\ell(z) \sim \frac{z^\ell}{(2\ell+1)!!} \qquad n_\ell(z) \sim -\frac{(2\ell-1)!!}{z^{\ell+1}}$$

$$\lim_{z\to\infty}: \quad j_\ell(z) \sim \frac{1}{z}\sin\left(z - \frac{\ell\pi}{2}\right) \qquad n_\ell(z) \sim -\frac{1}{z}\cos\left(z - \frac{\ell\pi}{2}\right)$$

(3) Compare the results with the classically expected ones.

(10 points)

H6.2 *Resonance scattering on an idealised spherical shell*

With the notation of H6.1, consider the potential $V(r) = \lambda\frac{\hbar^2}{2m}\delta(r-a)$.

(1) Use the matching conditions at $r = a$ in order to find an equation for $\tan\delta_\ell$ fixing the scattering phases. For $r < a$ take the ansatz $A_\ell(r) \sim j_\ell(kr)$ and argue why the $n_\ell(kr)$ are not allowed to appear for $r < a$. Show that the limit $\lambda \to \infty$ leads to the expression for the scattering on an infinitely hard sphere.

(2) Restrict the following considerations on s wave scattering in the low energy limit, i.e. $\lambda a \gg ka$. The energy of the scattering particles is $E_k = \frac{\hbar^2 k^2}{2m}$. Show that in this limit resonance behaviour will appear for certain energies. In this limit one has $|\tan ka| \ll 1$ and resonance appears for vanishing $\cot\delta_0$.

(3) Expand the scattering amplitude $f_0 = \frac{1}{k\cot\delta_0 - \mathrm{i}k}$ around the resonance energy E_{res} and express the result by the *resonance width* $\Gamma = -\frac{1}{\frac{\mathrm{d}}{\mathrm{d}E}\cot\delta_0|_{E=E_{\mathrm{res}}}}$.

(4) State the total cross section. How does this resonance behave in the limit $\lambda \to \infty$?

(20 points)