# Exercises VII

December 3rd

E7.1 *Algebra of fermion operators*

Consider a pair of an anticommuting fermionic creation and annihilation operator, i.e. $\{a, a^+\} = 1$. The occupation number operator is $N = a^+ a$.

(1) Let $|n\rangle$ be an eigenstate of $N$ with eigenvalue $n$. Is $n$ real?

(2) Show that $n \geq 0$. Also show that $n$ is bounded from above.

(3) Show that $a^+|n\rangle$ and $a|n\rangle$ are eigenstates of $N$. Compute the eigenvalues.

(4) Figure out which values $n$ can take. Discuss Pauli's principle.

(5) Which statements on $n$ can be made in the bosonic case $[a, a^+] = 1$?

E7.2 *Low energy states of an interacting Bose gas*

We would like to consider the ground state of a many particle system as a vacuum, its lowest excitations are called quasi particles. The so-called Bogoliubov transformation connects the creation and annihilation operators of quasi and elementary particles. The Hamiltonian for bosons with interaction potential $u$ in a volume $V$ reads

$$H = \sum_{\vec{p}} \frac{\vec{p}^2}{2m} a^+_{\vec{p}} a_{\vec{p}} + \frac{1}{2} \sum_{\vec{p}_1, \vec{p}_2, \vec{p}'_1, \vec{p}'_2} \delta_{\vec{p}_1 + \vec{p}_2, \vec{p}'_1 + \vec{p}'_2} U_{\vec{p}'_1 \vec{p}'_2 | \vec{p}_1, \vec{p}_2} a^+_{\vec{p}'_1} a^+_{\vec{p}'_2} a_{\vec{p}_2} a_{\vec{p}_1},$$

where the interaction term $U_{\vec{p}'_1 \vec{p}'_2 | \vec{p}_1 \vec{p}_2} = \frac{1}{V} \int \mathrm{e}^{i\vec{p}\vec{r}/\hbar} u(\vec{r}) \mathrm{d}^3 r$ for momentum transfer $\vec{p} = \vec{p}_2 - \vec{p}'_2 = \vec{p}'_1 - \vec{p}_1$ is the Fourier transform of a potential $u(\vec{r})$. The delta function for the conservation of momentum is due to the dependence of the potential on the relative distance of the two particles (which can thus be expressed by $u(\vec{r})$) only. For low temperatures only small momenta will arise, thus in lowest order we have

$$H = \sum_{\vec{p}} \frac{\vec{p}^2}{2m} a^+_{\vec{p}} a_{\vec{p}} + \frac{u_0}{2V} \sum_{\vec{p}_1, \vec{p}_2, \vec{p}'_1, \vec{p}'_2} \delta_{\vec{p}_1 + \vec{p}_2, \vec{p}'_1 + \vec{p}'_2} a^+_{\vec{p}'_1} a^+_{\vec{p}'_2} a_{\vec{p}_2} a_{\vec{p}_1}$$

where the interaction term is approximated by $u_0 = \int u(\vec{r}) \mathrm{d}^3 r$. Looking at the lowest excitations of the interacting Bose gas, almost all particles should be in the ground state such that the particle number $N$ is approximately $N_0$. Now we expand the interaction term in such a way that at most two of the four momenta do not vanish.

(1) Show $a^+_0 a^+_0 a_0 a_0 = N_0^2 - N_0 \approx N^2 - 2N \sum_{\vec{p} \neq 0} a^+_{\vec{p}} a_p$ .

(2) Compute how $a_0$ and $a^+_0$ act on an $n-$particle state $\Phi_n = \frac{1}{\sqrt{n!}} (a^+_0)^n |0\rangle$: $a_0 \Phi_n = \sqrt{n} \Phi_{n-1}$, $a^+_0 \Phi_n = \sqrt{n+1} \Phi_{n+1}$. Deduce that $a_0$ and $a^+_0$ can be represented as multiplication by $\sqrt{n}$ for large $n$. Show that

$$H \approx \frac{u_0 N^2}{2V} + \frac{u_0 N}{2V} \sum_{\vec{p} \neq 0} \left( a^+_{\vec{p}} a^+_{-\vec{p}} + a_{\vec{p}} a_{-\vec{p}} + 2 a^+_{\vec{p}} a_{\vec{p}} \right) + \sum_{\vec{p} \neq 0} \frac{\vec{p}^2}{2m} a^+_{\vec{p}} a_{\vec{p}} .$$

(3) For $\alpha_p \in \mathbb{R}$ with $|\alpha_p| < 1$ consider the *Bogoliubov transformation* from $a_p$ to $b_p$:

$$b_{\vec{p}} = \frac{a_{\vec{p}} + \alpha_{\vec{p}}\ a^+_{-\vec{p}}}{\sqrt{1 - \alpha^2_{\vec{p}}}} \quad \text{und} \quad b^+_{\vec{p}} = \frac{a^+_{\vec{p}} + \alpha_{\vec{p}} a_{-\vec{p}}}{\sqrt{1 - \alpha^2_{\vec{p}}}}.$$

Show

$$[b_{\vec{p}}\,, b^+_{\vec{p}'}] = \delta_{\vec{p},\vec{p}'}\,,\;\; [b_{\vec{p}}\,, b_{\vec{p}'}] = [b^+_{\vec{p}}\,, b^+_{\vec{p}'}] = 0,$$

i.e. the $b_{\vec{p}}\,, b^+_{\vec{p}}$ correspond to bosonic *quasi particles*, so-called bogolons.

(4) Show that by an appropriate choice of $\alpha_p$ the Hamiltonian becomes diagonal:

$$H = E_0 + \sum_{\vec{p}\neq 0} \varepsilon(\vec{p}) b^+_{\vec{p}} b_{\vec{p}}\,.$$

Compute $E_0$ and $\varepsilon(\vec{p})$ and interpret these quantities.

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# Homework VII

Return: December 10th

**H7.1** *Hamiltonian of the one dimensional Hubbard model* (20 points)

$N$ Elektronen placed on a chain with sites $x_i$ (with distance $a$) are described by

$$H_0 = -t \sum_{i=1}^{N} \sum_{\sigma} (a^+_\sigma(x_i) a_\sigma(x_i + a) + a^+_\sigma(x_i) a_\sigma(x_i - a)),$$

where the two terms correspond to the motion to the left and to the right. The $a(x_i)$ satisfy canonical anticommutation relations $\{a_\sigma(x_i), a^+_{\sigma'}(x_j)\} = \delta_{\sigma,\sigma'} \delta_{x_i,x_j}$. The spin indices $\sigma$ can take the values $\uparrow \equiv \frac{1}{2}$ and $\downarrow \equiv -\frac{1}{2}$.

(1) Show that in the momentum representation $H_0$ has the following form:

$$H_0 = -\frac{1}{2\pi} \sum_\sigma \int \mathrm{d}k\, \varepsilon(k) a^+_\sigma(k) a_\sigma(k).$$

State the explicit form of the dispersion relation $\varepsilon(k)$. For that purpose, make the transition (in the limit of a long chain) from the discrete Fourier transform $a_\sigma(x) = \frac{1}{L} \sum_k \mathrm{e}^{\mathrm{i}kx} a_\sigma(k)$ to the continuous one $a_\sigma(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}kx} a_\sigma(k)$.

(2) Consider now an additional interaction (what does that mean physically?)

$$H' = \frac{U}{2} \sum_{i,\sigma} a^+_\sigma(x_i) a_\sigma(x_i) a^+_{-\sigma}(x_i) a_{-\sigma}(x_i).$$

Show that the $M_\sigma = \sum_i a^+_\sigma(x_i) a_\sigma(x_i)$ have good quantum numbers with respect to the total Hamiltonian $H = H_0 + H'$. What is the physical meaning of $M_\sigma$?

(3) The three components of the spin operator are defined by ($\sigma^i$ are Pauli's matrices)

$$S^i_n = \frac{\hbar}{2} \sum_{\sigma,\sigma'} a^+_\sigma(x_i) \sigma^i_{\sigma\sigma'} a_{\sigma'}(x_i).$$

Do these operators satisfy the angular momentum algebra?

**H7.2** *Distribution function* (10 points)

Consider non-interacting fermions or bosons which can take states $\lambda$, with operators $\{a_\lambda, a^+_{\lambda'}\} = \delta_{\lambda\lambda'}$ or $[a_\lambda, a^+_{\lambda'}] = \delta_{\lambda\lambda'}$. Let $N_\lambda = a^+_\lambda a_\lambda$ be the number operator for $\lambda$.

(1) State the Hamiltonian $H$ and the occupation number operator $N$.

(2) Let $\varrho = \mathrm{e}^{-\beta(H-\mu N)}$ be the density operator of the underlying grand canonical ensemble. First, compute the *partition function* $Z = \mathrm{tr}(\varrho)$ and then the average occupation number $n_\lambda = \frac{1}{Z}\mathrm{Spur}(\varrho N_\lambda)$ in the state $\lambda$ for both kinds of particles.