Exercises IX

December 16th

E9.1 *Contra and Covariance*

We consider the space $\mathbb{R}^4$ with the *Minkowski metric* $g = \mathrm{diag}(1, -1, -1, -1)$ and

$$\langle x, x' \rangle := \sum_{\mu,\nu=0}^{3} g_{\mu\nu} x^\mu x'^\nu = c^2 t t' - x^1 x'^1 - x^2 x'^2 - x^3 x'^3 =: x^0 x'^0 - \vec{x} \cdot \vec{x}'$$

for *four vectors*

$$x = \begin{pmatrix} c\,t \\ \vec{x} \end{pmatrix} = \begin{pmatrix} x^0 \\ \vec{x} \end{pmatrix}.$$

We call maps $x \mapsto \Lambda x$, in components: $x^\mu \mapsto \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu =: \Lambda^\mu{}_\nu x^\nu$ Lorentz transformations, if $\langle \Lambda x, \Lambda x \rangle = \langle x, x \rangle$, leading to the condition $\Lambda^T g \Lambda = g$ on Λ. The four vectors defined in this way are called *contravariant*. The corresponding *covariant* "vectors" (with lower indices) are defined as

$$x_{\mathrm{cov}} = (c\,t, -x^1, -x^2, -x^3) = (x^0, -\vec{x}^T) = x^T g,$$

in components: $x_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} x^\nu =: g_{\mu\nu} x^\nu$. Then we have

$$\langle x, x' \rangle = x_{\mathrm{cov}} \cdot x' = x_\mu x'^\mu = x'_\mu x^\mu = x'_{\mathrm{cov}} \cdot x.$$

(1) Expand the condition $\Lambda^T g \Lambda = g$ in components: $\Lambda^\mu{}_\kappa g_{\mu\nu} \Lambda^\nu{}_\lambda = g_{\kappa\lambda}$.

(2) Show $|\Lambda^0{}_0| \geq 1$ and $\det\Lambda = \pm 1$, leading to four branches. The branch with $\mathbb{1}$ ($\Lambda^0{}_0 \geq 1$, $\Lambda^0{}_0 \geq 1$) is called $\mathcal{L}_+^\uparrow$, the proper (no reflection) orthochrone (conserving the time direction) transformations. Show that they indeed form a group.

(3) Show $\langle x, x \rangle = x_{\mathrm{cov}} g^{-1} x_{\mathrm{cov}}^T = g^{\mu\nu} x_\mu x_\nu$, where $g^{\mu\nu} := (g^{-1})_{\mu\nu}$, i.e. $g^{\mu\nu} = g_{\mu\nu}$ here.

(4) Show that under a Lorentz transformation $x_{\mathrm{cov}} \mapsto x_{\mathrm{cov}} \Lambda^{-1}$ holds. Show for the inverse matrix in components: $(\Lambda^{-1})^\mu{}_\nu = g_{\nu\beta} \Lambda^\beta{}_\alpha g^{\alpha\mu} =: \Lambda_\nu{}^\mu$, d.h. $x_\mu \mapsto \Lambda_\mu{}^\nu x_\nu$.

(5) We define the derivatives $\partial^\mu = \frac{\partial}{\partial x_\mu}$ and $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Show that the derivative with respect to the contravariant coordinates transforms as a covariant four vector ("an upper index in the denominator is a lower index") and vice versa.

(6) Show for the wave operator that $\Box = \partial^\mu \partial_\mu = \partial_\mu \partial^\mu$ and deduce its Lorentz invariance. One can interpret the wave operator as Laplacian in Minkowski space, the rotational invariance of Δ is transferred to the Lorentz invariance of $\Box$.

E9.2 *Covariant electrodynamics*

(1) For a charge and current distribution we define $j^0 = c\varrho$, $j^i = (\vec{\jmath})^i$. Show that the continuity equation can be written as $\partial_\mu j^\mu = 0$. Why does this imply that j is a four vector, i.e. under Lorentz transformations $j'^\mu(x') = \Lambda^\mu{}_\nu j^\nu(x)$ holds.

(2) Let $\vec{j} = 0$ in a system. What is j in a system moving with the velocity $\vec{v}$ relative to it? Deduce that the total charge $q = \frac{1}{c}\int \mathrm{d}^3r\, j^0(x)$ is Lorentz invariant, although this quantity is not manifestly covariant.

(3) For the electromagnetic potentials we define $A^0 = \Phi$, $A^i = (\vec{A})^i$. Show that the Maxwell equations read $\Box A^\mu(x) = \frac{4\pi}{c} j^\mu(x)$, using the Lorentz gauge $\partial_\mu A^\mu = 0$. Why does this imply that A is a four vector, i.e. under Lorentz transformations $A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x)$ holds. Why does the Lorentz gauge hold in each system?

(4) We define the electromagnetic field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Why is it invariant under gauge transformations $A^\mu \to A^\mu - \partial^\mu \chi$? Show that the inhomogeneous Maxwell equations read $\partial_\mu F^{\mu\nu}(x) = \frac{4\pi}{c} j^\nu(x)$. How does $F^{\mu\nu}$ transform under Lorentz transformations? Write F as a matrix and express its entries by the electromagnetic fields $\vec{E}$ and $\vec{B}$.

(5) How does a Coulomb field transform to a boosted system?

(6) Write the homogeneous Maxwell equations with the dual field strength tensor $\widetilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} g_{\alpha\gamma} g_{\beta\delta} F^{\gamma\delta}$. Write $\widetilde{F}$ as matrix, too.

E9.3 *Relativistic kinematics*

The four momentum of a particle with velocity $\vec{v}$ is $p^\mu = m\dot{x}^\mu(\tau)$, where the eigen time τ is related to x^0 via $\mathrm{d}\tau = \frac{1}{c\gamma}\mathrm{d}x^0$ ($\gamma = (1-\beta^2)^{-\frac{1}{2}}$, $\beta = \frac{|\vec{v}|}{c}$).

(1) Show that the Minkowski length of the four momentum is the “mass” mc.

(2) Motivate the definiton $\vec{p} = \gamma m \vec{v}$. Show $p^0 = \frac{E}{c}$ with $E = \sqrt{\vec{p}^2c^2 + m^2c^4}$.

(3) Why can a photon not decay into an elektron–positron pair in the vacuum? In which manner can such a pair creation happen anyway?

(4) When a proton collides with a proton at rest, a proton–antiproton pair may be created in addition. Both created particles have the same mass. Find out, how large the initial energy has to be such that the reaction will take place. Use four momentum conservation and deduce a condition for the energy.

(5) How large is γ at the energy threshold? Why is the threshold not $2mc^2$?

Homework IX

Return: January 7th

H9.1 *Covering of the Lorentz group*

(1) For a four vector p with $\langle p, p\rangle = 1$ and $p^0 > 0$, consider the map

$$\Lambda_p\, x := x - 2\langle p, x\rangle p.$$

Show that Λ_p is Lorentz transformation.

(2) Show that $g_p = \Lambda_p \Lambda_{e_0}$ with $(e^\mu)_0 = \delta^\mu{}_0$ is proper and orthochrone.

(3) Show for each orthochrone Λ: $\Lambda e_0 = g_p e_0$ with a unique $p \in \mathbb{R}^{1,3}$.

(4) Deduce from (3) that each $\Lambda \in \mathcal{L}$ can be expressed as

$$\Lambda = g_p \mathcal{O} P^\ell T^k,$$

using parity P and time reversal T, where $\mathcal{O} \in SO(3)$ and $k, \ell \in \{0, 1\}$.

For the rotational group we defined a *covering map* $\tilde{h} : SU(2) \to SO(3)$ in such a way (TP II, E9.1), that for given $g \in SU(2)$ the identity

$$g\sigma(\vec{a})g^{+} = \sigma(\tilde{h}(g)\vec{a}) \qquad (*)$$

holds for all $\vec{a} \in \mathbb{R}^3$. Here, $\sigma(\vec{a}) = \sum_{i=1}^{3} a^i \sigma_i$ with the Pauli matrices σ_i is the most general description for a hermitian matrix with vanishing trace.

We generalise the condition $(*)$ by setting $\sigma(a) = a^0 \mathbb{1} + \sigma(\vec{a})$ to four vectors a, yielding a general hermitian matrix. As we will see in the following, this defines a new covering $h : SL(2,\mathbb{C}) \to \mathcal{L}_+^\uparrow$. $SL(2,\mathbb{C})$ is the group of the complex 2×2 matrices with determinant one, this group includes $SU(2)$.

(5) Show that h is a homomorphism: $h(g_1 g_2) = h(g_1)h(g_2)$ and $h(\mathbb{1}) = \mathbb{1}$.

(6) Show that $h(g)$ is a Lorentz transformation only for $|\det g| = 1$ and that one can restrict the considerations to $g \in SL(2,\mathbb{C})$. First, show that $\det \sigma(a) = \langle a, a \rangle$.

(7) Why can one define $\sqrt{A^+A}$ for an invertable matrix A in a unique way?

(8) Deduce from (7) that each invertable matrix can be written as a product of a unitary and a positive definite hermitian matrix.

(9) Show that $\sigma(p)\sigma(a)\sigma(p) = \sigma(g_p a)$.

(10) With (8) and (9) show that Im $h = \mathcal{L}_+^\uparrow$.

(11) From the known fact Ker $\tilde{h} = \{\pm\mathbb{1}\}$, deduce that Kern $h = \{\pm\mathbb{1}\}$.

(12) Show that $h(\mathrm{e}^{\frac{u}{2}\sigma_i})$ is a boost with rapidity u ($\tanh u = \beta$) in i-th direction.

(15 points)

H9.2 *Planck's radiation formula*

For a photon gas one has $\varepsilon(\vec{p}) = c|\vec{p}| = \hbar ck$. The spin has only two directions, since electromagnetic waves are transversal (or the photon has zero mass, respectively).

(1) Why does the chemical potential for photons vanish, i.e. $\mu = 0$?

(2) Show that one has for the partition function in the box: $Z = \prod_{\vec{p}\neq 0}(1 - \mathrm{e}^{-\beta c|\vec{p}|})^{-2}$. Whay are there no complications concerning the ground state (like Bose condensation), although the fugacity always has the critical value $z = 1$ due to $\mu = 0$?

(3) Show that $\Phi = -\frac{4\sigma}{3c} VT^4$, where σ is the Stefan–Boltzmann constant (cf. (6)):

$$\sigma = \frac{\pi^2 k^4}{60\hbar^3 c^2}.$$

(4) Why is $\frac{V}{\pi^2 c^3} \frac{\omega^2 \mathrm{d}\omega}{\mathrm{e}^{\hbar\omega/kT} - 1}$ the number of occupied states in the frequency interval $[\omega, \omega + \mathrm{d}\omega]$? The spectral energy density $u(\omega)$ is this expression multiplied by the energy per volume and frequency unit. Show

$$u(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{\mathrm{e}^{\hbar\omega/kT} - 1}.$$

(5) Show for the maximum of u: $\hbar\omega_{\max} = 2.82kT$ (*Wien's shift law*).

(6) Show that the radiation energy leaving the box per time unit and square unit is given by $I(\omega, T) = \frac{c}{4\pi} \int \mathrm{d}\Omega\, u(\omega) \cos\vartheta = \frac{c}{4} u(\omega)$. Deduce that the radiated power per square unit is $I(T) = \sigma T^4$ (*Stefan–Boltzmann law*).

This law was found in 1900 by M. Planck using heuristic assumptions. It was the beginning of quantum theory, since classical physics (Rayleigh/Jeans) totally failed. The shift law was observed by M. Wien earlier.

(7) What is the average temperature on earth from (6)? Why is it higher in reality?

(8) Compute the temperature of heaven, if you take the follwing vers Isaiah 30,26 literally: "At the time, when the Lord will heal the sufferings of His people and will dress its wounds, the light of the moon shall be as the light of the sun and the light of the sun shall be sevenfold, as the light of seven days." In Revelations, the "Lake which burneth with fire and brimstone" is mentioned several times. From that, estimate the temperature of hell.

(10 points)

H9.3 *Angular momentum tensor*

For a system of N particles with spacetime coordinates x_i and four momenta p_i, $1 \leq i \leq N$ we define the tensor $M^{\mu\nu} = \sum_{i=1}^{N}(x_i^\mu p_i^\nu - x_i^\nu p_i^\mu)$.

(1) What does this tensor have in common with the field strength tensor $F^{\mu\nu}$?

(2) What are the spatial components M^{jk} of this tensor?

(3) Show that the components M^{0j} form the vector $c\sum_{i=1}^{N}(t\vec{p}_i - \frac{1}{c^2}E_i\vec{r}_i)$.

(4) One can show that $M^{\mu\nu}$ is a conserved quantity. Interpret M^{0i} being constant. What does one obtain in the nonrelativistic limit?

(5 points)

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first check list for the final exam

We would like to suggest, that you recapitulate the following topics:

Clebsch–Gordan coefficients, addition of angular momenta: E1.1

Wigner–Eckart theorem: E2.1 (especially selection rules), H2.1 (application of the projection theorem)

time-dependent perturbation theory: E3.1, E4.1 (mainly concepts like Fermi's golden rule, interpretation of the delta distribution as density of states, selection rules)

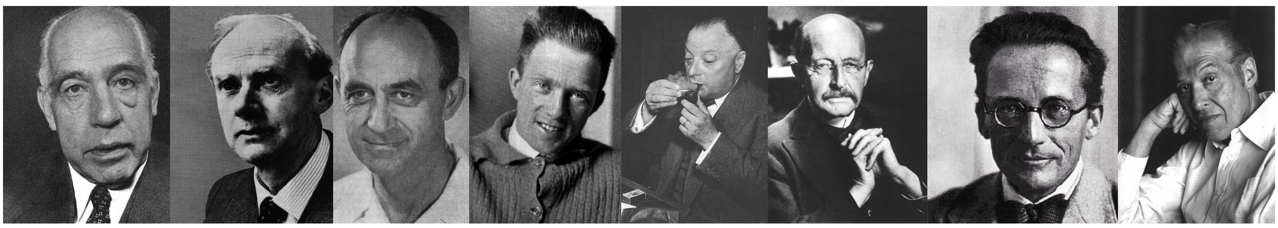
scattering theory: E5.1 (Born approximation), E6.2 (partial waves)

many particle theory: E7.1, E7.2 (again the concepts like momentum representation, one and two particle operators)

statistical physics: E8.1 (things like the definition of the partition function, distribution function for bosons and fermions)

theory of relativity: E9.1, E9.2, E9.3, H9.1 (only the statements)

Niels, Paul, Enrico, Werner, Wolfgang, Max, Erwin und Julian wish you



MERRY CHRISTMAS AND A HAPPY NEW YEAR !