Exam

February 8th, 2003

 # 1 *Addition of angular momenta* (15 points)

Do the addition of angular momenta for $j_1 = j_2 = \frac{1}{2}$ explicitly. Start with the states $|\frac{1}{2}, m_1\rangle \otimes |\frac{1}{2}, m_2\rangle$ and use the action of the raising and lowering operators.

 # 2 *Wigner-Eckart theorem* (4+6+5+3=18 points)

(1) What is a spherical tensor operator of rank r? State an example for $r = 1$.

(2) Give the statements of the Wigner-Eckart theorem and illustrate them briefly.

(3) Which operator is responsible for electric dipole transitions? Show which values for $\Delta\ell$ are possible due to the Wigner-Eckart theorem.

(4) Deduce a further restriction on $\Delta\ell$ from the invariance under parity.

 # 3 *Hyperfine structure* (4+6+8+2+6+5=31 points)

The hyperfine interaction in the hydrogen atom is given by the Hamiltonian

$$H' = \underbrace{-8\pi\mu_e\mu_p\delta^{(3)}(\vec{r})\vec{S}_e \cdot \vec{S}_p}_{H_{\mathrm{HF}}} + 2\,\frac{\mu_e\mu_p}{r^3}(\vec{S}_e \cdot \vec{S}_p - \frac{3}{r^2}(\vec{r} \cdot \vec{S}_e)(\vec{r} \cdot \vec{S}_p)),$$

where $\mu_{e,p}$ are the gyromagnetic ratios of electron and proton, $\vec{S}_{e,p}$ their spins, respectively. $\vec{r}$ denotes their relative distance. These interactions couple the angular momentum $\vec{J}$ of the electron with the proton spin to $\vec{F} = \vec{J} + \vec{S}_p = \vec{L} + \vec{S}_e + \vec{S}_p$.

(1) Why are the states of the full Hamiltonian characterised by quantum numbers ℓ, j, f, m_f, where $\hbar^2 f(f+1)$ is the eigenvalue of $\vec{F}^2$ and $\hbar m_f$ the one of F_3?

(2) Using the projection theorem, show that

$$\langle \ell j f m_f|\vec{S}_e|\ell j f m_f\rangle = \frac{1}{2j(j+1)}(j(j+1) + \frac{3}{4} - \ell(\ell+1))\langle \ell j f m_f|\vec{J}|\ell j f m_f\rangle.$$

(3) Compute the energy shift by H_{HF} in first order perturbation theory.

(4) Which states are split energetically by H_{HF} in this approximation?

(5) Which values for f are possible? Which of them has lower energy?

(6) By symmetry considerations, show that the second term in H' does not contribute for S states in first order perturbation theory.

 # 4 *Scattering theory* (3+9+12+2=26 points)

(1) How does the differential cross section for elastic scattering on a potential $V(\vec{r})$ look like in Born approximation?

(2) Let $V(\vec{r})$ be the Coulomb potential of a charge density $\varrho(\vec{r}\,')$. Show

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{32\pi^3 m^2}{\hbar^4}\frac{1}{|\vec{q}|^4}|(\mathcal{F}\varrho)(\vec{q})|^2$$

in Born approximation ($\mathcal{F}$ Fourier transform). Find the result for a point-charge.

(3) A particle of mass m is scattered on a spherically symmetric potential well $V(r) = V_0\Theta(R-r)$. We consider S wave scattering only. Compute the scattering phase $\delta_{\ell=0}(k)$ by solving the Schrödinger equation exactly.

(4) State scattering amplitude and cross section for (3).

 # 5 *Many particle theory* (3+4+6=13 points)

(1) State explicitly the wave function for a system of N identical, non interacting fermions with one particle states $|\psi_i\rangle$.

(2) Which algebraic relations do the creation and annihilation operator for bosons satisfy? How does the occupation number operator look like?

(3) Consider a many particle system with dispersion relation $\varepsilon(\vec{p})$ and a two-particle interaction V, i.e. V depends only on the relative distance of the particles. State the Hamiltonian, written in terms of creation and annihilation operators.

6 *Quantum statistics* (3+6=9 points)

A particle with spin $\frac{1}{2}$ is in thermal equilibrium with absolute temperature T, under the influence of a constant magnetic field $\vec{B}$.

(1) How does the statistical operator ϱ describing the interaction with $\vec{B}$ look like?

(2) Compute the expectation value of S_3, where $\vec{B}$ may be directed in z-direction.

7 *Questions on relativistic quantum mechanics* (2+4+4+8+4+6+14+3=45 points)

(1) State the free Klein-Gordon equation. Why is it Lorentz invariant?

(2) Which algebra do the γ matrices satisfy? Show that $\operatorname{Tr}\gamma^\mu = 0$.

(3) How do the two inequivalent representations of the Lorentz group for spin $\frac{1}{2}$ on two-component spinors look like?

(4) Deduce from the Lorentz covariance of the Dirac equation: $\psi'(x') = S\psi(x)$ and the properties of the matrix S. Give S explicitly in Weyl representation.

(5) State the Pauli equation. What does that equation predict for the energy of a particle with spin $\frac{1}{2}$ in a magnetic field?

(6) State the parity operator for the Dirac equation. Give reasons for the answer.

(7) Give the Dirac scalar and the Dirac axial vector current. Show how they transform under proper orthochrone Lorentz transformations and under parity.

(8) Sketch the energy levels for the solution of the Dirac equation for the hydrogen atom. Which degeneracy is not broken until the Lamb shift is taken into account?

8 *Gauge invariance of the Dirac equation* (8 points)

Consider a charged particle with spin $\frac{1}{2}$ in an electromagnetic potential $A^\mu(x)$. Show how the Dirac spinor of the particle has to transform under gauge transformations $A^\mu(x) \to A^\mu(x) + \partial^\mu\lambda(x)$ such that the Dirac equation remains valid.

9 *Electron in a constant magnetic field* (9+9+10+4+3=35 points)

Consider an electron ($q{=}{-e}$) in a constant magnetic field $\vec{B} = B\vec{e}_z$. Choose $\vec{A} = Bx\vec{e}_y$ and the γ^μ in standard representation. We look for stationary states with energy E.

(1) Take the ansatz $\psi(\vec{r},t) = \mathrm{e}^{-\mathrm{i}Et/\hbar}\binom{\xi(\vec{r})}{\eta(\vec{r})}$ and show that

$$\left(-\hbar^2\Delta + \frac{e^2}{c^2}B^2x^2 + \frac{e}{c}B\left(\hbar\sigma_z + 2x\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial y}\right)\right)\xi(\vec{r}) = \left(\frac{E^2}{c^2} - m^2c^2\right)\xi(\vec{r}).$$

(2) Why can one take the ansatz $\xi(\vec{r}) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p_y y + p_z z)}f(x)$? With $w = \sqrt{\frac{eB}{\hbar c}}\left(x + \frac{p_y c}{eB}\right)$, show

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}w^2} + w^2 + \sigma_z\right)f(w) = \frac{c}{eB\hbar}\left(\frac{E^2}{c^2} - m^2c^2 - p_z^2\right)f(w).$$

(3) Why can one take the ansatz $f(w) = \binom{f_+(w)}{0}$ or $\binom{0}{f_-(w)}$ as a solution? To which problem is the solution traced back that way? Show for the energy eigenvalues that $E^2 = m^2c^4 + p_z^2c^2 + eB\hbar c(2n+1\pm1)$ with $n \in \mathbb{N}$ and interpret them.

(4) Show

$$\eta(\vec{r}) = \frac{1}{\frac{E}{c} + mc}\left(\pm p_z + \sqrt{\frac{eB\hbar}{c}}\left(ew\sigma_y - \mathrm{i}\sigma_x\frac{\partial}{\partial w}\right)\right)\xi(\vec{r}).$$

(5) The equation for E in (3) is quadratic. Discuss the consequences of that.

Good Luck!