The Lorenz Attractor

Theory of strange Attractors and the chaotic Butterfly-Effect

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Table of contents:

- The physical problem statement
- Dynamic systems in phase-space
- Attractors and their characteristics
- Maps (Hénon-Map - Poincaré-Map)
- Examples
- Lorenz-Equations (shortcut)
- Lorenz-Attractor

Problem statement:

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- dynamical system given by a set of equations
- knowledge of it's previous history determines the system
- in chaotic systems: the use of statistical description is more efficient

Phase-space:

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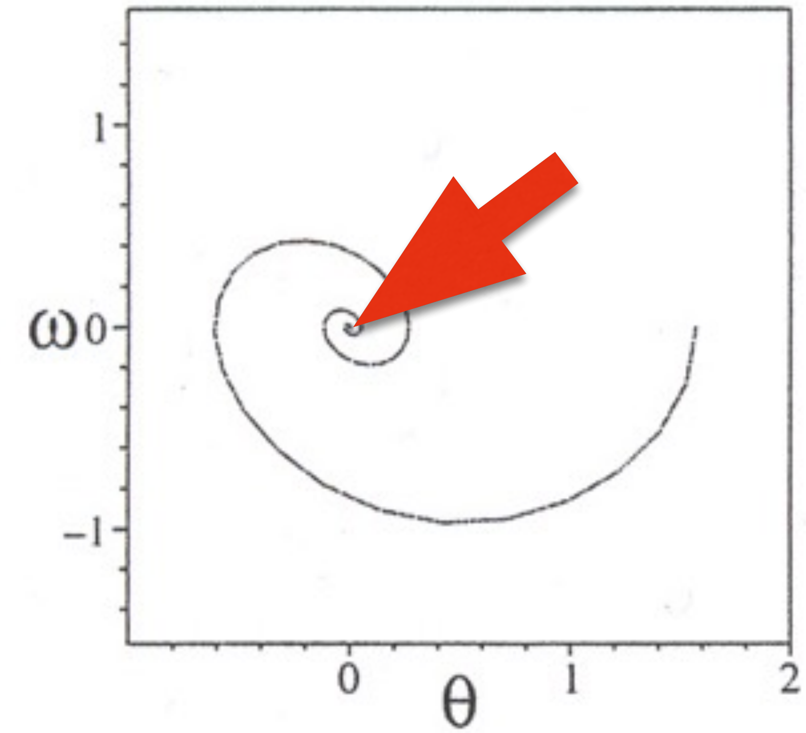
- Phase space represents all possible states of a system
- Useful to analyze characteristic structure of dynamical systems
- coordinates of a system at any time are composed of all the variables -> possible to calculate state of a system at any time
- set of all points is called trajectories

Dynamic systems in phase-space:

Dissipative harmonic oscillator:

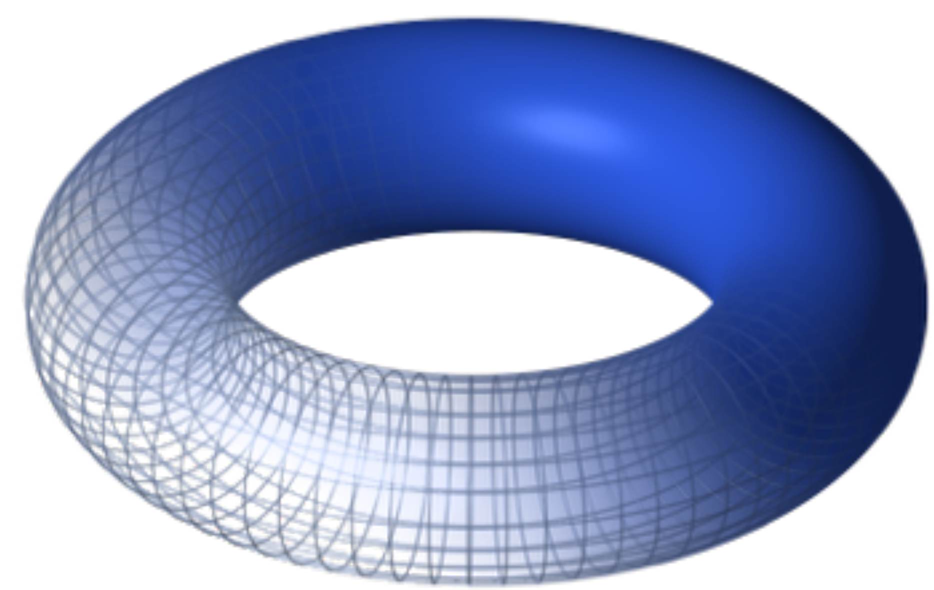
$$m\ddot{x} + \rho\dot{x} + kx = 0$$



Double pendulum:

$$x_1 = \frac{l}{2}\sin(\theta_1)$$

$$y_1 = -\frac{l}{2}\cos(\theta_1)$$



Attractors:

- Attractor: A set of values in phase-space into which a system tends to evolve
- An attractor can be: a point, a set of points, a curve, a manifold, or a fractal
- Systems which values get close to an attractor remain close to it
- the opposite is a „repellor“

Mathematical definition:

- Let $f(t,*)$ be the function which specifies the dynamics of the system
- If a equals the initial point, then $f(0,a)=a$
- And $f(t,a)$ describes the systems evolution in time

- For example: A free particle in $\mathbb{R}^2$ with:

  v = velocity, x = location

$$f(t,(x,v)) = (x+tv,v) \qquad \text{with} \quad a=(x,v)$$

Mathematical definition:

An attractor is a subset A of the phase-space characterized by the conditions:

- A is forward invariant under f $\boxed{\text{If } \ a \in A \Rightarrow f(t,a) \in A}$

- There exists a neighborhood of A (the **basin of attraction**) and denotes B(A) which consists all points b that enter A in the limit $t \longrightarrow \infty$

- There is no proper (not empty) subset of A having the first two properties

$$f(t,(x,v)) = (x + tv, v)$$

Types of attractors:

- Fix point(s) $\Rightarrow$ Gregor and Alexander
- Limit cycles
- Limit torus

$\left.\begin{array}{l}\text{Limit cycles}\\ \text{Limit torus}\end{array}\right\} \Rightarrow$ Robin and Kai

- Strange attractor $\Rightarrow$ Our main topic

Strange attractors:

- An attractor is called **strange**, if it’s dimension isn’t a natural number
- Most (not all!) strange attractors describe a chaotic movement
- It’s has sensitive dependence in it’s initial conditions
- Locally unstable (chaotic), but globally stable (attractor)

Maps:

In case of dynamics:

- A new way of looking at the phase-space
- Can transforms chaotic structures to easier structures

The term „mapping“ is used for:

- A normal function which maps some points to another
- A morphism in category theory i.e. topology

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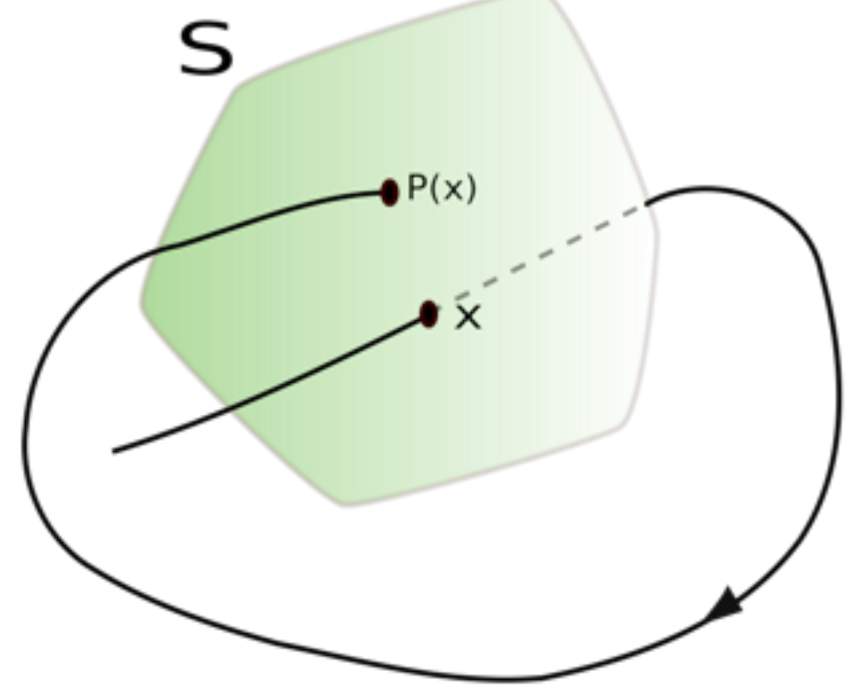
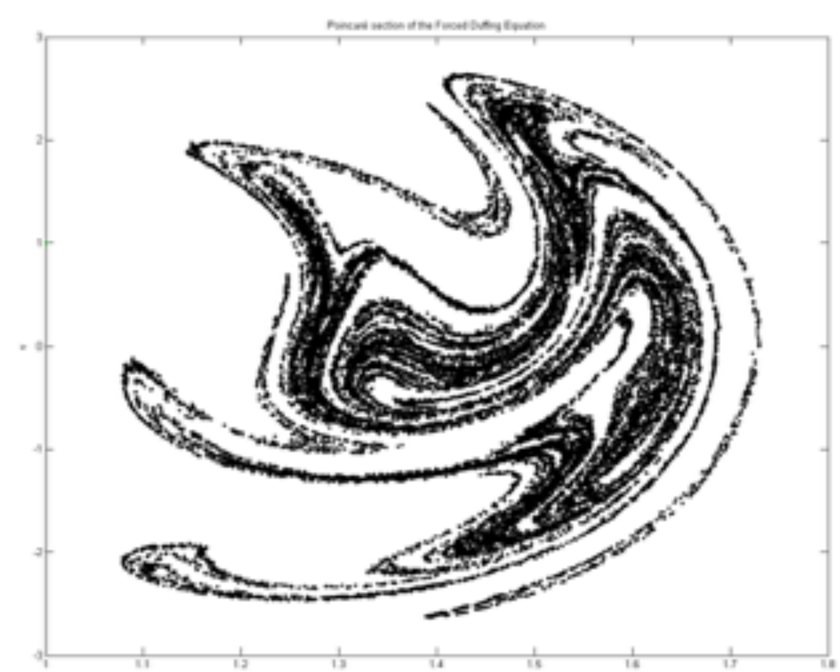
Poincaré Map:

- As described in a previous talk: It's an easy way to visualize a dynamic system
- It's defined in the state-space (phase-space + time)
- It reduces the dimension of the system from n to n-1



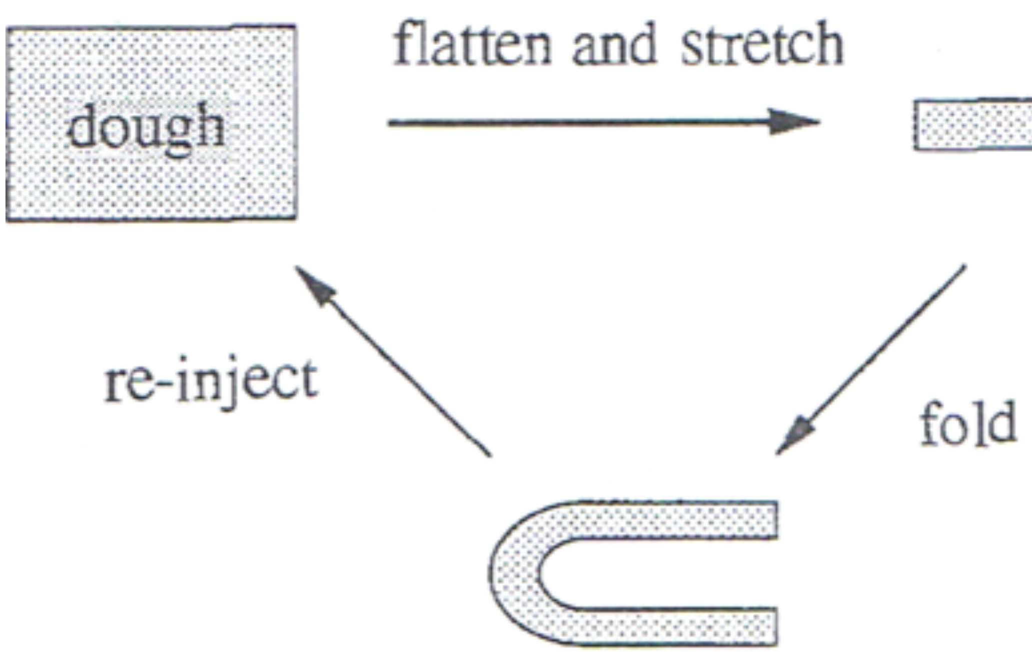
Example: Duffing equations for damped and driven oscillators:

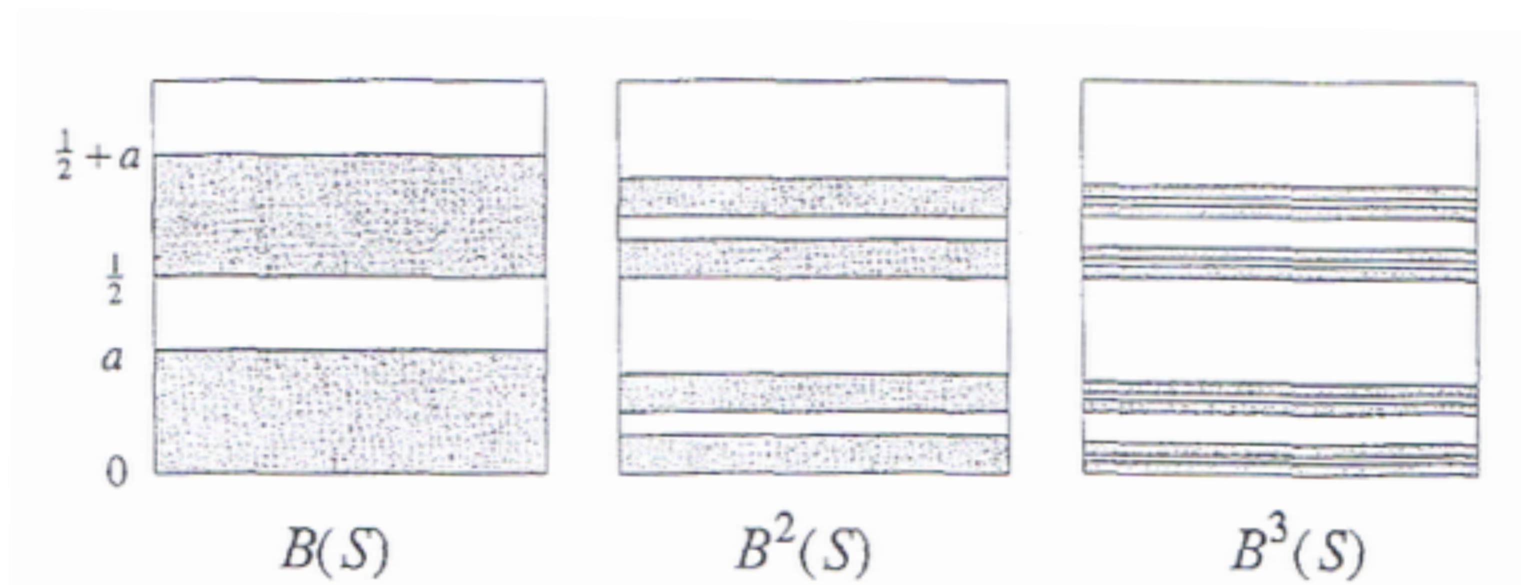
$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma \cos(\omega t)$$



The pastry map:

- Hénon and Rössler supplied deeper understanding by intuitive concept of stretching and folding
- How can trajectories diverge endlessly and stay bounded?
- Example of dough (pastry or bakers map):





Hénon map:

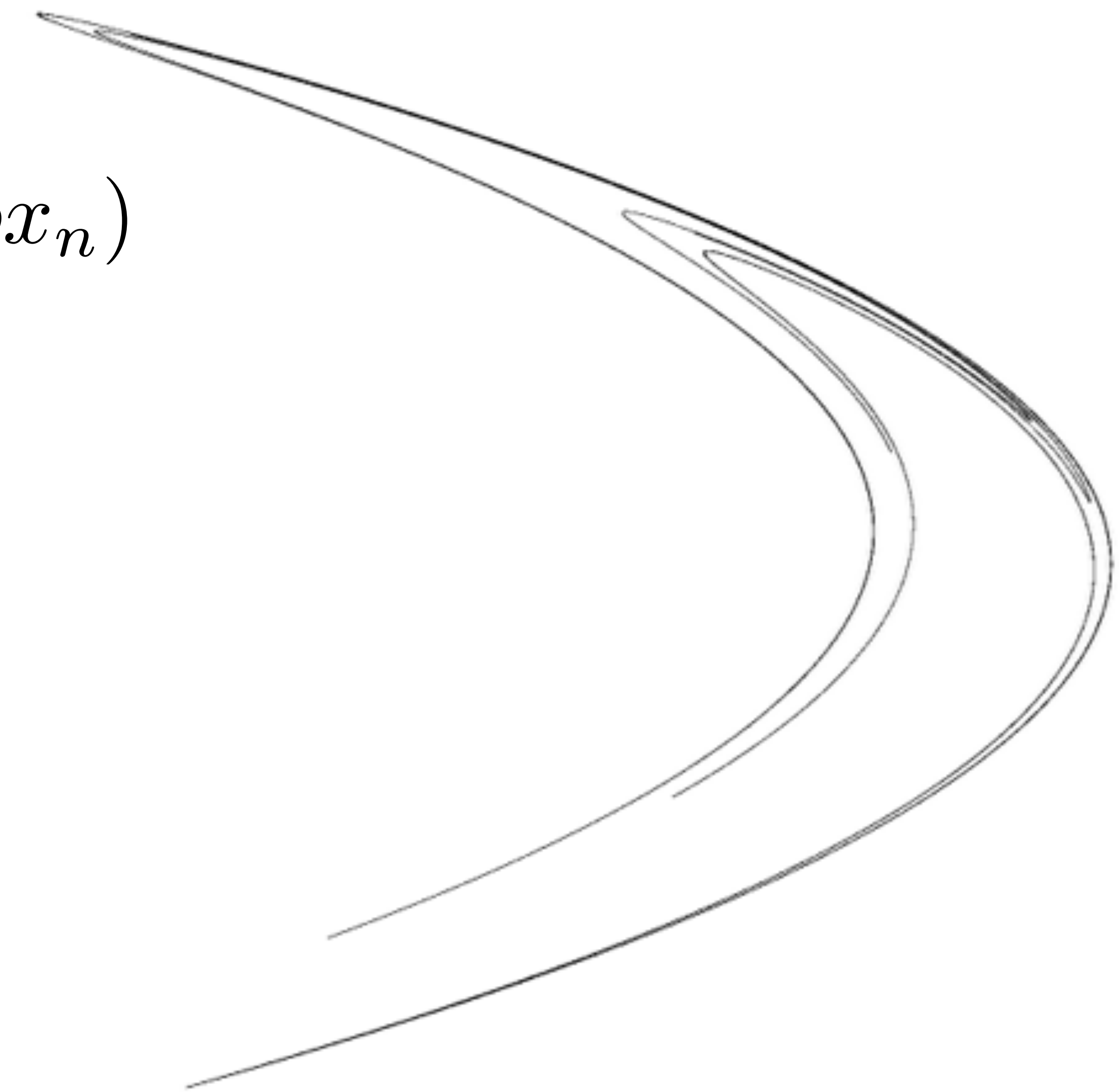
- The Hénon map takes a point (x_n, y_n) and maps it to the new point:

$$(x_{n+1} = 1 - ax_n^2 + y_n \ , \ y_{n+1} = bx_n)$$

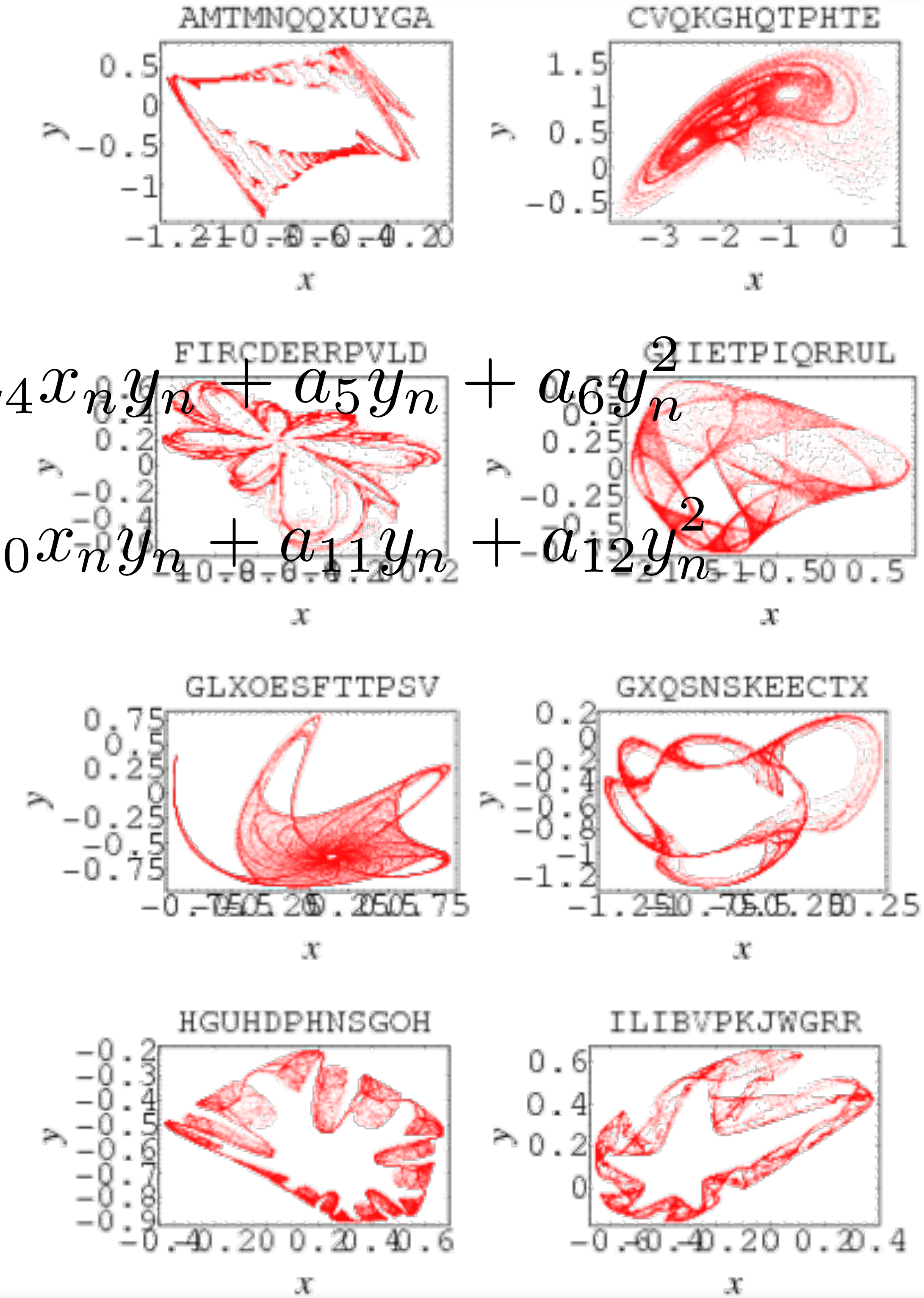
- It depends on two Parameters a and b, but not for every value it is chaotic

- The classical values for chaotic systems are a = 1.4 and b = 0.3

- Represents a strange attractor, it's Hausdorff dimension is 1.261 ± 0.003



General quadratic map:



$$x_{n+1} = a_1 + a_2 x_n + a_3 x_n^2 + a_4 x_n y_n + a_5 y_n + a_6 y_n^2$$

$$y_{n+1} = a_7 + a_8 x_n + a_9 x_n^2 + a_{10} x_n y_n + a_{11} y_n + a_{12} y_n^2$$

$\Rightarrow 25^{12} \approx 6 \cdot 10^{16}$ opportunities

- about 1.6% of all those maps are chaotic
- A-Y illustrate the coefficients from -1.2 to 1.2 in 0.1 steps
- many different shapes are the result of little changes in coefficients

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Examples:

- Rössler system
- Chemical reactions

Rössler system:

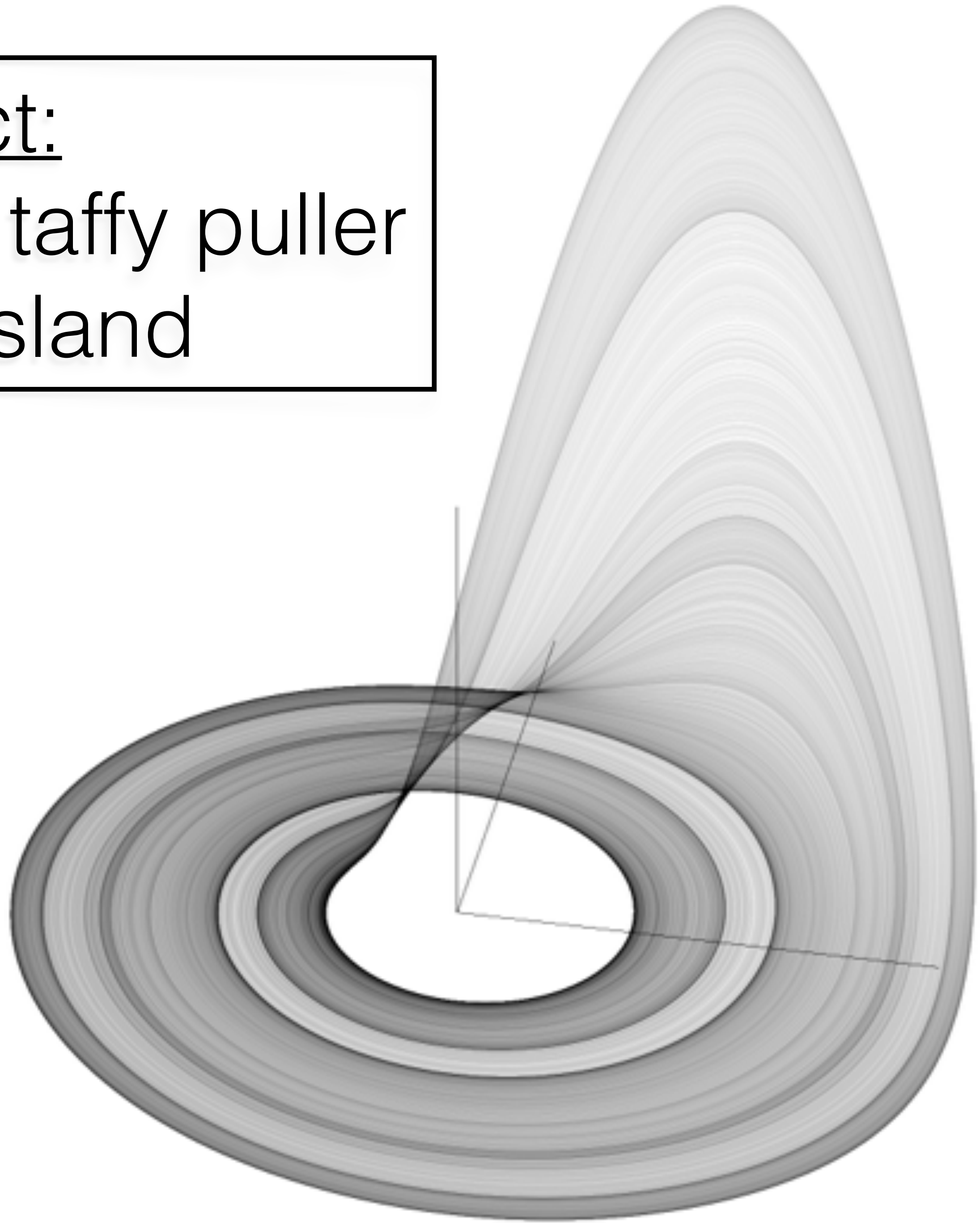
$$\frac{dx}{dt} = -y - z$$
$$\frac{dy}{dt} = x + ay$$
$$\frac{dz}{dt} = b + z(x - c)$$

Fun Fact:
It's inspired by a taffy puller on Coney Island

Fixpoints: Set the equations to zero and solve to (x,y,z)

$$x = \frac{c \pm \sqrt{c^2 - 4ab}}{2}$$
$$y = -\frac{c \pm \sqrt{c^2 - 4ab}}{2a}$$
$$z = \frac{c \pm \sqrt{c^2 - 4ab}}{2a}$$

Up to 6 fixpoints!
(„equilibrium point")



with a = 0.2, b = 0.2, c = 5.7

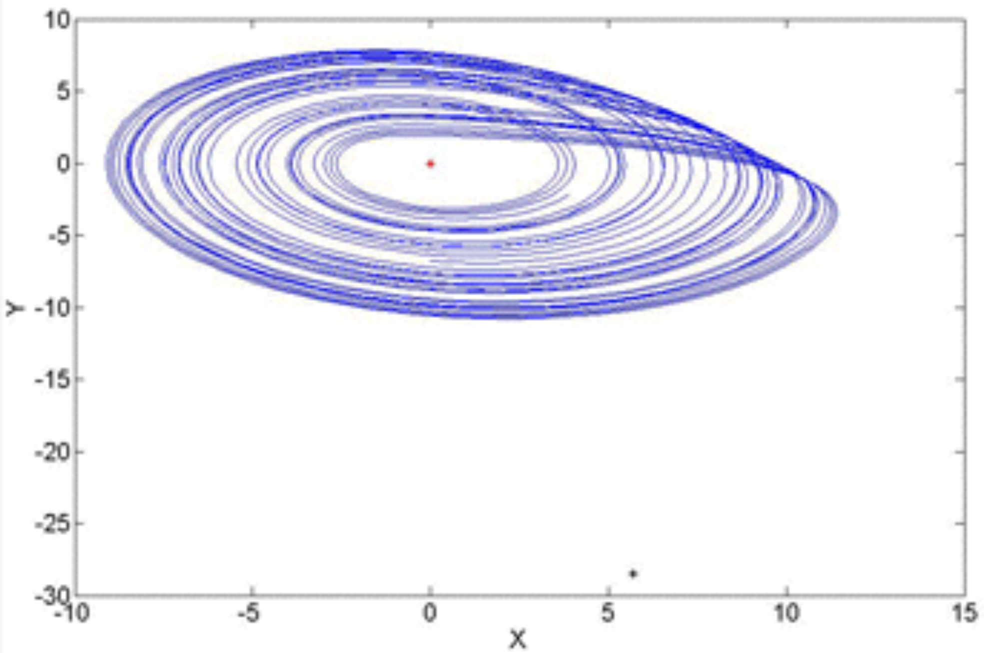
Rössler system:

Set z=0: ODE’s are linear

$$\left.\begin{aligned}\frac{dx}{dt} &= -y\\ \frac{dy}{dt} &= x + ay\end{aligned}\right\} \Rightarrow \text{Unstable for } 0 < a < 2$$

known by calculating the eigenvalues of the Jacobian:

$$\begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix} \Rightarrow (a \pm \sqrt{a^2 - 4})/2$$



XY-plane of the system

$$\left.\begin{aligned}\frac{dx}{dt} &= -y - z\\ \frac{dy}{dt} &= x + ay\\ \frac{dz}{dt} &= b + z(x - c)\end{aligned}\right\} \Rightarrow$$

The movement in z-direction is determined by x and c

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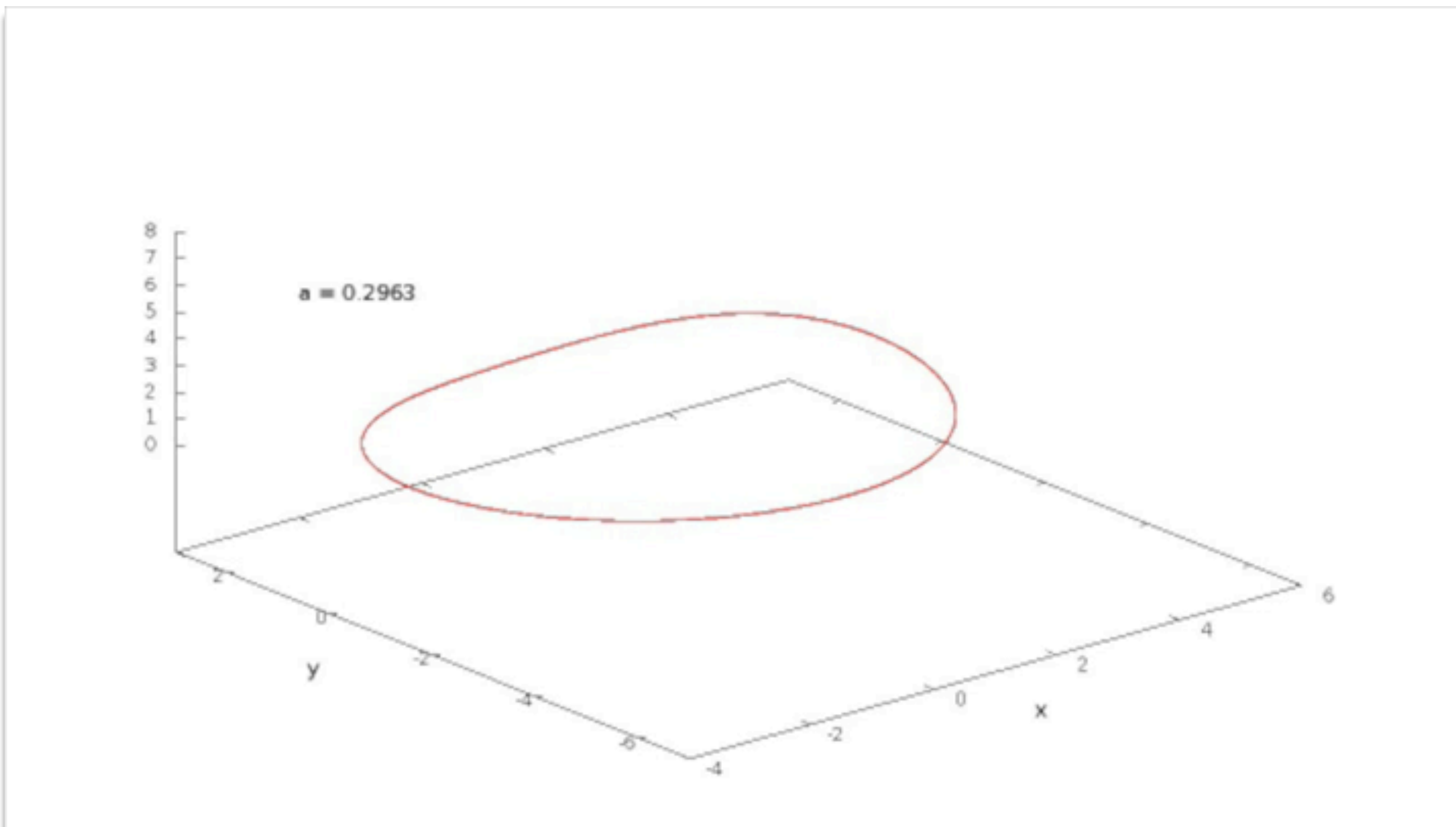
Rössler system:

- changing a shows the stability of the whole system:
- The difference between **chaotic** and **periodic** is pretty clear:



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Poincaré map of Rössler:

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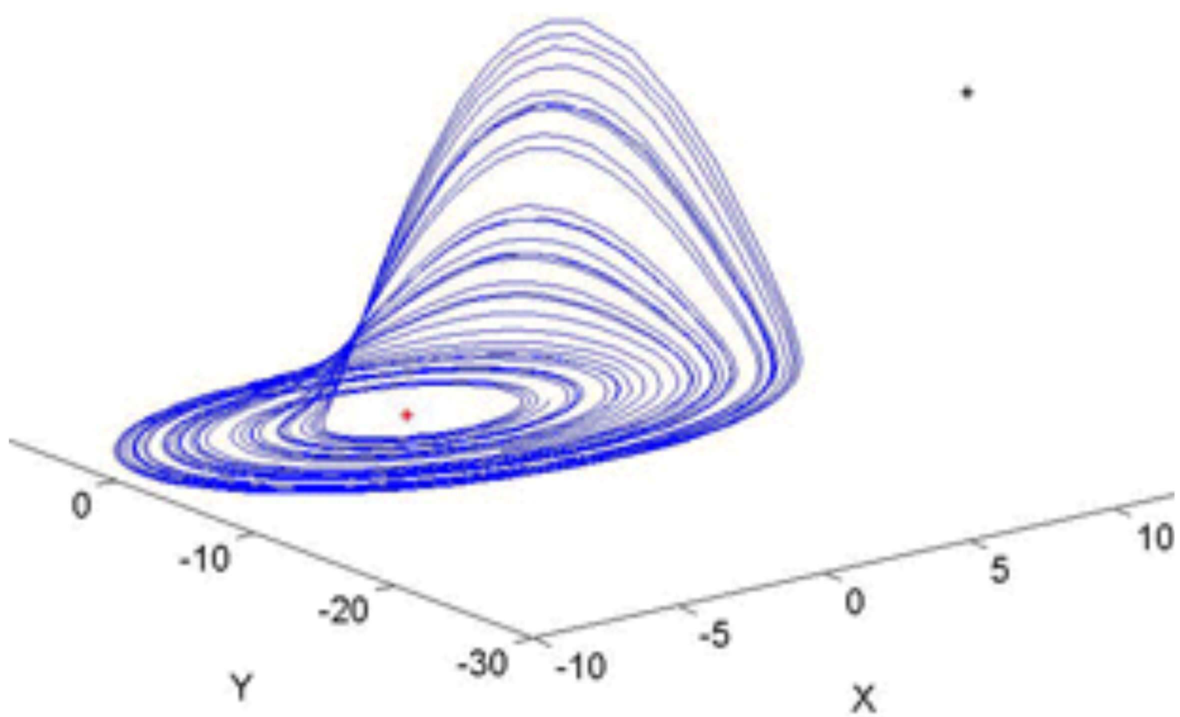
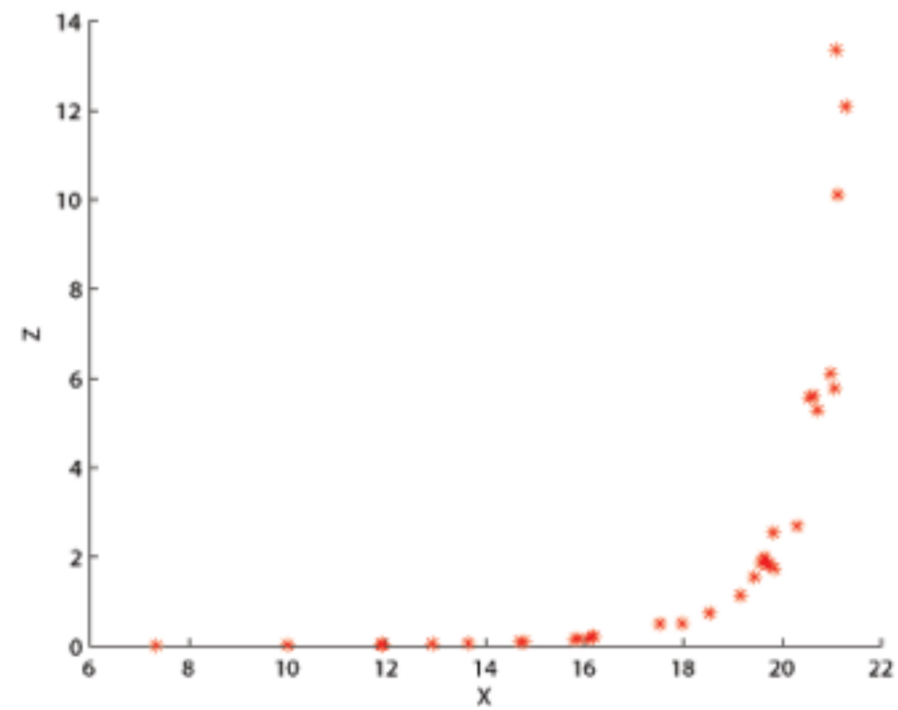
The Poincaré map helps us to describe the systems behavior, so take a look:

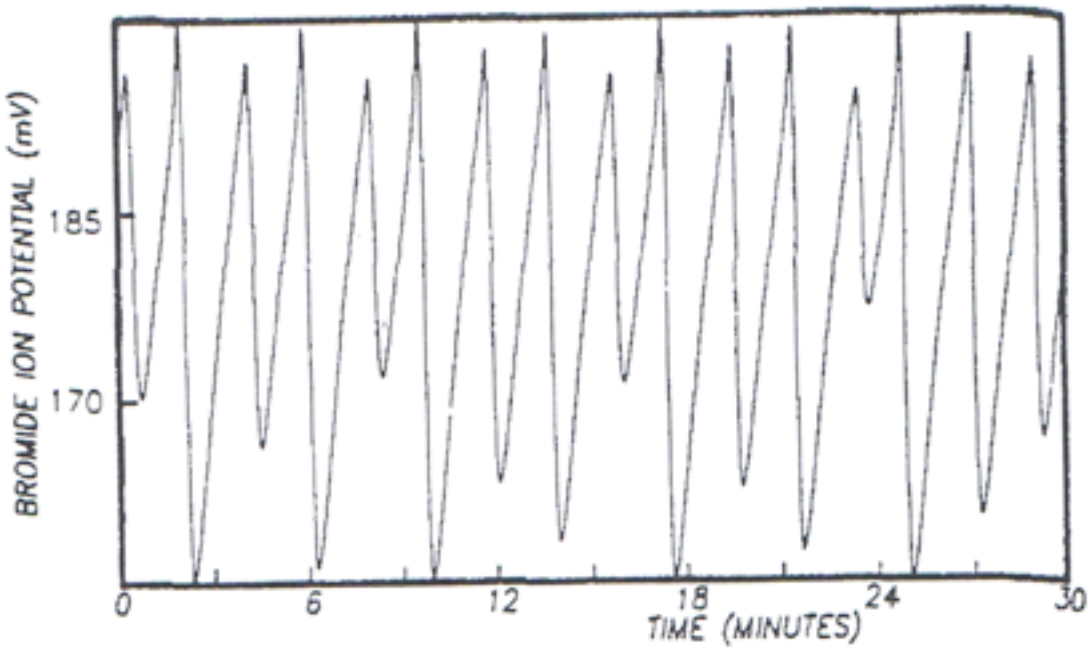




Poincaré map in X-Z-plane:

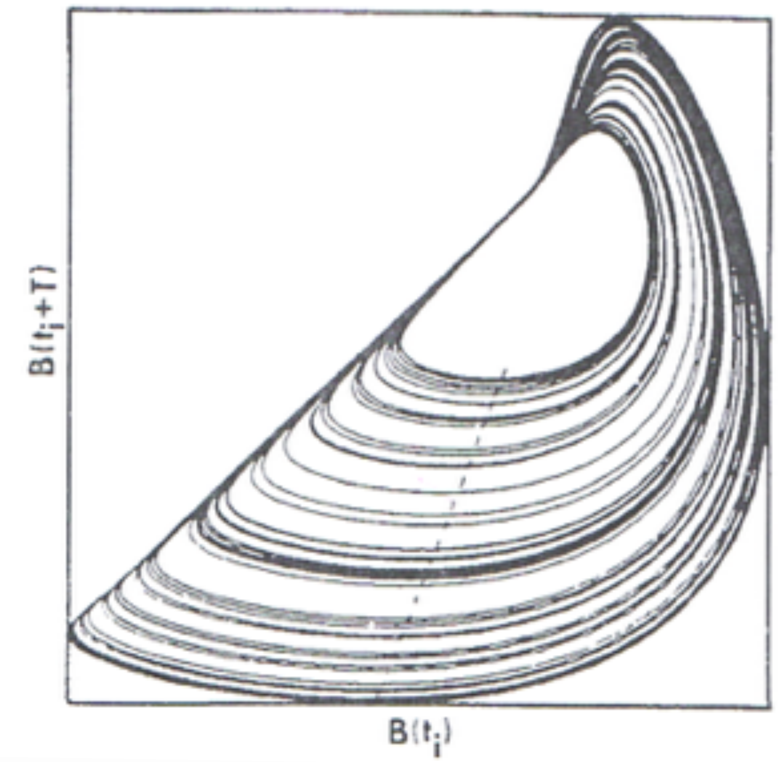
Chemical chaos:

- Belousov Zhabotinsky reaction (BZ)
- Can it become chaotic under suitable conditions?
- „continous flow stirred tank reactor“
- How to demonstrate presence of an attractor?



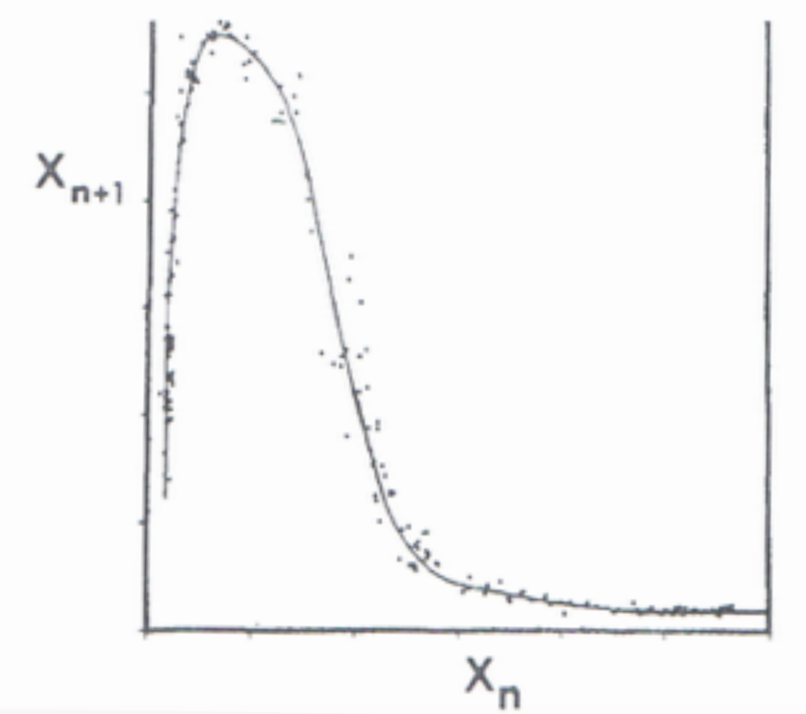
attractor reconstruction:

- in phase space reconstructed by measurements of one time series



- all data falls on one-dimensional curve

$X_1, X_2, ..., X_n, X_{n+1}$ are successive values of $B(t+\tau)$



- results of the system can be understood as one-dimensional —> universality theory

Lorenz-Equations:

- lorenz system is system of ODE
- lorenz attractor is set of chaotic solutions for the lorenz system

$$\boxed{\begin{aligned} \frac{dx}{dt} &= \sigma(y - x) \\ \frac{dy}{dt} &= x(\rho - z) - y \\ \frac{dz}{dt} &= xy - \beta z \end{aligned}}$$

Lorenz-Equations:

How to you get the equations out of a physical problem?

—> Next week!

Here two systems, which lead us to those equations:

- The water wheel
- Heated fluid: The „Rayleigh-Bénard convection“

Water wheel:

- analog to origin Lorenz observation of convection in meteorology

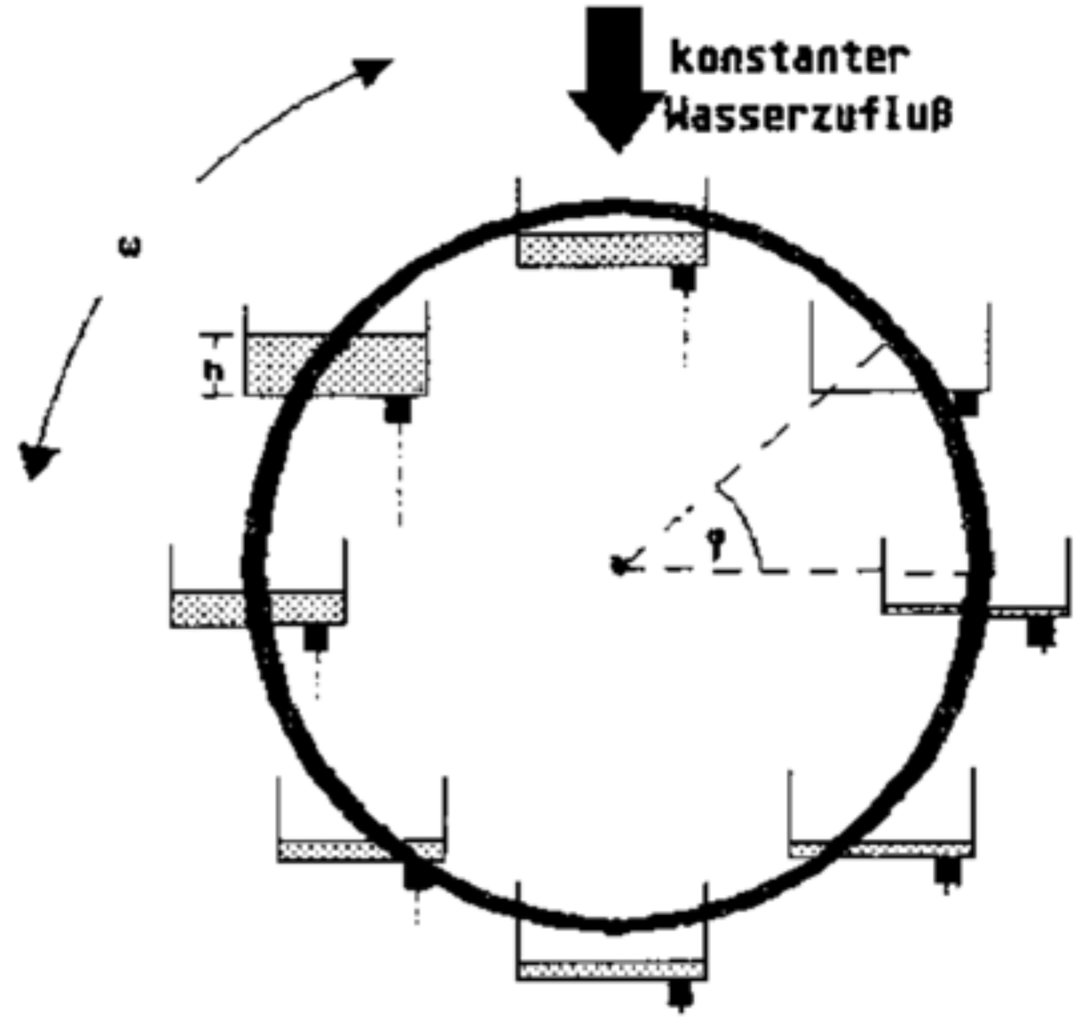
$$\frac{d(r^2\,\overline{m}\omega)}{dt} = -gr\,\overline{m\ cos(\phi)} - k\omega$$

$$\frac{dm(\phi)}{dt} = \frac{\partial m(\phi)}{\partial t} + \omega\frac{\partial m(\phi)}{\partial \phi}$$

$$\dot{X} = \sigma(Y - X)$$

$$\dot{Y} = RX - Y - XZ$$

$$\dot{Z} = XY - Z$$



- simplify equations —> lorenz equations

Rayleigh-Bénard convection:

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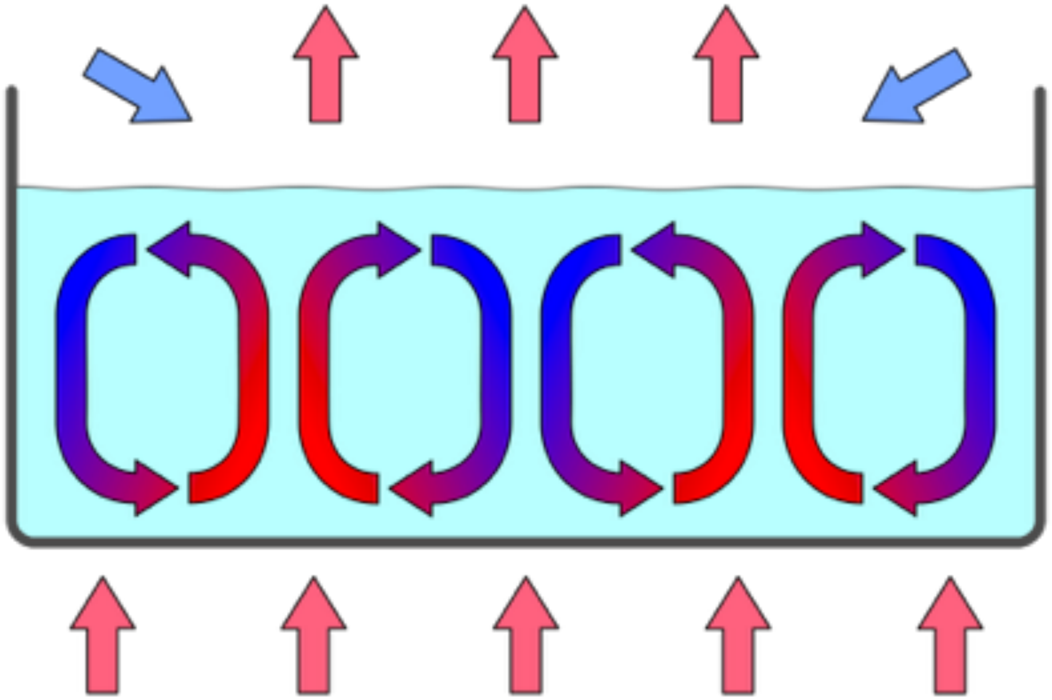
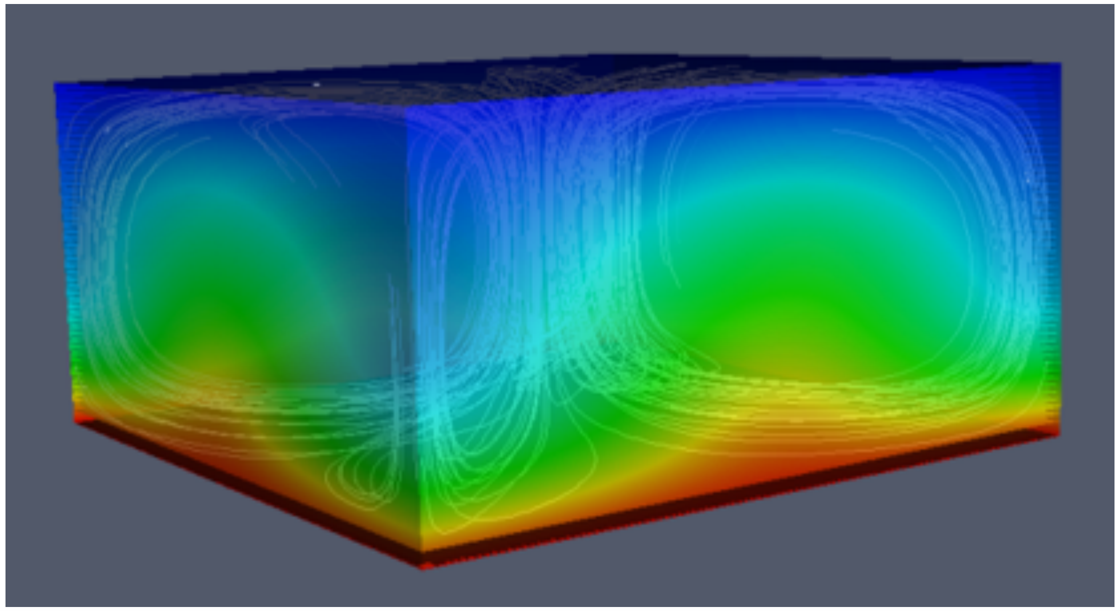
- A layer of liquid between two parallel planes
- Heating the bottom; cooling the top
- „Bénard-Cells“ show up
- The whole system is sensitive in it’s initial conditions —> chaotic
- The higher the temperature, the higher the chaotic movement (turbulent flow)





Lorenz-Attractor:

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$

Analysis:

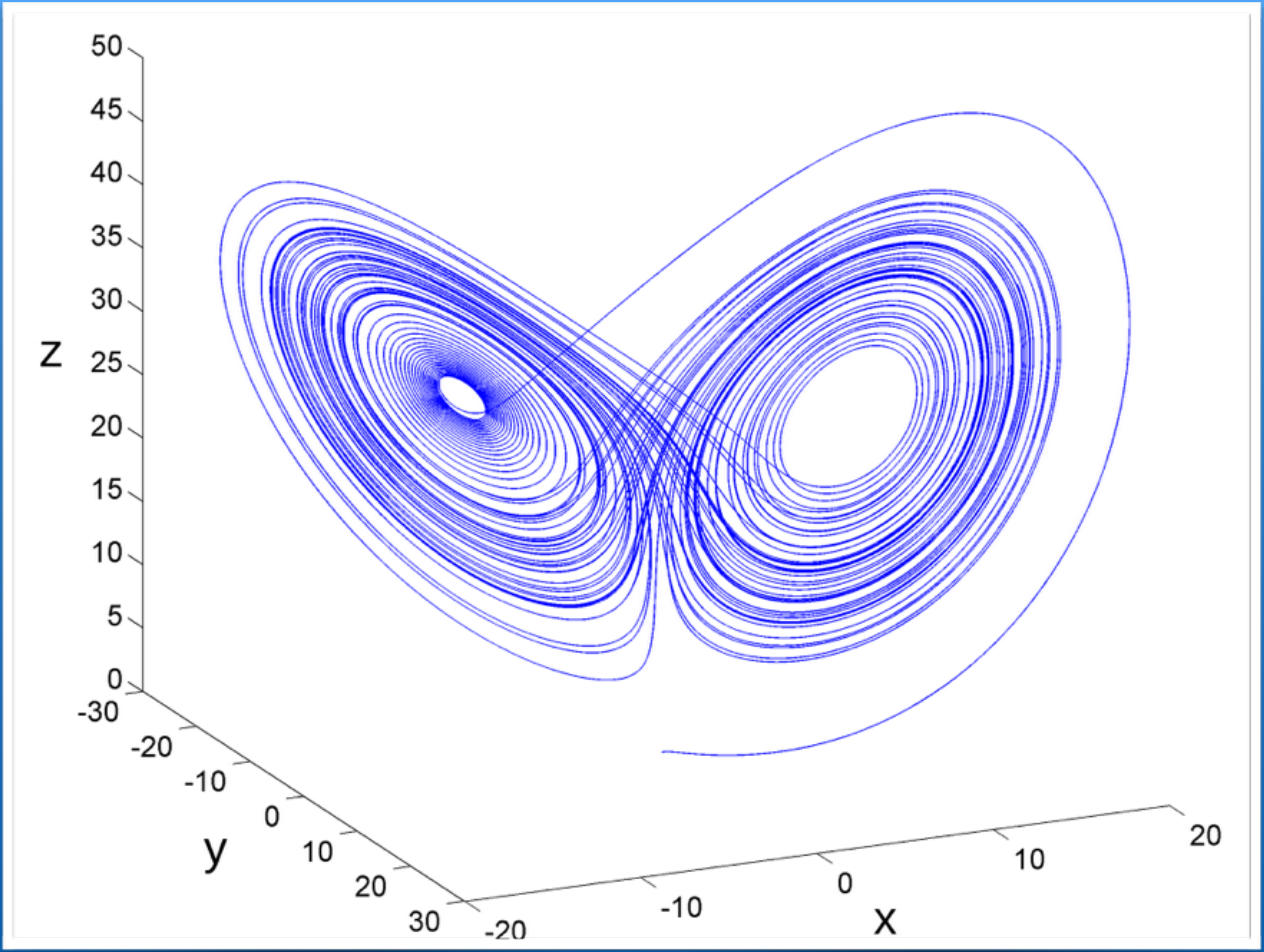
$\rho < 1 \quad \Rightarrow$ just one equilibrium point at the origin

$\rho \geq 1 \quad \Rightarrow$ Pitchfork bifurcation

two new critical points at: $(\pm\sqrt{\beta(\rho - 1)}, \rho - 1)$

just stable if: $\rho < \sigma \frac{\sigma + \beta + 3}{\sigma - \beta - 1} \Rightarrow \sigma > \beta + 1$

- not every set of parameter values creates a chaos
- When chaos: Hausdorff dimension of 2.06 ± 0.01
- simplified model for: lasers, dynamos, DC motors, electric circuits, chemical reactions, …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$\rho = 28$

$\sigma = 10$

$b = 2.67$

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Sources:

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Literature:

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- H. Joachim Schlichting „Chaos beim Wasserrad - ein einfaches Mechanisches Modell für das Lorenzsystem“, Physik und Didaktik 19/3 (1999)
- https://en.wikipedia.org/wiki/Attractor#Strange_attractor
- http://mathworld.wolfram.com/StrangeAttractor.html
- http://mathworld.wolfram.com/QuadraticMap.html
- https://en.wikipedia.org/wiki/Rayleigh–Bénard_convection
- https://en.wikipedia.org/wiki/Rössler_attractor

Mathematica Notebooks from:
Wolfram Demonstrations Project

- http://demonstrations.wolfram.com/index.html

Videos:

- Lorenz Attractor Simulation: https://www.youtube.com/watch?v=dP3qAq9RNLg
- Poncaré map of Rössler-attractor (Drexel University): https://www.youtube.com/watch?v=IhDSPByJW8I
- Evolution of Rössler-attractor: https://www.youtube.com/watch?v=o6w9CR7fk8s