The Lorenz Attractor

Theory of strange Attractors and the chaotic Butterfly-Effect
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Problem statement:

- dynamical system given by a set of equations
- knowledge of it’s previous history determines the system
- in chaotic systems: the use of statistical description is more efficient
Phase-space:

- Phase space represents all possible states of a system.
- Useful to analyze characteristic structure of dynamical systems.
- Coordinates of a system at any time are composed of all the variables -> possible to calculate state of a system at any time.
- Set of all points is called trajectories.

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Dynamic systems in phase-space:

Dissipative harmonic oscillator:

\[ m\ddot{x} + \rho \dot{x} + kx = 0 \]

Double pendulum:

\[ x_1 = \frac{l}{2}\sin(\theta_1) \]
\[ y_1 = -\frac{l}{2}\cos(\theta_1) \]
Attractors:

- Attractor: A set of values in phase-space into which a system tends to evolve

- An attractor can be: a point, a set of points, a curve, a manifold, or a fractal

- Systems which values get close to an attractor remain close to it

- the opposite is a „repellor“

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Mathematical definition:

- Let $f(t, \ast)$ be the function which specifies the dynamics of the system.
- If $a$ equals the initial point, then $f(0, a) = a$.
- And $f(t, a)$ describes the system's evolution in time.

For example: A free particle in $\mathbb{R}^2$ with:

$v = \text{velocity, } x = \text{location}$

$$f(t, (x, v)) = (x + tv, v) \quad \text{with} \quad a = (x, v)$$
Mathematical definition:

An attractor is a subset $A$ of the phase-space characterized by the conditions:

1. $A$ is forward invariant under $f$.
   \[
   \text{If } a \in A \implies f(t, a) \in A
   \]

2. There exists a neighborhood of $A$ (the **basin of attraction**) and denotes $B(A)$ which consists all points $b$ that enter $A$ in the limit $t \to \infty$.

3. There is no proper (not empty) subset of $A$ having the first two properties.

Mathematical definition:

\[
f(t, (x, v)) = (x + tv, v)
\]
Types of attractors:

- Fix point(s) \( \Rightarrow \) Gregor and Alexander
- Limit cycles \( \Rightarrow \) Robin and Kai
- Limit torus
- Strange attractor \( \Rightarrow \) Our main topic

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Strange attractors:

- An attractor is called **strange**, if it’s dimension isn’t a natural number

- Most (not all!) strange attractors describe a chaotic movement

- It’s has sensitive dependence in it’s initial conditions

- Locally unstable (chaotic), but globally stable (attractor)
Maps:

In case of dynamics:

- A new way of looking at the phase-space
- Can transforms chaotic structures to easier structures

The term „mapping“ is used for:

- A normal function which maps some points to another
- A morphism in category theory i.e. topology
Poincaré Map:

- As described in a previous talk: It’s an easy way to visualize a dynamic system

- It’s defined in the state-space (phase-space + time)

- It reduces the dimension of the system from $n$ to $n-1$

Example: Duffing equations for damped and driven oscillators:

$$\ddot{x} + \delta \dot{x} + \alpha x + \beta x^3 = \gamma \cos(\omega t)$$
The pastry map:

- Hénon and Rössler supplied deeper understanding by intuitive concept of stretching and folding.

- How can trajectories diverge endlessly and stay bounded?

- Example of dough (pastry or bakers map):

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Hénon map:

• The Hénon map takes a point \((x_n, y_n)\) and maps it to the new point:

\[
(x_{n+1} = 1 - ax_n^2 + y_n, \quad y_{n+1} = bx_n)
\]

• It depends on two Parameters \(a\) and \(b\), but not for every value it is chaotic

• The classical values for chaotic systems are \(a = 1.4\) and \(b = 0.3\)

• Represents a strange attractor, it's Hausdorff dimension is \(1.261 \pm 0.003\)
General quadratic map:

$$x_{n+1} = a_1 + a_2 x_n + a_3 x_n^2 + a_4 x_n y_n + a_5 y_n + a_6 y_n^2$$

$$y_{n+1} = a_7 + a_8 x_n + a_9 x_n^2 + a_{10} x_n y_n + a_{11} y_n + a_{12} y_n^2$$

$\Rightarrow 25^{12} \approx 6 \cdot 10^{16}$ opportunities

- about 1.6% of all those maps are chaotic
- A-Y illustrate the coefficients from -1.2 to 1.2 in 0.1 steps
- many different shapes are the result of little changes in coefficients
Examples:

- Rössler system
- Chemical reactions
Rössler system:

\[
\begin{align*}
\frac{dx}{dt} &= -y - z \\
\frac{dy}{dt} &= x + ay \\
\frac{dz}{dt} &= b + z(x - c)
\end{align*}
\]

**Fun Fact:**
It’s inspired by a taffy puller on Coney Island

**Fixpoints:** Set the equations to zero and solve to \((x, y, z)\)

\[
\begin{align*}
x &= \frac{c \pm \sqrt{c^2 - 4ab}}{2} \\
y &= -\frac{c \pm \sqrt{c^2 - 4ab}}{2a} \\
z &= \frac{c \pm \sqrt{c^2 - 4ab}}{2a}
\end{align*}
\]

Up to 6 fixpoints! ("equilibrium point")

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with \(a = 0.2, b = 0.2, c = 5.7\)
Rössler system:

Set $z=0$: ODE’s are linear

\[
\begin{align*}
\frac{dx}{dt} &= -y \\
\frac{dy}{dt} &= x + ay
\end{align*}
\]

\Rightarrow\text{ Unstable for } 0 < a < 2

known by calculating the eigenvalues of the Jacobian:

\[
\begin{pmatrix}
0 & -1 \\
1 & a
\end{pmatrix} \Rightarrow (a \pm \sqrt{a^2 - 4})/2
\]

\[
\begin{align*}
\frac{dx}{dt} &= -y - z \\
\frac{dy}{dt} &= x + ay \\
\frac{dz}{dt} &= b + z(x - c)
\end{align*}
\]

\Rightarrow\text{ The movement in } z\text{-direction is determined by } x \text{ and } c

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Rössler system:

- changing $a$ shows the stability of the whole system:

- The difference between chaotic and periodic is pretty clear:
Poincaré map of Rössler:

The Poincaré map helps us to describe the system's behavior, so take a look:

Poincaré map in X-Z-plane:

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Chemical chaos:

- Belousov Zhabotinsky reaction (BZ)
- Can it become chaotic under suitable conditions?
- „continuous flow stirred tank reactor“
- How to demonstrate presence of an attractor?
attractor reconstruction:

• in phase space reconstructed by measurements of one time series

• all data falls on one-dimensional curve

\[ X_1, X_2, \ldots, X_n, X_{n+1} \text{ are successive values of } B(t + \tau) \]

• results of the system can be understood as one-dimensional —> universality theory
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Lorenz-Equations:

- lorenz system is system of ODE
  \[
  \frac{dx}{dt} = \sigma (y - x) \\
  \frac{dy}{dt} = x (\rho - z) - y \\
  \frac{dz}{dt} = xy - \beta z
  \]

- lorenz attractor is set of chaotic solutions for the lorenz system
Lorenz-Equations:

How to you get the equations out of a physical problem?

—> Next week!

Here two systems, which lead us to those equations:

• The water wheel

• Heated fluid: The „Rayleigh-Bénard convection“
Water wheel:

- analog to origin Lorenz observation of convection in meteorology

\[
\frac{d(r^2 \bar{m}\omega)}{dt} = -gr m \cos(\phi) - k\omega
\]

\[
\frac{dm(\phi)}{dt} = \frac{\partial m(\phi)}{\partial t} + \omega \frac{\partial m(\phi)}{\partial \phi}
\]

\[
\dot{X} = \sigma(Y - X)
\]

\[
\dot{Y} = RX - Y - XZ
\]

\[
\dot{Z} = XY - Z
\]

- simplify equations —> lorenz equations

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Rayleigh-Bénard convection:

- A layer of liquid between two parallel planes
- Heating the bottom; cooling the top
- „Bénard-Cells“ show up
- The whole system is sensitive in its initial conditions —> chaotic
- The higher the temperature, the higher the chaotic movement (turbulent flow)
Lorenz-Attractor:

Analysis:

\[ \frac{dx}{\rho} \geq \frac{1}{\sigma} (y - x) \]

Pitchfork bifurcation

\[ \frac{dy}{dt} = x(\rho - z) \frac{y}{\sigma} + z \frac{3}{\beta + 1} \]

just stable if: \( \rho < \sigma \frac{y}{\beta + 1} \Rightarrow \sigma > \beta + 1 \)

\[ \frac{dz}{dt} = xy - \beta z \]

- not every set of parameter values creates a chaos
- When chaos: Hausdorff dimension of 2.06 ± 0.01
- simplified model for: lasers, dynamos, DC motors, electric circuits, chemical reactions, …
\[ \rho = 28 \]
\[ \sigma = 10 \]
\[ b = 2.67 \]

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Sources:

Literature:

- https://en.wikipedia.org/wiki/Attractor#Strange_attractor

Mathematica Notebooks from: Wolfram Demonstrations Project


Videos:

- Lorenz Attractor Simulation: https://www.youtube.com/watch?v=dP3qAq9RNLg
- Poncaré map of Rössler-attractor (Drexel University): https://www.youtube.com/watch?v=IhDSPByJW8I
- Evolution of Rössler-attractor: https://www.youtube.com/watch?v=o6w9CR7fk8s

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