Discussion of the Lorenz Equations

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Leibniz Universität Hannover

Proseminar Theoretische Physik SS/2015

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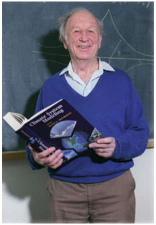
Outline

Edward N. Lorenz

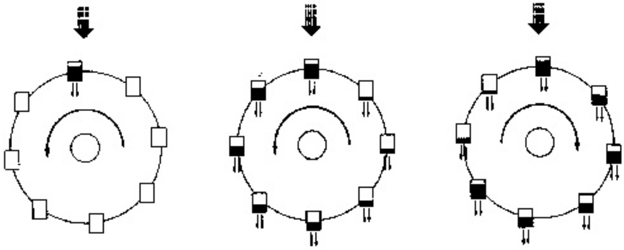


- $*$ May 23 1917
- Studied Mathematics
- After 1946: Focused on Meterology and introduced non linear statistical models to weather theory
- 1962 Deterministic Nonperiodic Flow
- 1972

  "Predictability: Does the Flap of a Butterfly's Wings in brazil set of a Tornado in Texas?"
- † April 16 2008

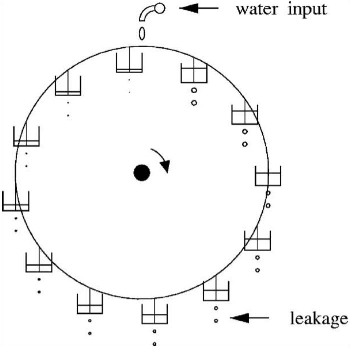
A Chaotic Waterwheel

- a mechanical model of the Lorenz equations was invented by Willem Malkus and Lou Howard
- the idealized mathematical model obeys a special case of the Lorenz equations



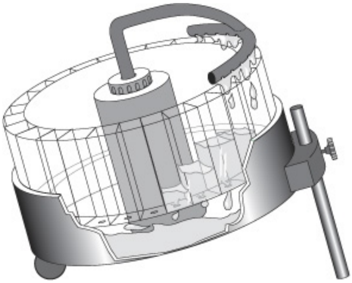
A Chaotic Waterwheel

- the simplest version consists of a few leaky cups
- single stream of water flows in the cup from the top
- when the top cup gets heavy enough the wheel starts to rotate



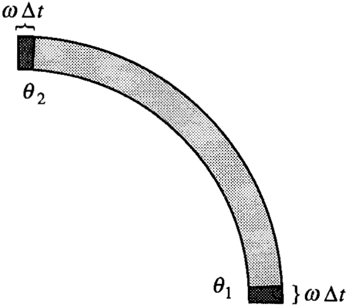
A Chaotic Waterwheel

- nearly horizontal so that the cups form a continuous ring without gaps
- the series of water streams reduces the propability of overflow
- a brake on the wheel can add more or less friction
- the tilt of the wheel can be varied to alter the effective strength of gravity



A Chaotic Waterwheel

- the water is pumped in by the rate Q
- the leakage occurs at a rate proportional to the mass

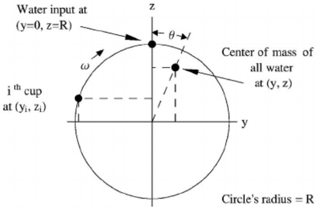


continuity equation

$$\frac{\partial m}{\partial t} = Q - Km - \omega \frac{\partial m}{\partial \theta}$$

Torque Balance

- two sources of damping: viscous damping and "inertial" damping
- the gravitational torque tends to increase ω when $sin\,(\theta) > 0$



Torque Balance

$$I\dot{\omega} = -v\omega + gr\int_0^{2\pi} m(\theta,t)sin(\theta)\,\mathrm{d}\theta$$

where v is the damping rate and g the effective gravitational constant

Amplitude Equations

- since $m(\theta, t)$ ist periodic in θ the equations can be written as a Fourier series

$$m(\theta, t) = \sum_{n=0}^{\infty} [a_n(t) sin(n\theta) + b_n(t) cos(n\theta)]$$

- no $sin(n\theta)$ terms because the water is added symmetrically

$$Q(\theta) = \sum_{n=0}^{\infty} q_n cos(n\theta)$$

Amplitude Equations

- matching coefficients of $sin\,(n\theta)$ and $cos\,(n\theta)$ yields to

$$\dot{a}_n = n\omega b_n - K a_n \qquad \dot{b}_n = -n\omega a_n - K b_n + q_n$$

- rewriting the torque balance equation eliminates all but one term by orthogonality

$$I\dot{\omega} = -v\omega + \pi g r a_1$$

waterwheel equations

$$\dot{a} = \omega b_1 - K a_1$$
$$\dot{b} = -\omega a_1 - K b_1 + q_1$$
$$\dot{\omega} = \left(-v\omega + \pi g r a_1\right)/I$$

Substitution

Substitutions:

$$
\begin{aligned}
a_1 &= \alpha Y \\
b_1 &= \beta Z + q_1/K \\
\omega &= \gamma X \\
t &= T\tau
\end{aligned}
$$

Lorenz equations

$$
\begin{aligned}
\dot{X} &= \sigma \left(Y - X\right) \\
\dot{Y} &= rX - Y - XZ \\
\dot{Z} &= XY - bZ
\end{aligned}
$$

Fixed Points

- Setting the derivatives equal to zero:

Fixed Points

$$a_1 = \omega b_1 / K$$
$$\omega a_1 = q_1 - K b_1$$
$$a_1 = v\omega / \pi g r$$

- solving for b_1

$$\frac{\omega b_1}{K} = \frac{v\omega}{\pi g r}$$

- $\omega = 0$ or $b_1 = Kv/\pi g r$

Fixed Points

there are two kinds of fixed points:

- if $\omega = 0$, then $a_1 = 0$ and $b_1 = q_1/K$

$$(a_1, b_1, \omega) = (0, q_1/K, 0)$$

- no rotation with inflow balanced by leakage
- if $\omega \neq 0$, then $b_1 = K/q_1/\left(\omega^2 + K^2\right) = Kv/\pi gr$

$$(\omega)^2 = \frac{\pi g r q_1}{v} - K^2$$

- when $\frac{\pi g r q_1}{v} > K^2$ there are two solutions $\pm\omega$
- steady rotation in either direction

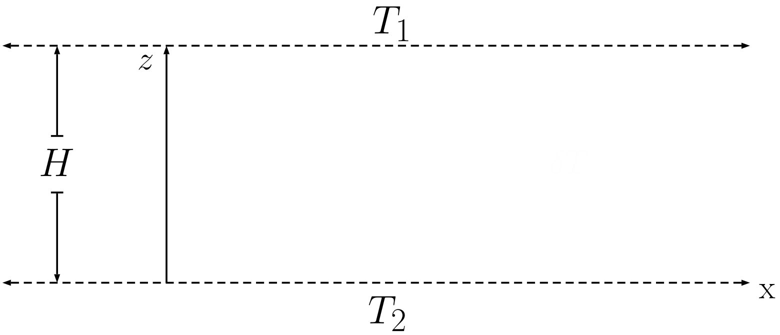
Rayleigh number

- the solutions only exist if

$$\frac{\pi g r q_1}{K^2 v} > 1$$

- this dimensionless group is called the Rayleigh number
- expresses a competition between a spinning and a stoping of the wheel

Convection crash course
Geometry of the Lorenz Model



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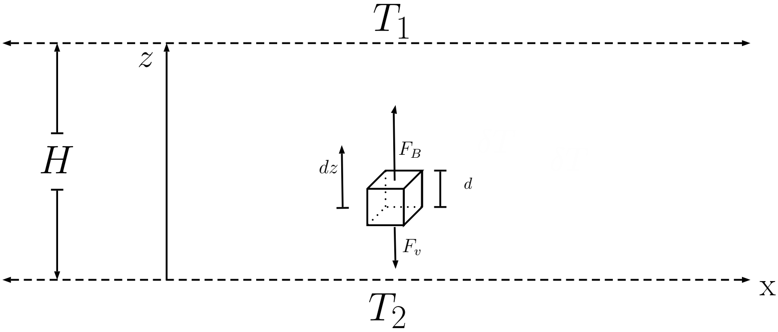

Heat Equation

$$\frac{\partial T}{\partial t} = D_T \Delta T \qquad (1)$$

Approximate the Laplacian operator by

$$\Delta T \approx \frac{\delta T dz}{h^3}$$

D_T	Thermal diffusivity
H	Height of the Box
α	Thermal Expansion Coefficient
ρ_0	Initial density
δT	Temperature difference
g	Gravitational acceleration
μ	Dynamic Viscosity

Find a Value for the Thermal displacement Time:

$$t_T \frac{\partial T}{\partial t} \stackrel{!}{=} dT = t_T D_T \Delta T$$
$$\approx t_T D_T \frac{dz}{H^3}$$

Calculate the bouyancy force:

$$F_B = \alpha \rho_0 g dT = \alpha \rho_0 g \frac{\delta T}{H} dz$$

D_T	Thermal diffusivity
H	Height of the Box
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ρ_0	Initial density
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Determine the viscous dissipation force:

$$F_v = \frac{\mu v}{H^2}$$

D_T	Thermal diffusivity
H	Height of the Box
α	Thermal Expansion Coefficient
ρ_0	Initial density
δT	Temperature difference
g	Gravitational acceleration
μ	Dynamic Viscosity

Set them equal:

$$F_v = F_B \iff v = \frac{\alpha \rho_0 g H \delta T}{\mu} dz$$

From this one obtains the displacement Time:

$$t_d = \frac{\mu}{\alpha \rho_0 g h \delta T}$$

D_T	Thermal diffusivity
H	Height of the Box
α	Thermal Expansion Coefficient
ρ_0	Initial density
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The ration between the displacement Times is the Rayleigh number:

$$R = \frac{t_T}{t_D} = \frac{\alpha \rho_0 g H^3 \delta T}{D_T \mu}$$

D_T	Thermal diffusivity
H	Height of the Box
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Derivation of the Equations

Navier-Stokes-Equations (Convective Form)

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}(\nabla \vec{u}) - \mu \nabla^2 \vec{u} = -\frac{\nabla p}{\rho_0} + \vec{g} \tag{1}$$

p is the Pressure, $\vec{g}$ is the gravitational acceleration, $\vec{u}$ is the stream field.

- Apply Bousinessq Approximation
- Substitute to dimensionless variables
- Introduce a Streamfunction Ψ
- $\rightarrow$ Saltzmann Equations

Saltzmann Equations

$$\frac{1}{\sigma}\left(\frac{\partial}{\partial t}(\Delta\Psi) - \frac{\partial}{\partial z}\left(\frac{\partial\Psi}{\partial z}\frac{\partial^2\Psi}{\partial^2 xz} - \frac{\partial\Psi}{\partial x}\frac{\partial^2\Psi}{\partial^2 z}\right)\right.$$
$$\left.- \frac{\partial}{\partial x}\left(\frac{\partial\Psi}{\partial z}\frac{\partial^2\Psi}{\partial^2 x} - \frac{\partial\Psi}{\partial x}\frac{\partial^2\Psi}{\partial^2 zx}\right)\right)$$
$$= R\frac{\partial T}{\partial x} + \Delta^2\Psi$$

Make the Ansatz:

$$\psi \propto X \sin(\frac{\pi a}{H}x)\sin(\frac{\pi z}{H})$$
$$\theta \propto (Y\cos(\frac{\pi a}{H}x) - Z\sin(\frac{2\pi}{H}z))$$

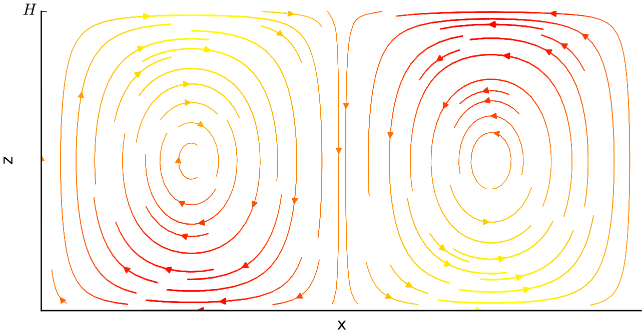
Where as: $\sigma = \frac{\mu}{D_T} \quad r = \frac{R}{R_c}$

and $R_c = 27\frac{\pi^4}{4}$ is the critical Rayleigh Number

Convection Rolls

Lorenz Ansatz: Fourier series truncation

$$\{x, z\} \propto X \left\{ a \cos\left(\frac{\pi a}{H}\right) \sin\left(\frac{\pi z}{H}\right), \sin\left(\frac{\pi a}{H}\right) - \cos\left(\frac{\pi z}{H}\right) \right\}$$



The actual Equations

From using the Lorenz Ansatz in the Saltzmann Equations:

Lorenz Equations

$$\dot{X} = \sigma(Y - X),$$
$$\dot{Y} = X(r - Z) - Y,$$
$$\dot{Z} = XY - bZ.$$

Properties of the Lorenz equations

Lorenz equations

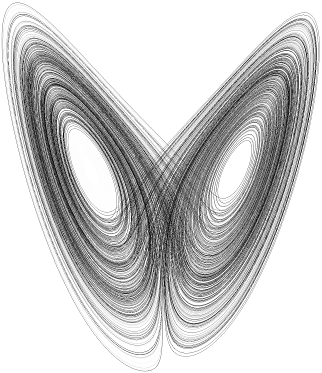
$$\dot{X} = \sigma\,(Y - X)$$
$$\dot{Y} = rX - Y - XZ$$
$$\dot{Z} = XY - bZ$$

- σ ist the Prandtl number, r is the Rayleigh number and b has no specific name
- the system has two nonlinearities, the quadratic terms XZ and XY

Symmetry

- important symmetry in the Lorenz equations
- all solutions are either symmetric or have a symmetric partner

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} -X \\ -Y \\ Z \end{pmatrix}$$



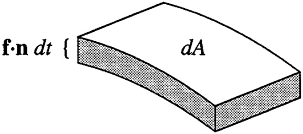
Volume Contraction

$$V(t+\mathrm{d}t) = V(t) + \int_S (f \cdot n\mathrm{d}t)\,\mathrm{d}A$$

$$\dot{V} = \int_S f \cdot n\,\mathrm{d}A = \int_V \nabla \cdot f\,\mathrm{d}V$$

$$\nabla \cdot f = -\sigma - 1 - b < 0$$

$$V(t) = V(0)\,e^{-(\sigma+1+b)t}$$



- volumes in phase space shrink exponentially fast
- shrinks to a limiting set of zero volume
- the Lorenz system is dissipative

Fixed Points

- the origin $(x, y, z) = (0, 0, 0)$ is a fixed point for all values of the parameters
- for $r > 1$ there is also a symmetric pair of fixed points

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \pm\sqrt{b\,(r-1)} \\ \pm\sqrt{b\,(r-1)} \\ r-1 \end{pmatrix}$$

- Lorenz called them C^+ and C^-
- they represent left- or right turning convection rolls or the steady rotation of the waterwheel
- as $r \to 1^+$, C^+ and C^- fuse to a pitchfork bifurcation

Stability of the Origin

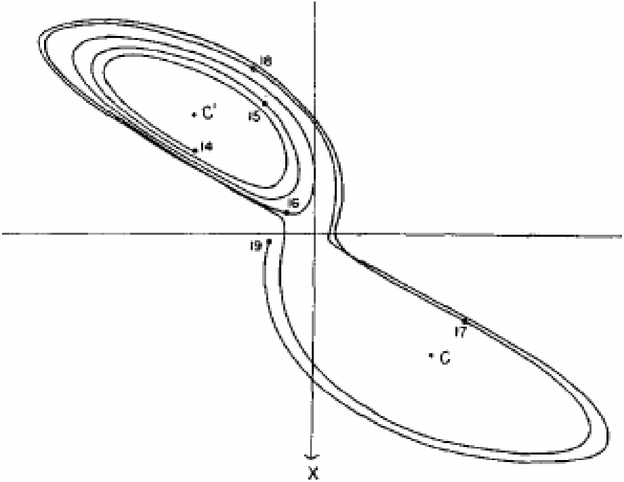
- the linearization is obtained by omitting the nonlinearities
- Jacobian at the Origin:

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \quad \det(J) = b\sigma(r-1)$$

- Z is decoupled and $Z(t) \to 0$
- if $r > 1$, the origin is a saddle point because $\det(J) < 0$
- if $r > 1$, the origin is a sink for all incoming directions

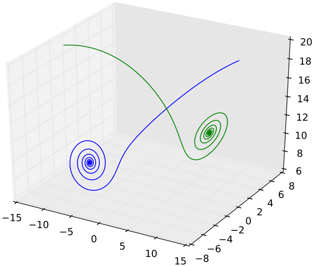
Numerical Integration for different Rayleigh numbers

Low Rayleigh Numbers $r < 40$

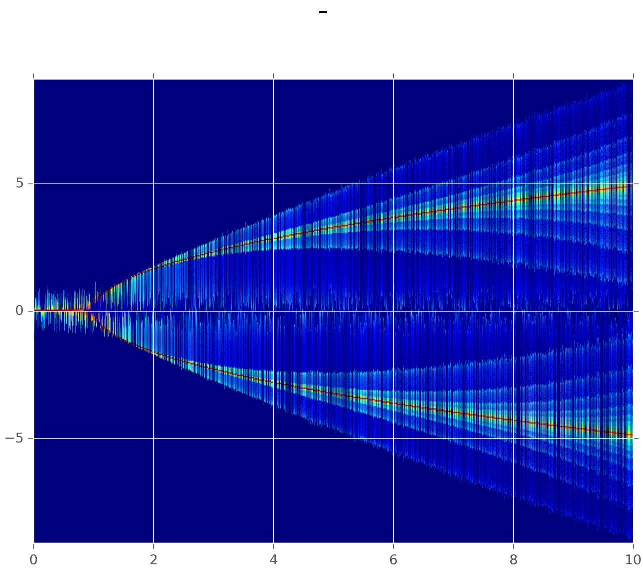


Low Rayleigh Numbers

- For $r < 1$, the origin is globally stable
- For $1 < r < r_a \approx 14$ there are two stable fixed points C_+ and C_-

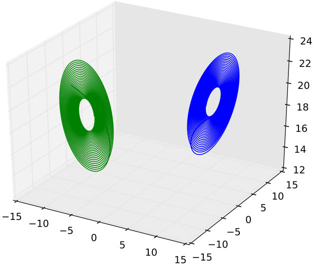


“Bifurcation Cloud”



Low Rayleigh Numbers

- For $r_a < r < r_H = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$ there are also unstable limit cycles.



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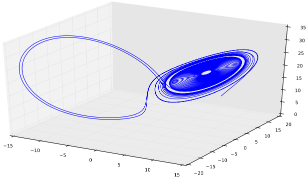
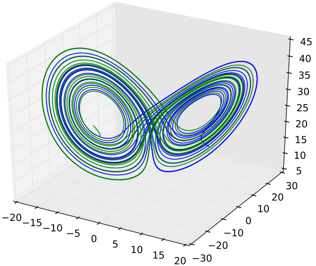


Semi-Chaotic

Low Rayleigh Numbers

- A subcritical Hopf bifurcation occurs at $r > r_H$.
- The fixed points disappear, leading to seemingly chaotic behaviour



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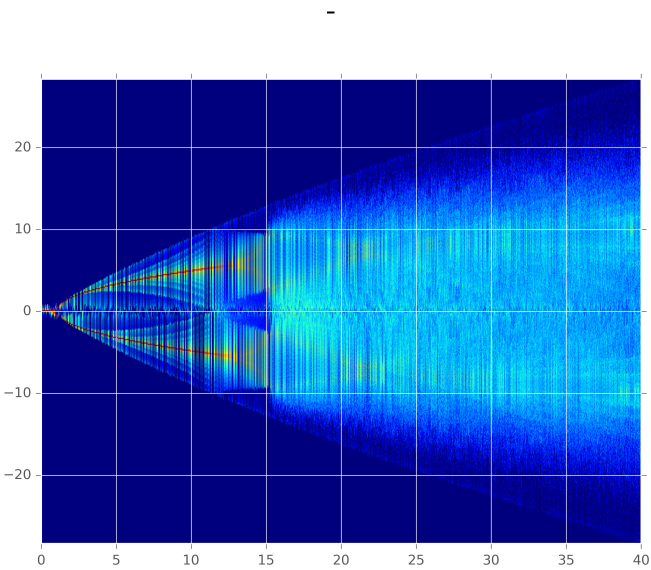

High Rayleigh Numbers

- For $r > 99$, there exist windows with stable limit cycles.
- For $r > 313$, there is only one global stable limit cycle

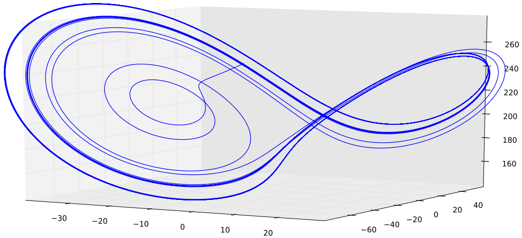

Figure: Solution for $r = 212$, Noisy Periodicity

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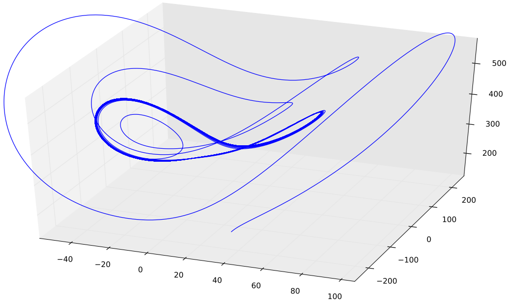
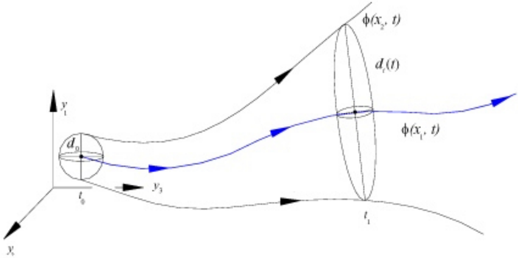


Figure: Solution for $r = 350$, Globally Stable limit Cycle

Exponential Divergence of Nearby Trajectories

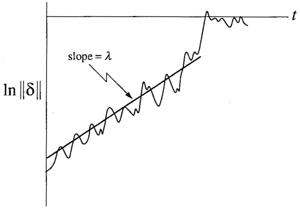
- sensitive dependence on initial conditions
- long-term prediction is impossible



Exponential Divergence of Nearby Trajectories

- $x(t)$ is a point on the attractor at time t
- $x(t) + \delta(t)$ is a nearby point with a tiny separation δ
- in numerical studies was found that

$$||\delta(t)|| \sim ||\delta_0|| e^{\lambda t} \qquad \text{where } \lambda \approx 0.9$$



- the divergence stops when the separation is comparable to the diameter of the attractor
- λ is often called the Ljapunov exponent

Time Horizon

- when a system has a positive Ljapunov exponent, there is a time horizon
- a is a measure of tolerance
- our prediction becomes intolerable when $||\delta(t)|| \leq a$

$$t_{horizon} \sim O\left(\frac{1}{\lambda} ln \frac{a}{||\delta_0||}\right)$$

- example:

original prediction: $$t_{horizon} \approx \frac{1}{\lambda} ln \frac{10^{-3}}{10^{-7}} = \frac{4\, ln 10}{\lambda}$$

improved prediction: $$t_{horizon} \approx \frac{1}{\lambda} ln \frac{10^{-3}}{10^{-13}} = \frac{10\, ln 10}{\lambda}$$

- after a millionfold improvement in our measurement we can predict only 2.5 times longer!
- $\Rightarrow$ big problem in weather prediction

Aperiodicity

Can we really tell if the solutions are aperiodic?

Figure: Royal McBee LGP-30, used by lorenz for numerical integration

Return Map

Consider a certain perspective of the trajectory:

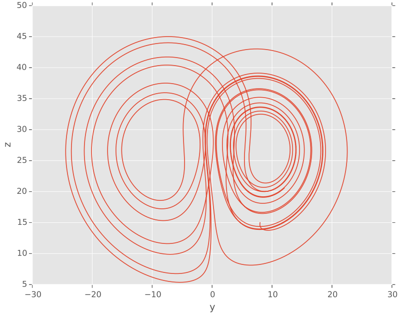
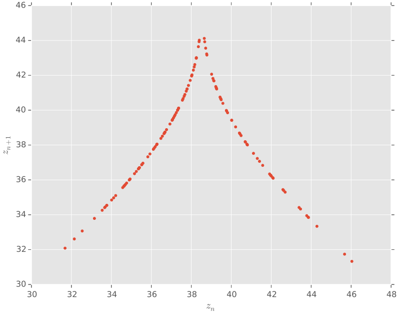


Figure: Projection of the Attractor on the x-z plane

Return Map

A approach to reduce the systems dynamics to a discrete 1D map.



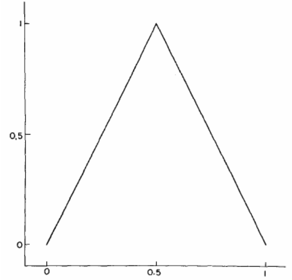
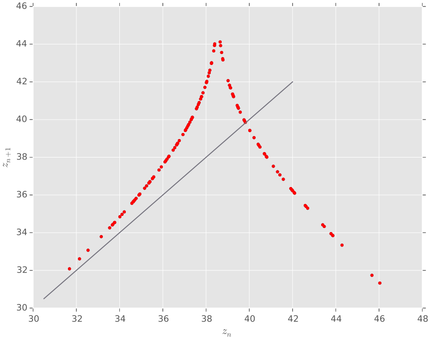


FIG. 5. The function $M_{n+1}=2M_n$ if $M_n<\frac{1}{2}$, $M_{n+1}=2-2M_n$ if $M_n>\frac{1}{2}$, serving as an idealization of the locus of points in Fig. 4.

Figure: Return Map for r=28, Left: Actual numerical results. Right: Idealized Function from Lorenz

Return Map

- The absolute value of the first derivative is $f'(z_n) > 1$
- For $z_n = z_n + 1$ the orbit is stable
- Pertubating the orbit by Δz,
- $\implies z'_{n+1} = z_n + \Delta z f'(z + \Delta z)$
- $\Delta z' = \Delta z f'(z + \Delta z) > \Delta z$
- This implies: The Cycle is unstable



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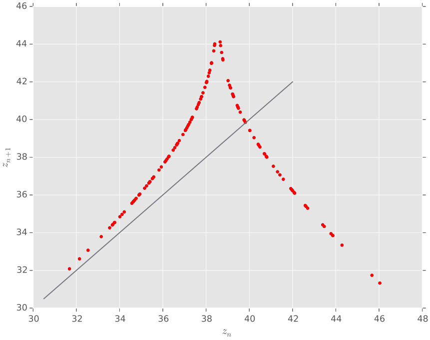

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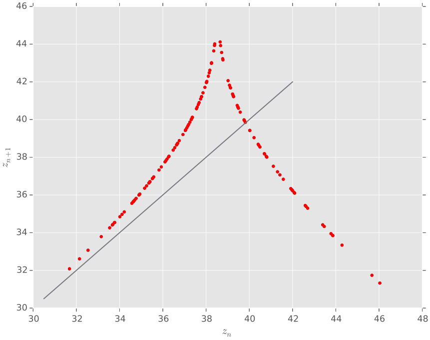

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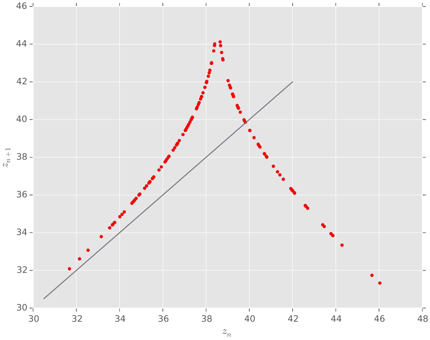

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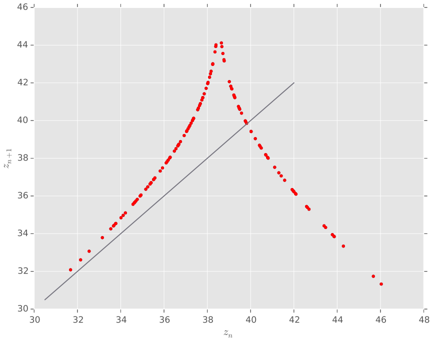
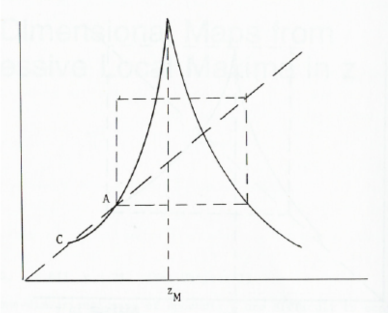


Return Map

The last argument does not hold for all values of r



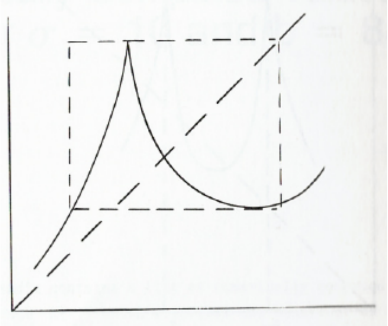


Figure: Return Map, Left: $r < 24.06$ Right: $r > 30.2$, Sparrow[1980]

Strangeness of the Lorenz Attractor

- Do the planes of attraction intersect at the origin?

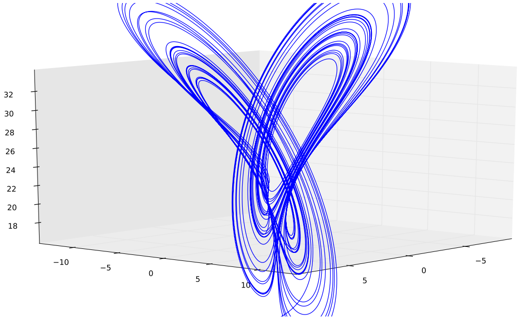


Figure: Lorenz Attractor at the origin for $r = 28$

Strangeness of the Lorenz Attractor

- Do the planes of attraction intersect at the origin?

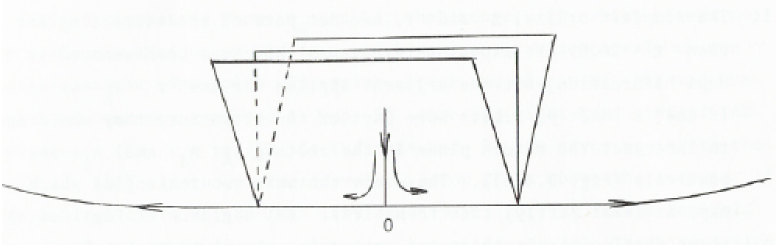


Figure: The Lorenz attractor manifold at the origin, Sparrow[1980]

“if the flap of a butterfly’s wings can be instrumental in generating a tornado, it can equally well be instrumental in preventing a tornado” *Lorenz 1972*

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