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The Sine-Gordon-Equation

- Nonlinear ODE
- Origin of the SGE
  - Transformation
- Scope of applications
  - Chain of pendulums
  - Gaussian curvature
- Solving the SGE
  - Case Analysis of the solutions
  - Solitons
    - Soliton Collision
- Modern Science

Nonlinear PDE

- $F(u+v) \neq F(u)+F(v)$

- KdV (simplified):

$$u_t + f(u)\cdot u = 0 \rightarrow u_t + v_t + f(u+v)\cdot u + f(u+v)\cdot v$$

- Solving methods:
  - Linearization (e.g. small-angle-approximation)
  - Inverse scattering method
  - Numerical solutions (e.g. Newton‘s method)

Origin of the SGE

- Energy-momentum-relation: $E^2 = p^2 + m^2$ with $c = 1$

- Schrödinger-Equation with $V = 0$

  $\hat{\mathbf{H}}\psi = E\psi$ with $E = i\hbar\frac{\partial}{\partial t}$

- Klein-Gordon-Equation (spinless particles!)

  $\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi + \left(\frac{m^2}{\hbar^2}\right)\psi = 0$ with $\boldsymbol{p} = -i\hbar\nabla$

- Klein-Gordon-Equation with any potential V

  $\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi + V' = 0$

Origin of the SGE

- Sine-Gordon-Equation:

$$\boxed{\psi_{tt} - \psi_{xx} + \sin(\psi) = 0}$$

with $\psi_{\alpha\alpha} = \frac{\partial^2}{\partial \alpha^2}\psi$ and $V'(\psi) = \sin(\psi)$

Transformation

$$\boxed{\psi_{tt} - \psi_{xx} + \sin(\psi) = 0}$$

- Transformation from (x,t) to (u,v)

$$u = \tfrac{1}{2}(x+t)$$
$$v = \tfrac{1}{2}(x-t)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$\psi_{xx} = \tfrac{1}{4}\left(\psi_{uu} + \psi_{vu} + \psi_{uv} + \psi_{vv}\right)$$

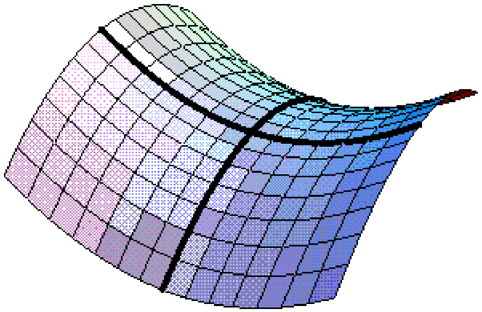
$$\psi_{tt} = \tfrac{1}{4}\left(\psi_{uu} - \psi_{vu} - \psi_{uv} + \psi_{vv}\right)$$

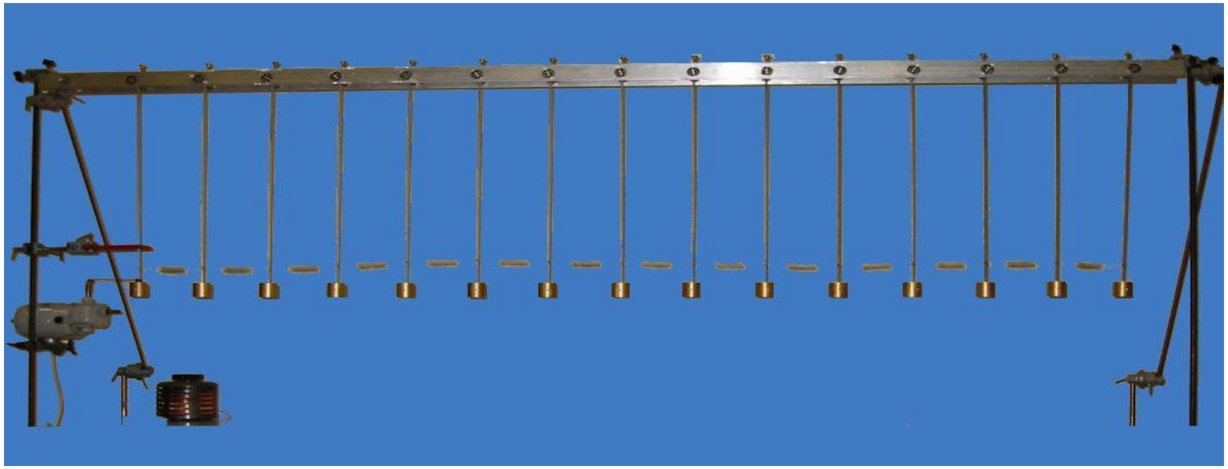
- SGE reduces to

$$\boxed{\psi_{uv} = \sin(\psi)}$$

Scope of applications

- Oscillation of a chain of pendulums
- Differential geometry: surface with Gaussian curvature $K = -1$
- Solid-state physics (crystals)
- Tentative model of an elementary particle
- Equivalent form of the Thirring model
- Nonlinear optics





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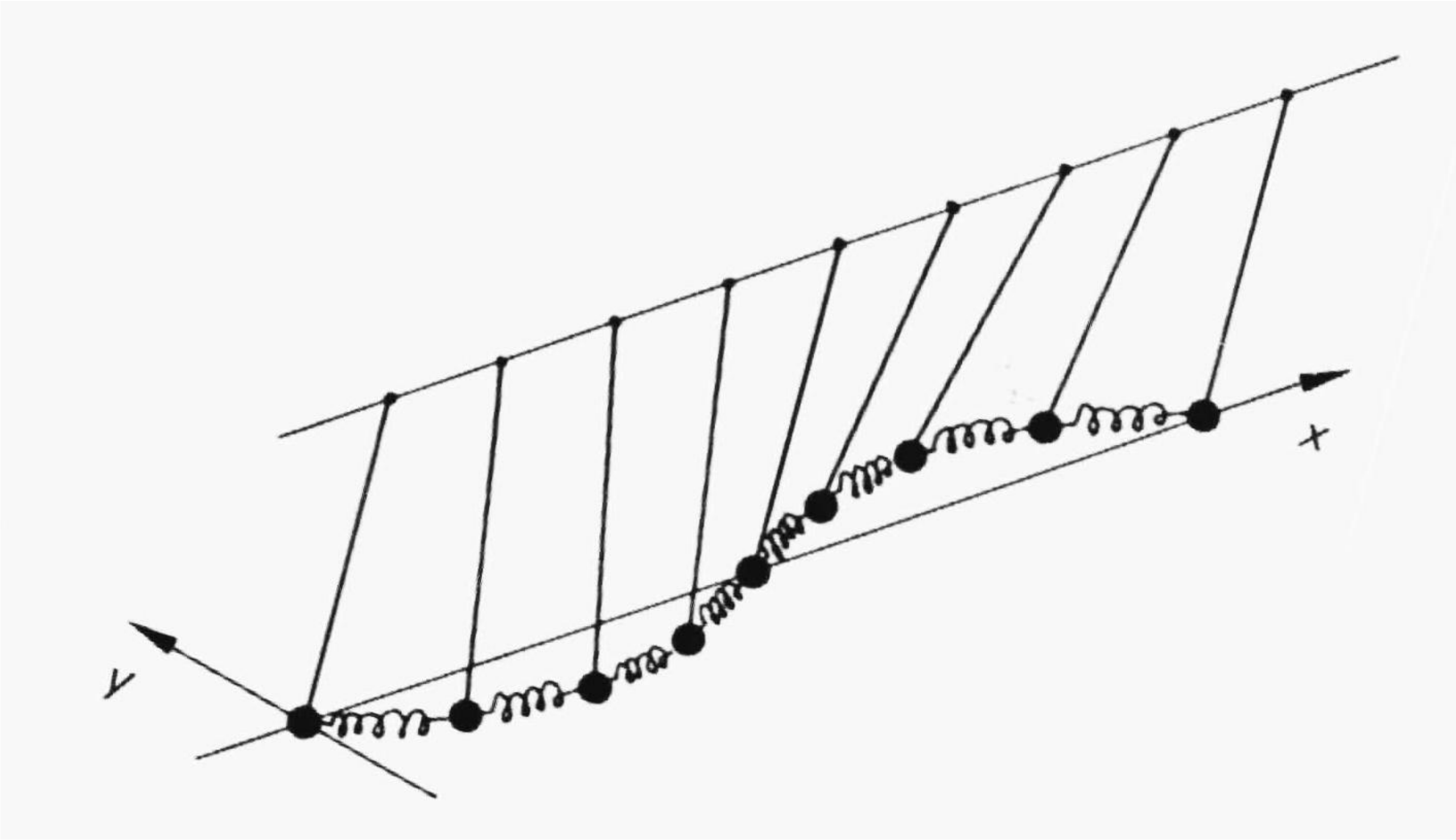
Chain of pendulums



Chain of pendulums

$$I\alpha = M_F + M_G$$

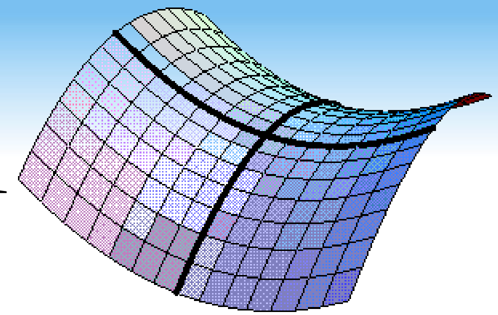
$$I\left(\frac{\partial^2}{\partial t^2}\phi_i(t)\right) = \kappa\left(\left[\phi_{i+1} - \phi_i\right] + \left[\phi_{i-1} - \phi_i\right]\right) - Mdg \cdot \sin(\phi_i)$$

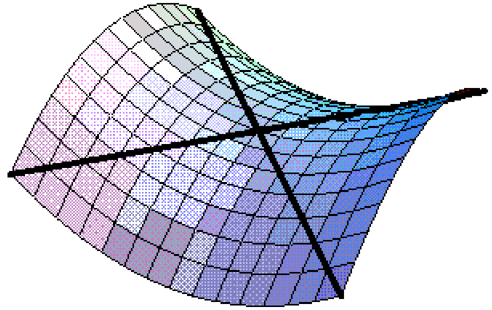
geeigneteKoordinaten :

$$X = x\sqrt{\frac{c_G}{\kappa}}, \quad T = t\sqrt{\frac{c_G}{I}}$$

$$\Rightarrow \phi_{TT} - \phi_{XX} + \sin(\phi) = 0 \quad \text{SGE}$$

Gaussian curvature





- surfaces with constant negative curvature $K=-1$
- Coordinate transformation:

$$ds^2 = E\cdot du^2 + 2F\cdot dudv + G\cdot dv^2$$

$$E = (\partial_u x)^2 = 1$$

$$F = (\partial_u x)(\partial_v x) = \cos(\theta)$$

$$G = (\partial_v x)^2 = 1$$

$$-1 = K = -\frac{\partial_{uv} F}{EG-F^2} + \frac{F}{(EG-F^2)^2}\partial_u F\cdot\partial_v F = -\frac{\partial_{uv}\theta}{\sin(\theta)}$$

$$\Rightarrow \partial_{uv}\theta = \sin(\theta) \text{ SGE!}$$

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Solving the SGE…

$$\psi(x,t) = \psi(z) = \psi(x - vt)$$

$$\overset{SGE}{\Rightarrow} (1 - v^2)\psi_{zz} = \sin(\psi)$$

$$z = \int d\psi \Big/ \pm\sqrt{\pm 2\left(A - 2\sin^2\left(\tfrac{1}{2}\psi\right)\right) / \left(1 - v^2\right)}$$

A : Integrationskonstante

v : Geschwindigkeit

Case Analysis of the solutions

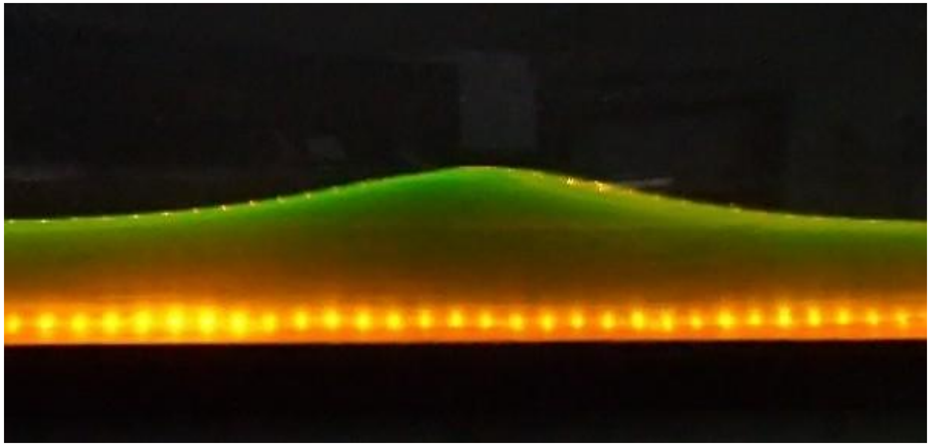
$$z = \int \frac{d\psi}{\pm\sqrt{\pm 2\left(A - 2\sin^2\left(\frac{1}{2}\psi\right)\right) / \left(1 - v^2\right)}}$$

1. Periodical Solution, unstable $0 < A < 2, v^2 > 1$
2. Periodical Solution, unstable $0 < A < 2, v^2 < 1$
3. Screw-shaped Solution, stable $A < 0, v^2 < 1$
4. Screw-shaped Solution, stable $A > 2, v^2 > 1$
5. Soliton-Solution, unstable $A = 2, v^2 > 1$
6. Soliton-Solution, stable $A = 0, v^2 < 1$
   (4 Subcases)

Solitons

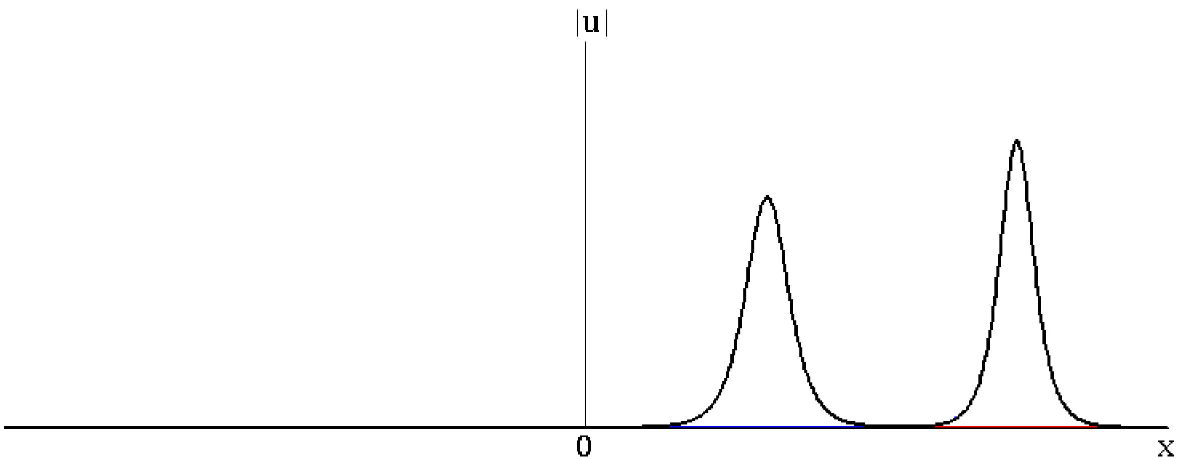
- Stable, localized solitary wave (wave packet / pulse)
- Definition is difficult to find but:
  - Solitons are of a permanent form (hump-shaped)
  - Solitons are localized within a region
  - Solitons emerge from the collision unchanged (except for a phase shift)
- E.g. water waves in a canal

  or smoke rings



Soliton collision

- Phase-shift (like KdV)



$$\psi^{2sol} = 4\arctan\left(\frac{v\sinh\left(x/\sqrt{1-v^2}\right)}{\cosh\left(vt/\sqrt{1-v^2}\right)}\right)$$

Soliton collision

- Bäcklund transformation (for SGE)

$$\frac{1}{2}\left(\varphi+\psi\right)_u = a\sin\left(\frac{\varphi-\psi}{2}\right)$$

$$\frac{1}{2}\left(\varphi-\psi\right)_v = a\sin\left(\frac{\varphi-\psi}{2}\right)$$

$$\frac{d\varphi}{du} = 2a\sin\left(\frac{\varphi}{2}\right) \Leftrightarrow \int \frac{d\varphi}{\sin\left(\frac{\varphi}{2}\right)} = 2au = 2\log\left(\tan\left(\frac{1}{4}\varphi\right)\right) + f\left(v\right)$$

$$\Rightarrow \varphi\left(u,v\right) = 4\arctan\left(\frac{f\left(v\right)}{g\left(u\right)}\exp\left(ax+\frac{t}{a}\right)\right)$$

Soliton collision

2 solitons collide with 2 antisolitons

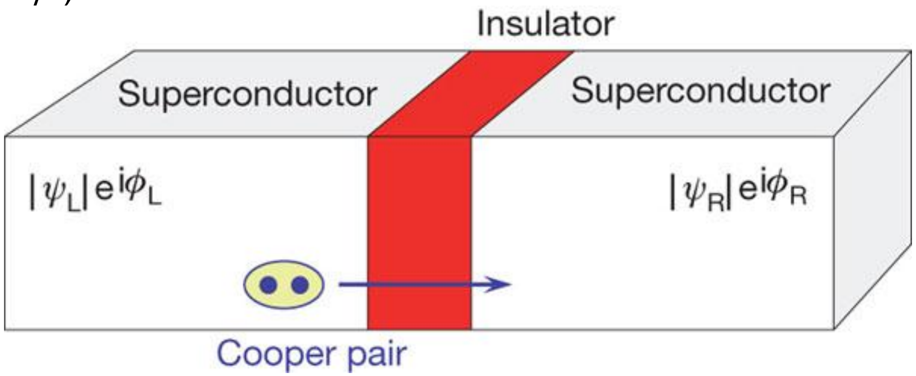
Breather

- Coupling of kink and antikink
- Special form of solitons
- Exact solution via inverse scattering

Modern Science

- Thirring Model
  - Equivalent to the quantum-SGE
  - Exactly solvable field equation

- Josephson junction
  - Extremely rapid Switching element
  - Coupling: $\psi(x,t) \leftrightarrow$ barrier
  - Current by tunneling: $I = I_c \sin(\Delta\varphi)$



Modern Science

- Einstein-Bose-condensate
  - Particle wave
  - Solitons in Rubidium-condensate (350 Atoms thickness)
    - Dispersion: non-linear Schrödinger-Equation

- Dislocation in crystals
  - Crystals that oszillate due to heat and pressure
  - Mass and energy wave as kinks

Modern Science

- Biophysics
  - DNA-proteins
    - Seperation of DNA
    - Double SGE:

$$\psi_{tt} - \psi_{xx} + \sin(\psi) = -\varepsilon \sin(\psi/2)\cos(\varphi/2)$$
$$\varphi_{tt} - \varphi_{xx} + \sin(\varphi) = -\varepsilon \sin(\varphi/2)\cos(\psi/2)$$

  - Epilepsy
    - nerv pulse $\leftrightarrow$ soliton
    - Medical: prognosis

Modern Science

- Data transfer in glass fibers
  - Small amplitude of light → Dispersion
  - $\lambda > 1{,}3\,\mu m$: solitary wave
    - pulse duration: PicoSec → TeraBits per Seconds

- Labor: 180M km of stable solitons

Conclusion

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