Chaos of planetary Motion

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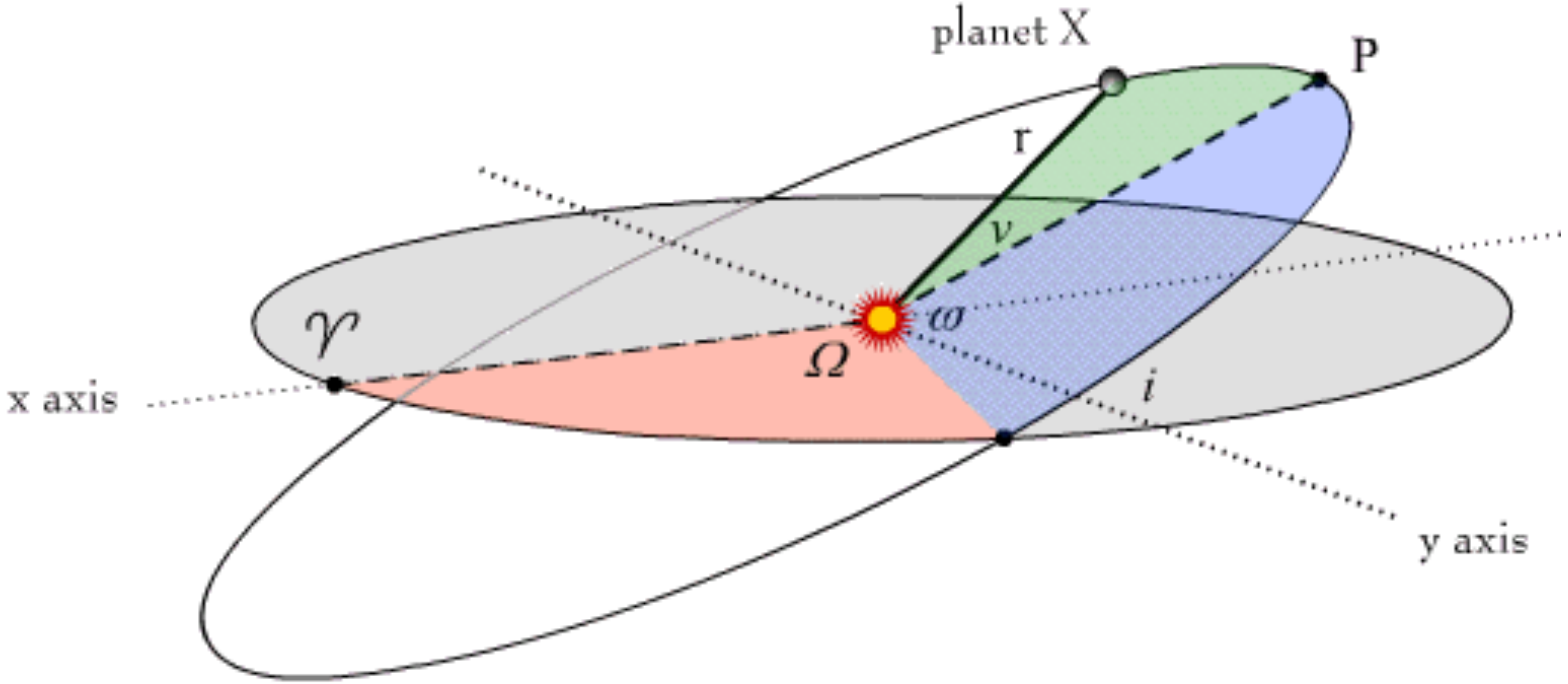
INCEPTION

History

- 1609 to 1618: Kepler fixed the Planet’s trajectories: He showed that the planets describe ellipses around the Sun.
- 1687: Newton announced the Law of universal gravitation.
- 18^{th} century: Laplace and Lagrange calculated that, long-term variations in the planets semi-major axes stay stable.
- 1892 to 1899: Poincaré said that eccentricity and inclination are unstable and there can be chaos. He used Hamilton-Jacobi-Theory and found out, that the equations of motion are not solvable analytically.

Eccentricity **e**: describes how " elliptical " the orbit is.



i is called inclination

- 1950s and 60s: the mathematicians Kolmogorov and Arnold, took up Poincaré’s work and showed, that for irrational ratios of the angular velocities in phase space the orbits stay stable.

- Because the equations of motion cannot be solved analytically, it is nowadays usual to use numerical methods (sometimes the relativistic corrections for the moon are also included).

- initial conditions are very important as the motion is often chaotic

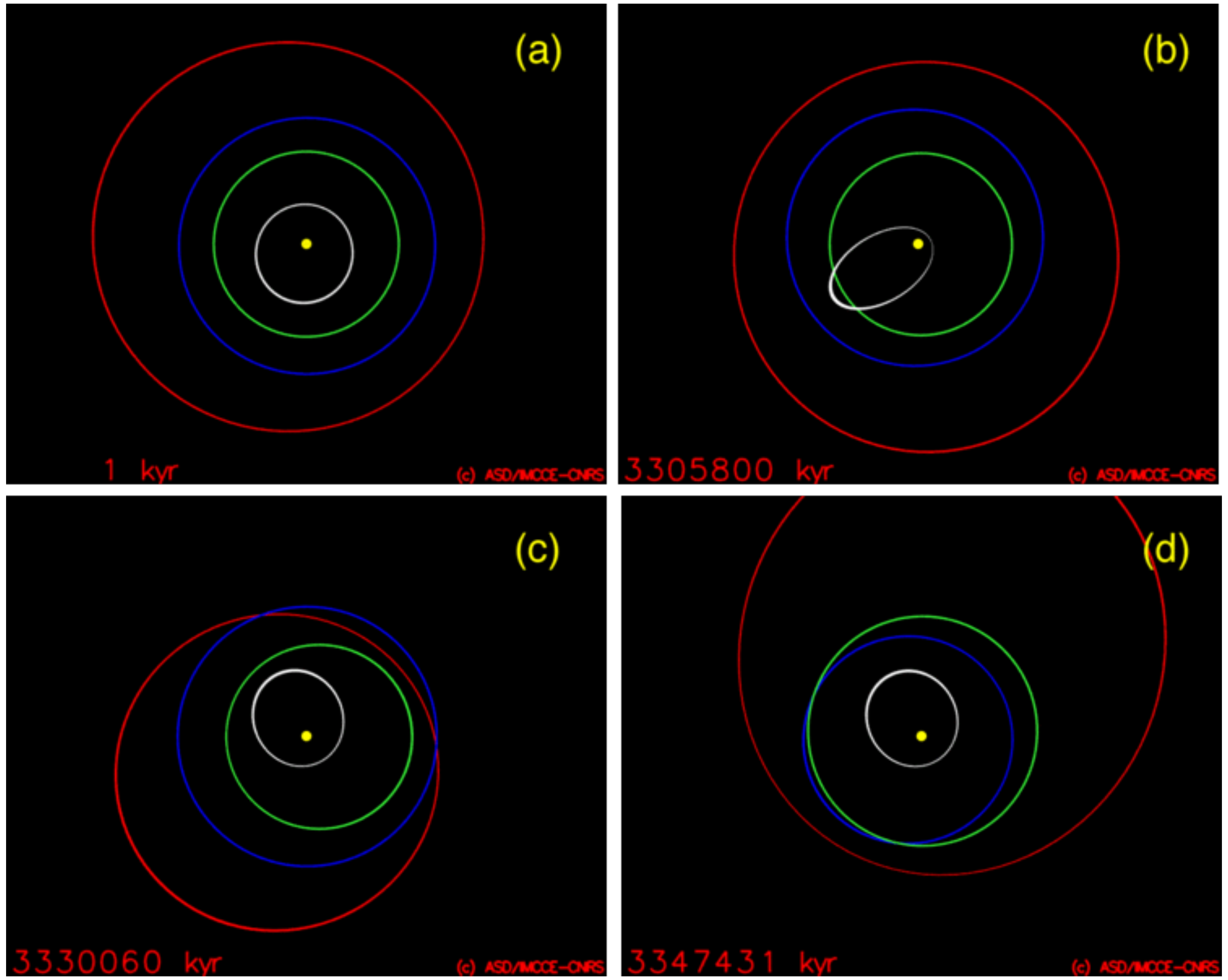
  e. g. 1994 and 2008: Laskar calculated the evolution of Solar System for different initial conditions

  a) the Solar System remains stable

  b) Mercury and Venus collide (1% chance)

  c) Earth and Mars collide

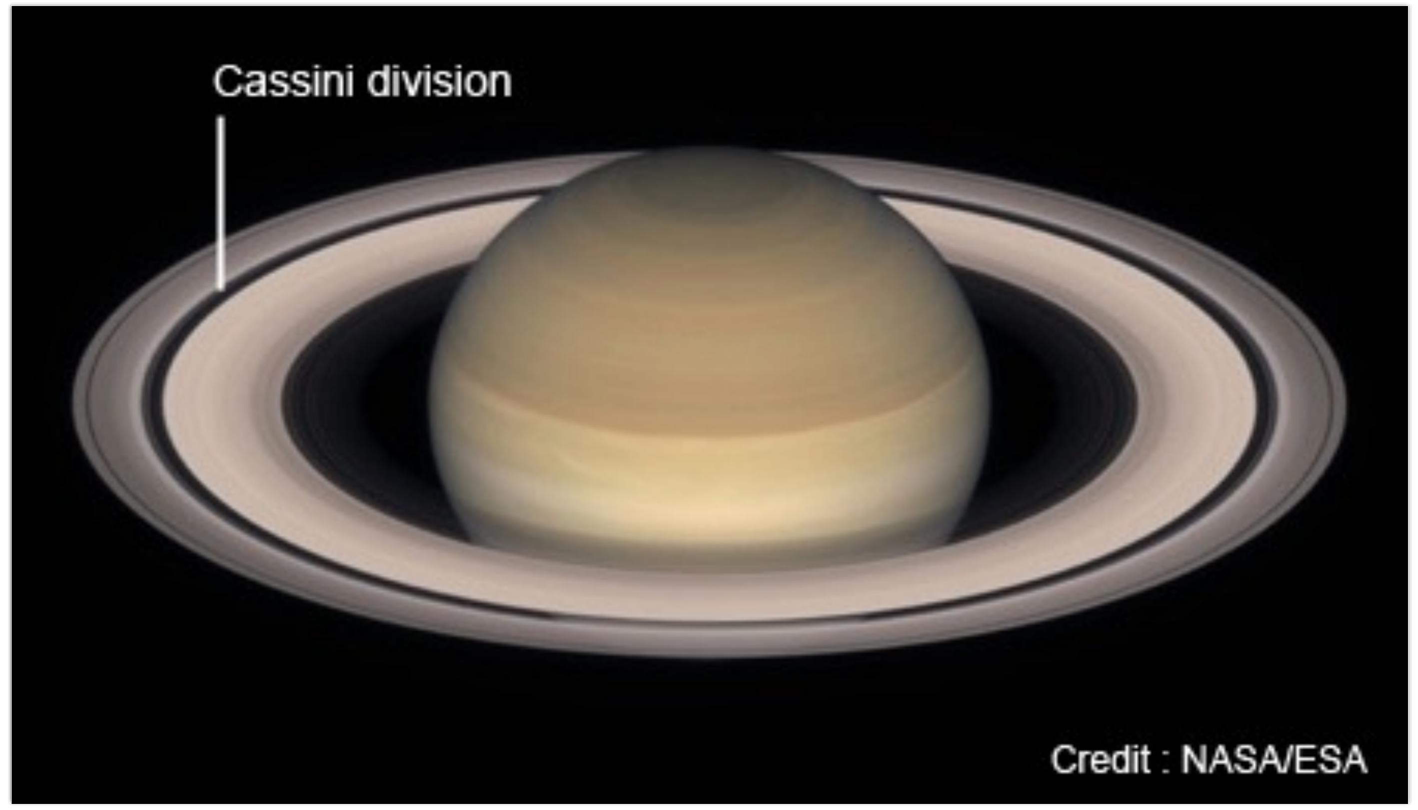
  d) Earth and Venus collide

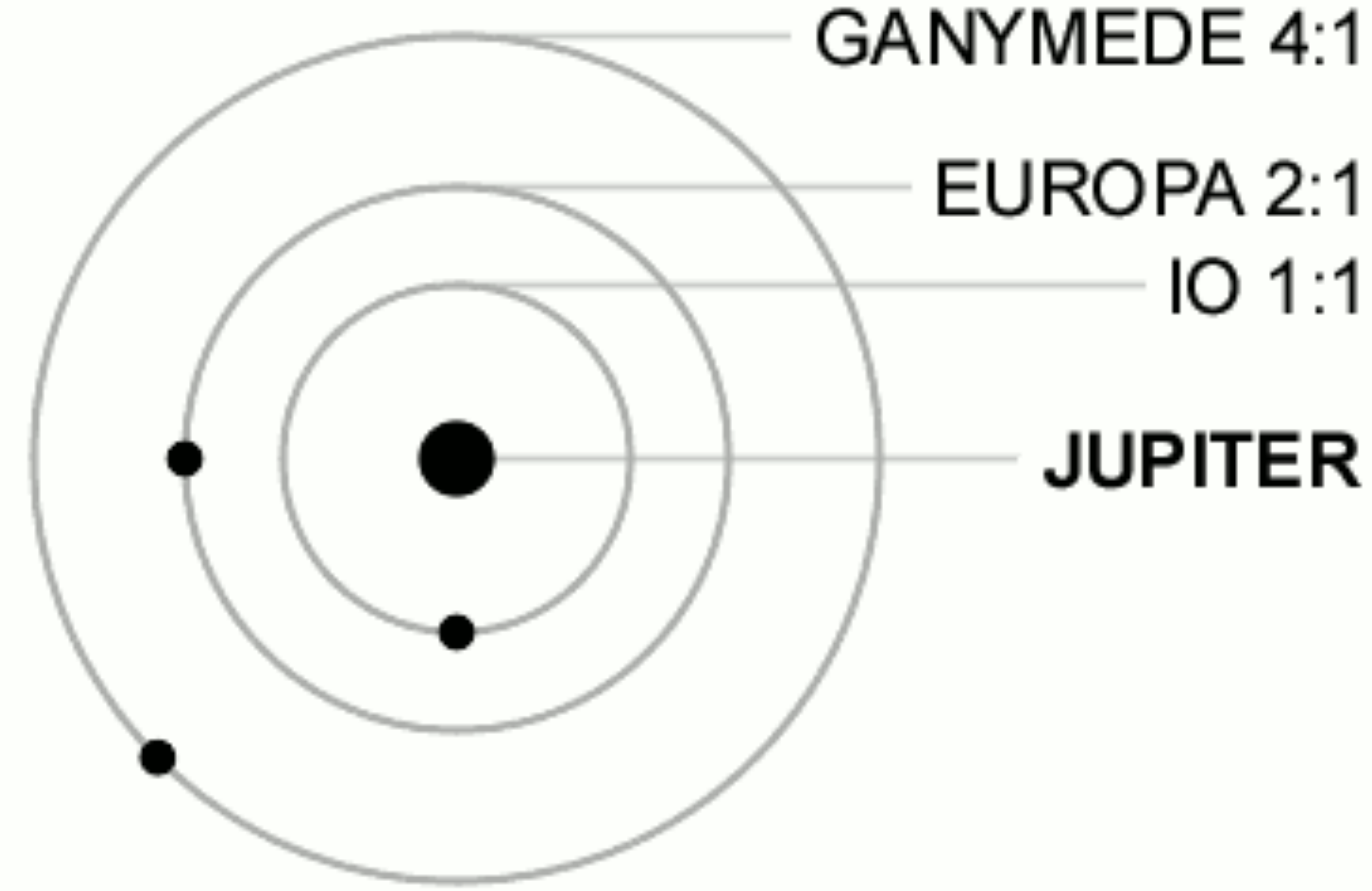


Orbital resonance

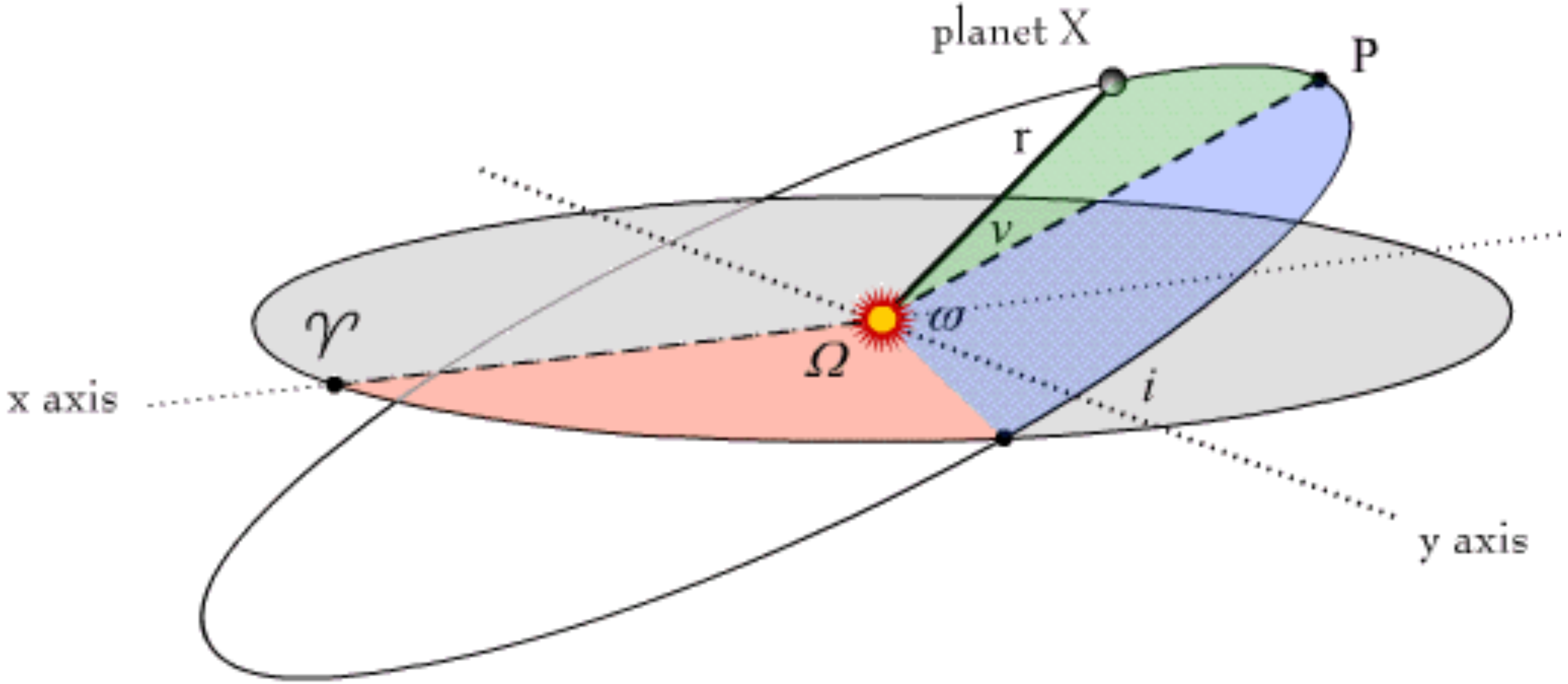
- 2 orbiting bodies exerting periodic gravitational forces on each other
- occur when orbital periods are related by a ratio of two small integers
- may have results for one or any combination of orbital parameters (e.g. eccentricity — inclination)
- can act on timescales between orbital periods and 10^4 to 10^6 years
- can either stabilize or destabilize the orbits

- Orbit of Pluto is stable despite crossing the orbit of Neptune due to a 2:3 resonance causing Neptune to be far away (Stabilization)
- Cassini-Division in the rings of Saturn caused by 2:1 resonance with moon Mimas (Destabilization)
- Laplace-Resonance: three or more bodies resonance (e.g. Ganymede, Europa, Io(moons of Jupiter) in 1:2:4 resonance)
- Secular resonance: precession (usually of perihelion or ascending node) of two orbits is synchronized
- Kozai-Resonance: inclination and eccentricity oscillate synchronously (while inclination is increasing, eccentricity decreases and vice versa)





- 2:1 resonance of Io - Europa
- orbital period of Io / Europa: $1.769\,\mathrm{d}\ /\ 3.551\,\mathrm{d}$
- but $\dfrac{360°}{1.769\,\mathrm{d}} - 2 \cdot \dfrac{360°}{3.551\,\mathrm{d}} \approx 0.745\,°\,\mathrm{d}^{-1} \neq 0$
- calculation does not contain precession of the orbits, correct formula would be $2n_{Eu} - n_{Io} - \dot{\omega}_{Io} = 0$ as Io's orbit precesses
- in some cases the calculation has to be expanded further as the point of conjunction (where the three bodies build a line) can oscillate, too



planet X
P
r
ν
ω
Ω
♈
i
x axis
y axis

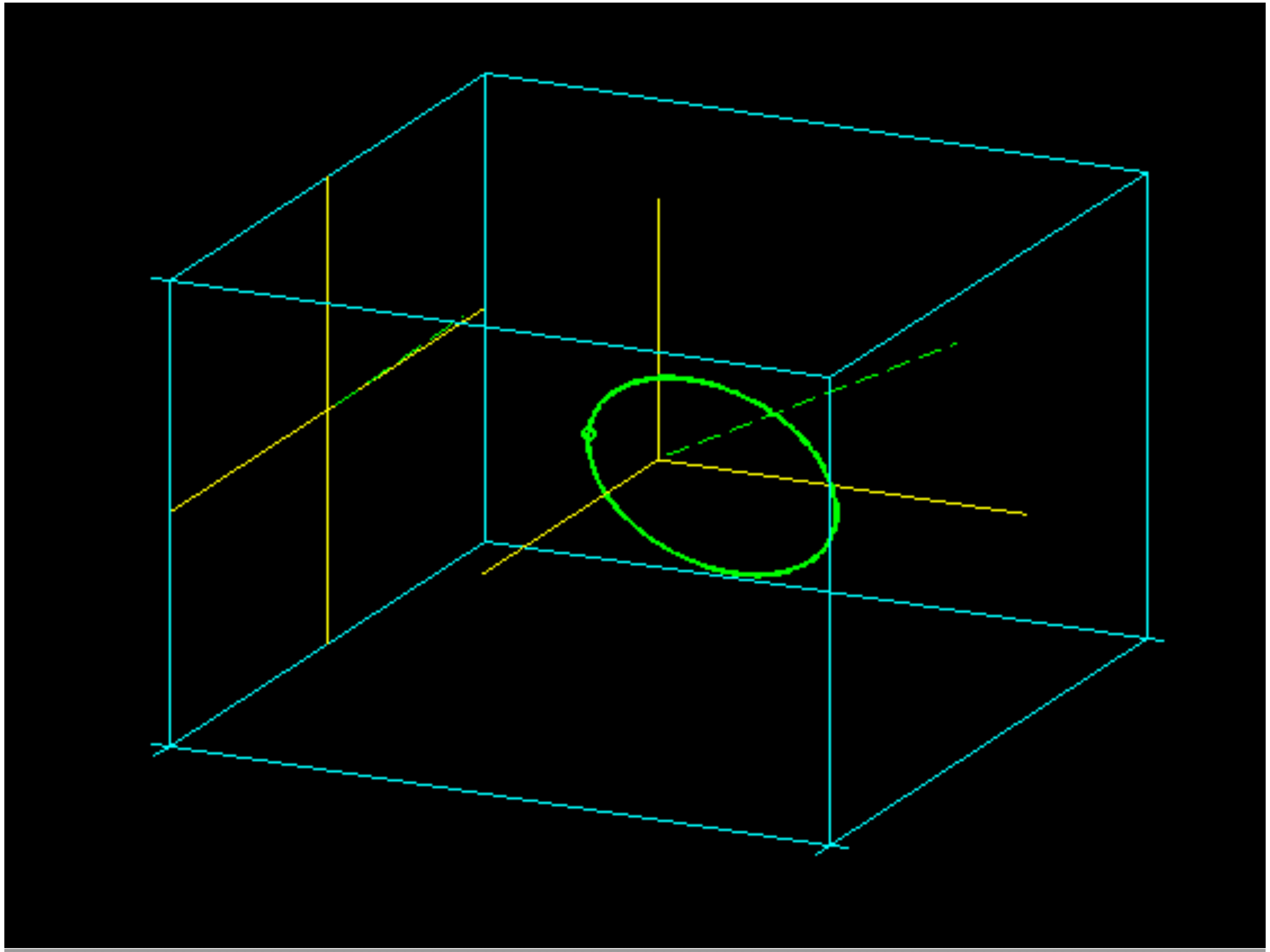
Kozai-Resonance

- two big bodies, a small mass orbiting one of the large bodies, the second body's gravitational force is small because of the distance
- average the gravitational potential of the companion over orbital periods
- after averaging the potential is constant, so the energy is conserved
- thus $a = -\frac{\gamma \cdot \mu}{2 \cdot \mathrm{E}} = const.$
- if the distant body orbits circular, the potential is axisymmetric, so L_z is conserved

$$L_z = \overrightarrow{e_\omega} \cdot \overrightarrow{L}, \quad e = \sqrt{1 - L^2} \quad \text{having set} \quad \gamma = \mu = -E = 1$$

$$\Rightarrow L = \sqrt{1 - e^2} \Rightarrow L_z = \sqrt{1 - e^2} \cdot \cos i$$

- it turns out, that $L_z \sim \sqrt{1-e^2}\ \cos i$ conserved even for elliptical orbits
- as $\sqrt{1-e^2}\ \cos i = \mathrm{const.}$ an increase of e results in a decrease of i and vice versa
- there is a limit for the initial inclination (for circular orbits $e=0$ $39.2°$) letting the orbit stable (e and i remain constant, but the orbit precesses)
- for larger initial inclinations the argument of perihelion oscillates around $90°$ and e and i oscillate remaining $\sqrt{1-e^2}\ \cos i$ conserved
- the amplitude of oscillation is independent of mass and separation of the bodies, only the period is affected by these parameters



- highly inclined orbits can become very eccentric, leading to tidal dissipation or collision
- irregular satellites (small bodies in far and eccentric orbits) only exist at small inclinations

Tidal force

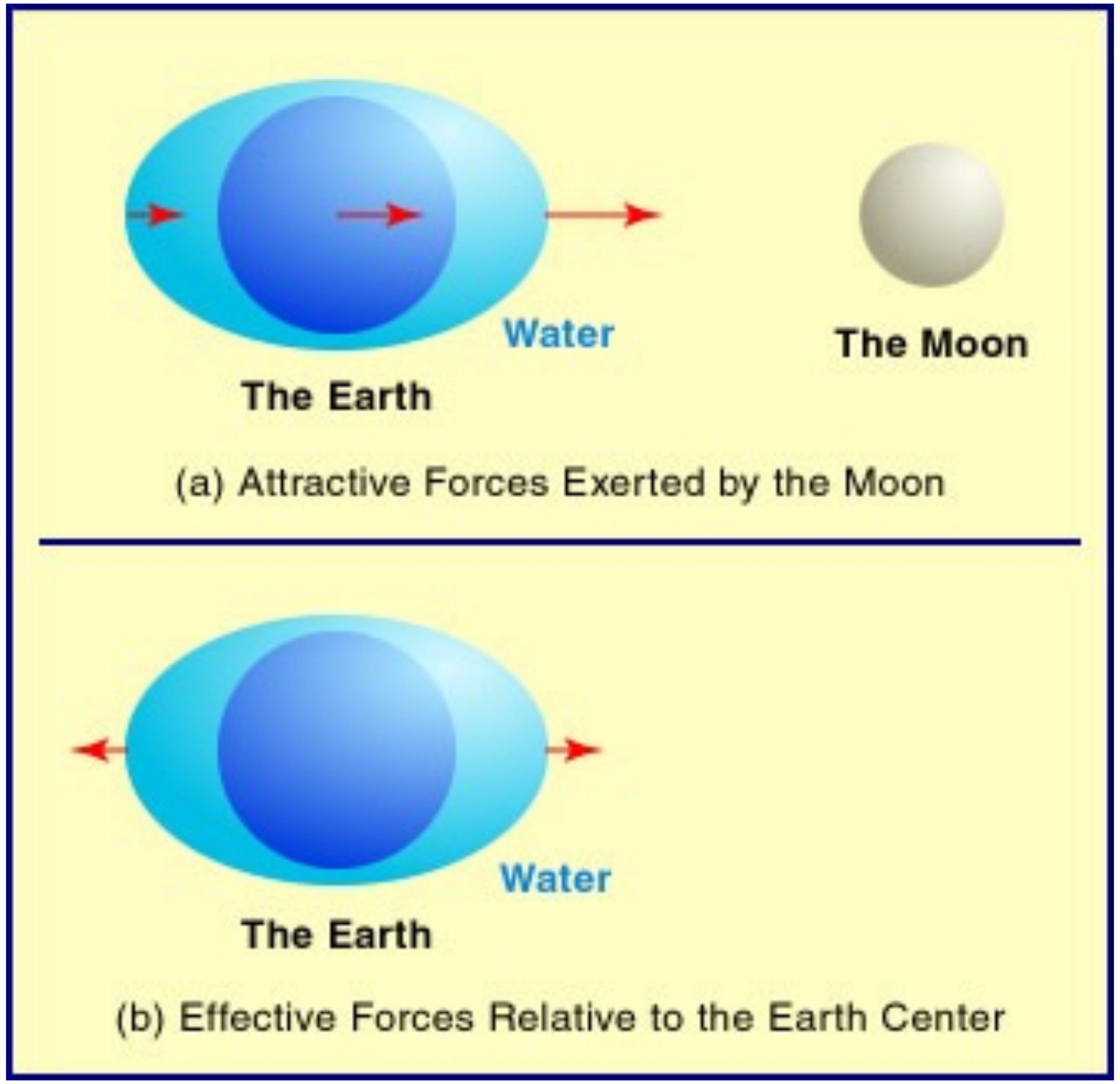
- secondary effect of the force of gravity.
- the gravitational force exerted by one body on another is not constant across it.

$$\vec{F}_g = -\hat{r}\, G \frac{Mm}{R^2} \quad ; \quad \vec{a}_g = -\hat{r}\, G \frac{M}{R^2}$$

- $\vec{a}_g = -\hat{r}\, G \dfrac{M}{(R \pm \Delta r)^2}$, M is itself a sphere of radius Δr
- $\vec{a}_g = -\hat{r}\, G \dfrac{M}{R^2} \dfrac{1}{(1 \pm \Delta r/R)^2}$, Maclaurin series: $1/(1 \pm x)^2 = 1 \mp 2x + 3x^2 \mp \ldots$

(a) $\vec{a}_g = -\hat{r}\, G\frac{M}{R^2} \pm \hat{r}\, G\frac{2M}{R^2}\frac{\Delta r}{R} + \ldots$

(b) $\vec{a}_t(\text{axial}) \approx \hat{r}\, 2\Delta r\ G\frac{M}{R^3}$

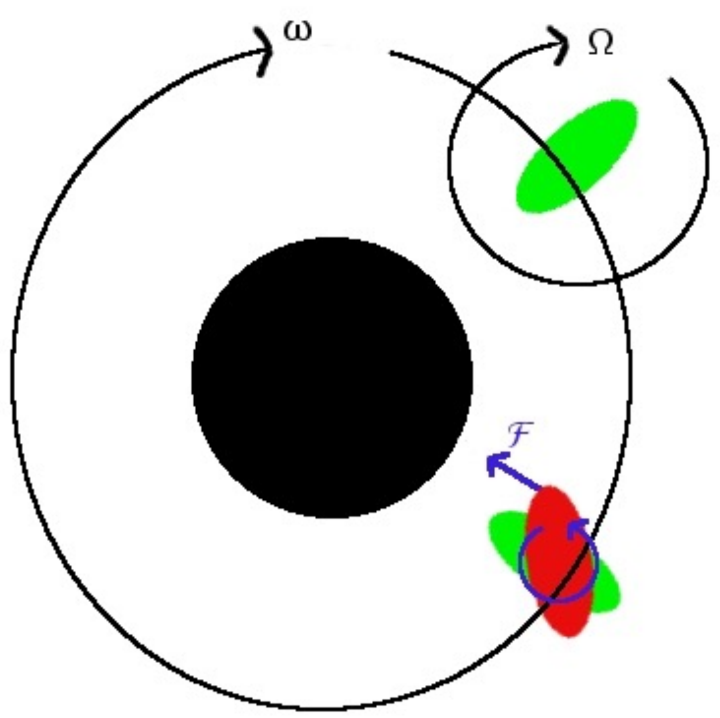


- this leads to Tidal locking

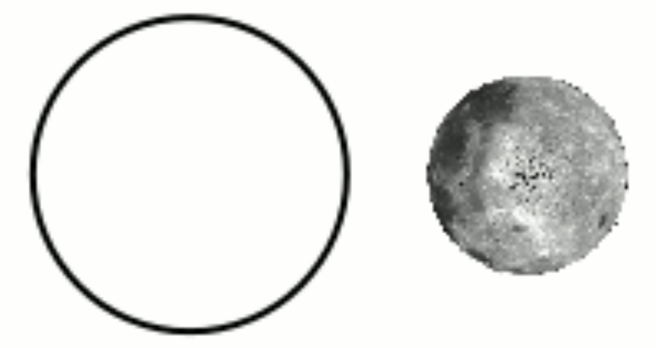
- because of the Moment of Force

  ω=Ω and the Moon is “locked” to Earth

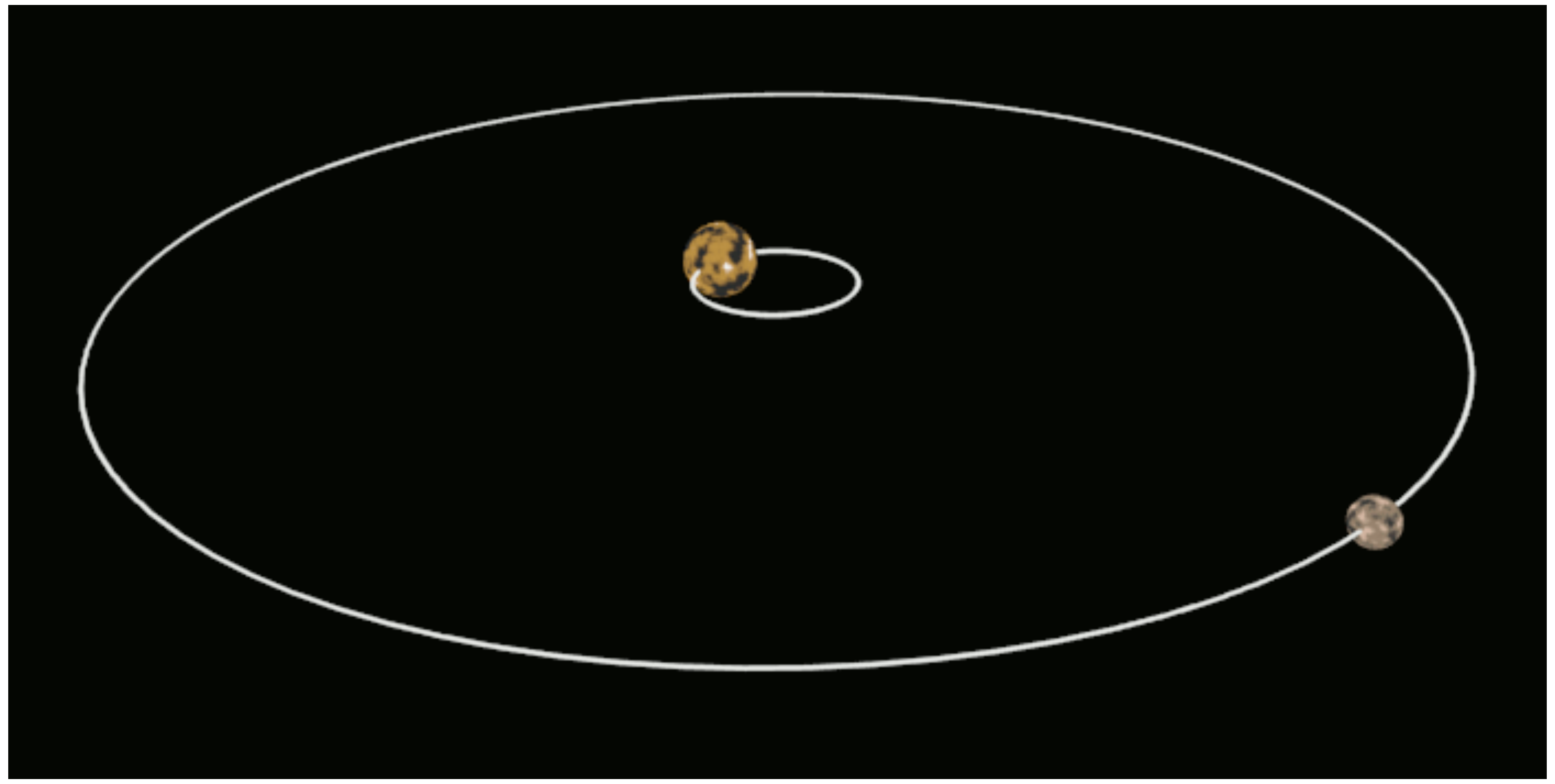
- <u>we see only one side of the Moon</u>

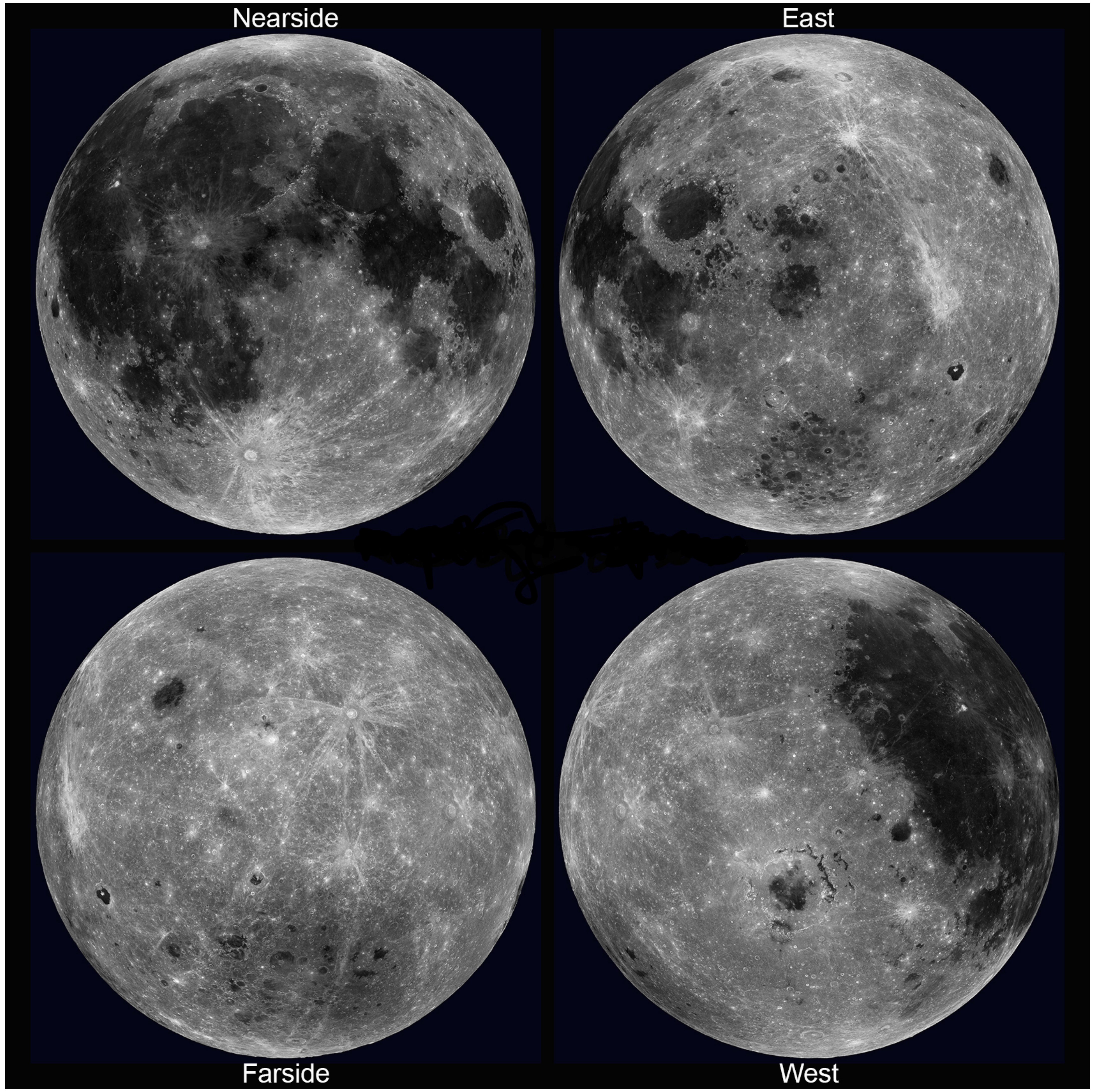


- $M \gg m$: as is the case between Earth and Moon



- $M \simeq m$: as is the case between Pluto and Charon




Nearside
East
Farside
West

Lagrangian Points

- appear in Circular Restricted Three-Body Problem
- three masses $m_1, m_2 \gg m_3$, m_1 and m_2 orbit circular in x,y-plane
- distance between m_1 and m_2, R=1, GM=1, $\mu_1 = Gm_1$, $\mu_2 = Gm_2 = 1 - \mu_1$, distance from m_3 to m_1, m_2 is ρ_1, ρ_2
- $\ddot{\vec{r}} + 2\vec{\omega} \times \dot{\vec{r}} = -\nabla U(\vec{r})$ with the effective potential

  $$U(\vec{r}) = -\frac{\mu_1}{\rho_1} - \frac{\mu_2}{\rho_2} - \frac{\omega^2}{2}\left(x^2 + y^2\right)$$
- points where $\ddot{\vec{r}} = \dot{\vec{r}} = 0 \Leftrightarrow \nabla U = 0$ in co-rotating frame are called Lagrange Points

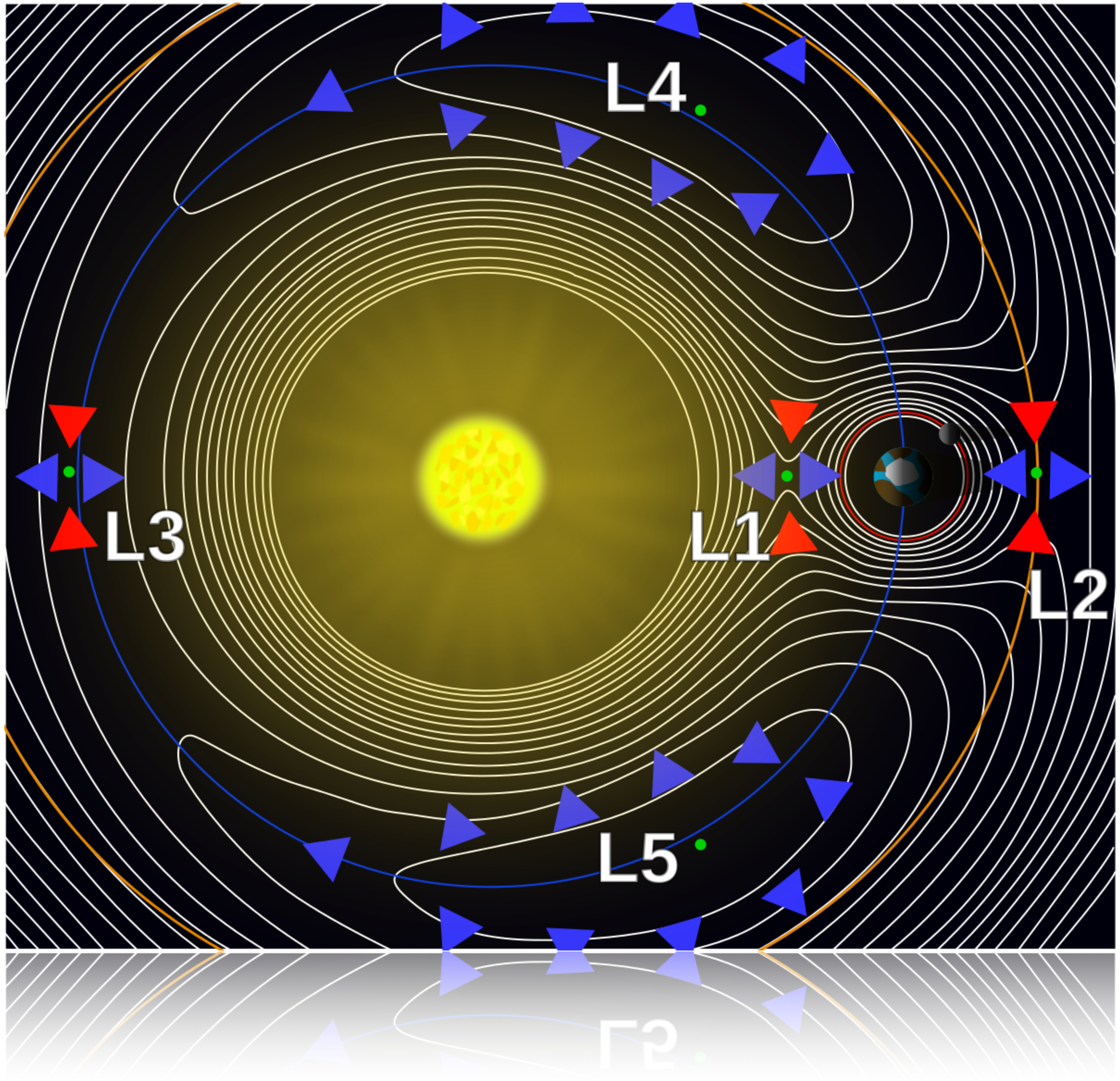
- $\frac{\partial U}{\partial z} = \left(\frac{\mu_1}{\rho_1^3} + \frac{\mu_2}{\rho_2^3}\right) z = 0 \Leftrightarrow z = 0$ shows that the Lagrangian Points lie in x,y-plane

- by rewriting $U = -\mu_1 \left(\frac{1}{\rho_1} + \frac{\rho_1^2}{2}\right) - \mu_2 \left(\frac{1}{\rho_2} + \frac{\rho_2^2}{2}\right) + \frac{\mu_1 \mu_2}{2}$ we see $\frac{\partial U}{\partial x} = \frac{\partial U}{\partial \rho_1}\frac{\partial \rho_1}{\partial x} + \frac{\partial U}{\partial \rho_2}\frac{\partial \rho_2}{\partial x}$ and the same for $\frac{\partial U}{\partial y}$

- $\frac{\partial U}{\partial y} = \left(\mu_1 \frac{1-\rho_1^3}{\rho_1^3} + \mu_2 \frac{1-\rho_2^3}{\rho_2^3}\right) y = 0$ has the obvious solution *y=0*, so there are Lagrangian Points on the x-axis

- $\frac{\partial U}{\partial x} = 0$ is more complicated to solve, but there are the approximations
$$x_1 \approx \mu_1 - \alpha, \quad x_2 \approx \mu_1 + \alpha, \quad x_3 \approx \mu_1 - 2 + 7\alpha^3/4 \quad \text{with} \quad \alpha = \left(\frac{\mu_2}{3\mu_1}\right)^{1/3}$$

- to find Lagrangian Points not lying on x-axis we search for solutions of $\dfrac{\partial U}{\partial \rho_1} = \dfrac{\partial U}{\partial \rho_2} = 0$
- this means $\mu_1 \cdot \left(\dfrac{1}{\rho_1^2} - \rho_1 \right) = \mu_2 \cdot \left(\dfrac{1}{\rho_2^2} - \rho_2 \right) = 0$
- possible solutions are $\rho_1 = \rho_2 = 1$
- this defines an equilateral triangle
- coordinates are $x_{4,5} = \dfrac{1}{2} - \mu_2, \quad y_{4,5} = \pm\dfrac{\sqrt{3}}{2}$


L4
L3
L1
L2
L5

Stability of Lagrangian Points

- $\left(\frac{\partial^2 U}{\partial z^2}\right)\Big|_{z=0} = \frac{\mu_1}{\rho_1^3} + \frac{\mu_2}{\rho_2^3} > 0$, so all Lagrangian Points are stable to small displacements to the z-axis

- now consider small displacements on the x- and y-axis

  $x = x_0 + \delta x \quad y = y_0 + \delta y$

- expand $U\left(\overrightarrow{r}\right)$ as a Taylor Series $U = U_0 + \frac{1}{2}U_{xx}\left(\delta x\right)^2 + U_{xy}\delta x \delta y + \frac{1}{2}U_{yy}\left(\delta y\right)^2$
  ($U_x = U_y = 0$ at a Lagrangian Point)

- substituting this into the equations of motion yields

  $$\delta\ddot{x} - 2\delta\dot{y} = -U_{xx}\delta x - U_{xy}\delta y$$

  $$\delta\ddot{y} + 2\delta\dot{x} = -U_{xy}\delta x - U_{yy}\delta y$$

  $$\text{having set } \omega = 1$$

- Ansatz: $\delta x\,(t) = \delta x_0 \cdot \mathrm{e}^{\gamma t} \quad \delta y\,(t) = \delta y_0 \cdot \mathrm{e}^{\gamma t}$
- this yields

$$\begin{pmatrix} \gamma^2+U_{xx} & -2\gamma+U_{xy} \\ 2\gamma+U_{xy} & \gamma^2+U_{yy} \end{pmatrix} \cdot \langle \delta \mathrm{x}_0\ \delta y_0 \rangle = \langle 0\ 0 \rangle$$

- solutions only exist for **det=0**
- stability criterion: ɣ is purely imaginary

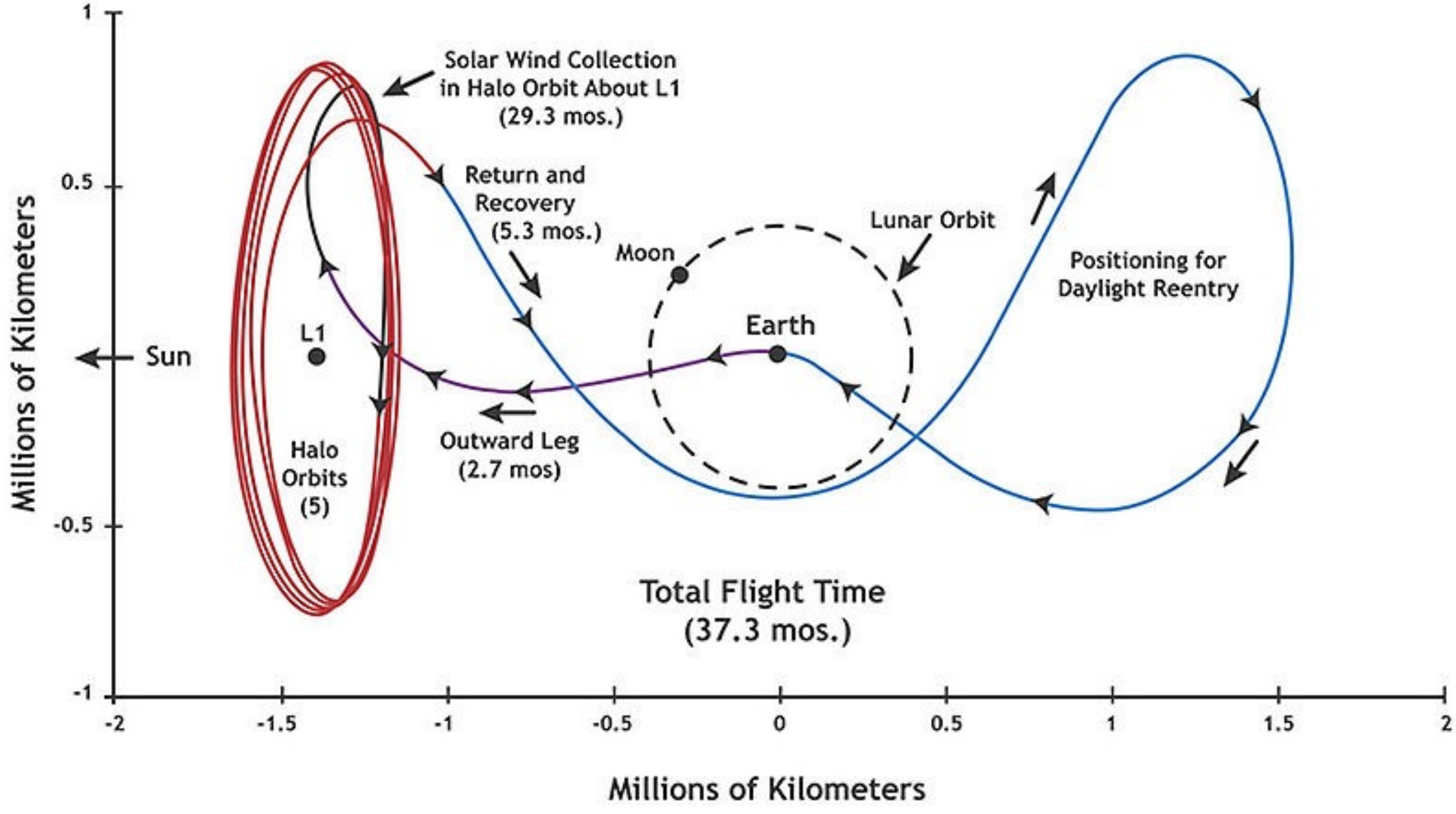
- for L_1, L_2, L_3 this means $\frac{8}{9} \leq \frac{\mu_1}{\rho_1^3} + \frac{\mu_2}{\rho_2^3} < 1$, but at L_1, L_2, L_3 $\frac{\mu_1}{\rho_1^3} + \frac{\mu_2}{\rho_2^3} > 1$ so L_1, L_2 and L_3 are unstable

- for L_4 and L_5 the stability criterion is $\mu_2 < 0.04$ which means that the smaller mass is less then about 4% of the bigger mass as $\mu_2 = \frac{m_2}{\mathrm{m}_1 + m_2}$

- L_1 (in Earth-Sun-System) is often used to observe the Sun

- L_2 for space-based observatories

- L_4 and L_5 contain interplanetary dust



Solar Wind Collection
in Halo Orbit About L1
(29.3 mos.)
Return and
Recovery
(5.3 mos.)
Moon
Lunar Orbit
Positioning for
Daylight Reentry
Earth
L1
Sun
Halo
Orbits
(5)
Outward Leg
(2.7 mos)
Total Flight Time
(37.3 mos.)
Millions of Kilometers
Millions of Kilometers
1
0.5
-0.5
-1
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

- https://en.wikipedia.org/wiki/Orbital_Resonance
- http://www.sns.ias.edu/ckfinder/userfiles/files/ipmu%2520kozai.pdf
- http://www.scholarpedia.org/article/Stability_of_the_solar_system
- http://farside.ph.utexas.edu/teaching/336k/Newtonhtml/node118.html
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- https://en.wikipedia.org/wiki/Orbital_inclination
- https://en.wikipedia.org/wiki/Lagrangian_point
- https://en.wikipedia.org/wiki/Tidal_locking
- https://en.wikipedia.org/wiki/Tidal_force
- https://en.wikipedia.org/wiki/Lagrangian_point
- https://de.wikipedia.org/wiki/Charon_(Mond)
- https://en.wikipedia.org/wiki/Genesis_(spacecraft)
- http://www.nasa.gov/mission_pages/genesis/media/jpl-release-071702.html
- Keynote, iMovie