Statistical Physics Exam

1 Series of questions

1.1 Fermions:

a) Write down the 2-particle wave function for two Fermions with single particle states $\phi_i(x_j), i, j = 1, 2$. (2 P.)

b) What is a Slater determinant? (2 P.)

c) Compute the Fermi-energy $\epsilon_F$ of a system of $N$ Fermions in its ground state as a function of $N$ (at zero temperature). (4 P.)

1.2 Second quantization:

a) Express the particle density and the total particle number in terms of the field operators. (2 P.)

b) Write down a generic Hamiltonian of a system of $N$ particles within a potential $U(x)$ and a 2-particle interaction $V(x - x')$ in second quantized form in the momentum representation. (4 P.)

1.3 Fermions II:

a) Express the pair-distribution function $g_{\sigma\sigma'}(x - x')$ in terms of particle-density operators. 

Hint: Use the definition $\langle \phi_0 | \Psi_{\sigma'}(x') \Psi_{\sigma'}(x') \Psi_\sigma(x) | \phi_0 \rangle = \left( \frac{n}{2} \right)^2 g_{\sigma\sigma'}(x - x')$. (2 P.)

b) Compute the pair-distribution function for free fermions, discuss separately the cases $\sigma = \sigma'$ and $\sigma \neq \sigma'$. 

Hint: Work in momentum representation. (6 P.)

1.4 Density matrix and correlators:

a) Write down the density matrix $\rho$ for a canonical ensemble. What is the partition function $Z$? (2 P.)

b) Express the expectation value $\langle O \rangle$ of an observable $O$ with the help of the density matrix. (2 P.)

c) Prove that, if the Hamiltonian $H$ is not explicitly time-dependent, the correlation functions satisfy $\langle A(t)B(t') \rangle = \langle A(t - t')B(0) \rangle$. 

Hint: Go to the Heisenberg picture. (4 P.)
2 Exercices

2.1 Heisenberg model:

The Heisenberg model of a ferromagnet is defined by the Hamiltonian

$$H = -\frac{1}{2} \sum_{l,l'} J(|l - l'|) \vec{S}_l \cdot \vec{S}_{l'},$$

where $l$ and $l'$ are nearest neighbor sites on a cubic lattice. In the large spin approximation ($S \gg 1$), the Holstein-Primakoff transformation

$$S_i^+ = \sqrt{2S} \phi(\hat{n}_i) a_i,$$
$$S_i^- = \sqrt{2S} a_i^\dagger \phi(\hat{n}_i),$$
$$S_i^z = S - \hat{n}_i,$$

with the number $S$ denoting the total spin, $\phi(\hat{n}_i) = \sqrt{1 - \hat{n}_i/2S}$ and $\hat{n}_i = a_i^\dagger a_i$, can be used to express the Hamiltonian in terms of Bose operators $a_i$.

a) Show that the commutation relations for the spin are satisfied. (4 P.)

b) Write down the Hamiltonian to second order (harmonic approximation) in terms of the Bose operators $a_i$ by regarding the square-roots in the above transformation as a short hand for the series expansion. (6 P.)

c) Diagonalize $H$ by means of a Fourier transform and determine the dispersion relation of the spin waves (magnons). (6 P.)

2.2 Bogoliubov theory of the Bose liquid:

In Bogoliubov’s theory, the Hamiltonian of the Bose liquid reads:

$$H_2 = \sum_k \left( \frac{\hbar^2 k^2}{2m} - \mu \right) a_k^\dagger a_k + \frac{n_0 g^2}{2} (a_k^\dagger a_{-k}^\dagger + 4a_k^\dagger a_k + a_{-k} a_k)$$

where $\mu = n_0 g$ being the chemical potential.

We introduce the Bogoliubov transformation:

$$a_k = u_k \alpha_k + v_k \alpha_{-k}^\dagger,$$
$$a_k^\dagger = u_k \alpha_k^\dagger + v_k \alpha_{-k},$$

where $u_k$ and $v_k$ are real and obey $u_k^2 - v_k^2 = 1$.

The idea is to rewrite the Hamiltonian in terms of these new operators and then to vary the parameters $u_k$ and $v_k$ to make it diagonal.

a) Writing the Bogoliubov transformation in the matrix form,

$$\begin{pmatrix} b_k \\ b_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} u_k & v_k \\ v_k & u_k \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix},$$

show that the pair of equations can be inverted to yield

$$\begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ -v_k & u_k \end{pmatrix} \begin{pmatrix} b_k \\ b_{-k}^\dagger \end{pmatrix}.$$ (4 P.)
b) Rewrite the Hamiltonian \( H_2 \) in the matrix form
\[
H_2 = \sum_k \begin{pmatrix} a^\dagger_k & a_{-k} \end{pmatrix} \begin{pmatrix} \epsilon_k + n_0 g & n_0 g / 2 \\ n_0 g / 2 & 0 \end{pmatrix} \begin{pmatrix} a_k \\ a^\dagger_{-k} \end{pmatrix}
\]
where \( \epsilon_k = \hbar^2 k^2 / 2m \). (4 P.)

c) Use the inverse of the Bogoliubov transformation to express \( H_2 \) in terms of the \( b' \)'s operators, namely you should find:
\[
H_2 = \sum_k \begin{pmatrix} b^\dagger_k & b_{-k} \end{pmatrix} M \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} b_k \\ b^\dagger_{-k} \end{pmatrix}
\]
where the coefficients \( M_{ij} \) have to be computed explicitly. (8 P.)

d) Show that the condition for the transformed matrix to be diagonal is that
\[
2u_k v_k = \frac{n_0 g}{\epsilon_k + n_0 g}
\]
must be satisfied. (4 P.)

e) Show that the trace of the \( M \) matrix is
\[
E = (\epsilon_k + n_0 g)(u_k^2 + v_k^2) - 2n_0 g u_k v_k.
\]
Using the representation \( u_k = \cosh(\theta) \) and \( v_k = \sinh(\theta) \), show from d) that
\[
\tanh(2\theta) = \frac{n_0 g}{\epsilon_k + n_0 g}
\]
and hence prove that
\[
E = \sqrt{\epsilon_k (\epsilon_k + 2n_0 g)},
\]
consistent with the Bogoliubov quasiparticle dispersion given in the lecture. (4 P.)

Total: (70 P.)