HomeWork 11 Linear Response Theory

Reminder: Read about the interaction picture ... (references and comments will be provided during the tutorial)

1 Response to a perturbation

We consider a system described by the Hamiltonian H'. This Hamiltonian is the sum of a time independent term H, with ground state Φ_0, and a perturbation explicitly time-dependent $\delta H(t)$.

$$H' = H + \delta H(t)$$

We want to calculate the mean value of an observable A. To do so, we'd better work within the interaction picture (described above). The mean value of A reads:

$$\langle A \rangle(t) = \langle \Phi_I(t)|A_I(t)|\Phi_I(t)\rangle.$$

We suppose that the perturbation is switched on adiabatically at $t = -\infty$ ($\delta H(-\infty) = 0$) then $\Phi_I(t) = U_I(t, -\infty)\Phi_I(-\infty)$ and $\Phi_I(-\infty) = \Phi_0$. In interaction picture the evolution operator U_I is:

$$U_I(t, -\infty) = T\left[\exp\left(-i\int_{-\infty}^{t} dt_1 \delta H_I(t_1)\right)\right]$$

1.1 Prove that, to first order in δH, the mean value of A is given by

$$\langle A \rangle(t) = A_0 + i\int_{-\infty}^{t} \langle \Phi_0|[\delta H_I(t_1), A_I(t)]|\Phi_0\rangle$$

A_0 has to be interpreted ...

1.2 Suppose that δH is of the form

$$\delta H(t) = BF(t) \Rightarrow \delta H_I(t) = B_I F(t)$$

with B being an operator, F a function of time. We then define the susceptibility χ by:

$$\langle A \rangle(t) - A_0 = \int_{-\infty}^{\infty} F(t')\chi_{BA}(t, t')$$

with $\chi_{BA}(t, t') = i\langle \Phi_0|[B_I(t'), A_I(t)]|\Phi_0\rangle\theta(t - t')$.

If H is independent of t, prove that χ_{BA} depends only on the difference $t - t'$. Write also the last relation in Fourier representation.

2 Application: The dielectric function

We consider a system of electrons. If there is a little charge fluctuation in the system, the associated potential will polarized the electrons, creating a response of the system. If the fluctuation is small enough, the linear response theory is appropriate. The induced density of charge created by the fluctuation $\delta\rho$ is:

$$\rho^{ind}(x,t) = \int dt' \int dx \phi_{\rho,\rho}(x-x', t-t')\delta\rho(x',t').$$

it is useful to introduce the so-called *dielectric function* defined by

$$\delta\rho(x,t) = \int dt' \int dx \epsilon(x-x', t-t')\rho^{tot}(x',t')$$

or in Fourier representation

$$\delta\rho(k,\omega) = \epsilon(k,\omega)\rho^{tot}(k,\omega)$$

2.1 Justify qualitatively the form of the perturbation

$$\delta H(t) = \int dr \int dr' \rho(r) \frac{e^2}{|r-r'|}\delta\rho(r',t)$$

2.2 Rewrite $\delta H(t)$ by introducing the momentum representation of ρ and $\delta\rho$ to find

$$\delta H(t) = \frac{4\pi e^2}{\Omega}\sum_{q'} \rho(-q')\frac{1}{q'^2}\delta\rho(q',t).$$

We see from the above relation that this perturbation is the sum over q' of elementary perturbations having the usual form (cf part **1**): $B = \rho(-q')$ and $F(t) = \frac{4\pi e^2}{q'^2\Omega}\delta(q',t)$.

2.3 Write the total response $\langle\rho(q,t)\rangle$ with the results of linear response theory.

2.4 Express, using the preceding results, the susceptibility χ_{ρ_{-q},ρ_q} first as a function of $\phi_{\rho,\rho}$ and then with the dielectric function making use of:

$$\epsilon^{-1} = \frac{\rho^{ind} + \delta\rho}{\delta\rho}$$