HomeWork 11 Linear Response Theory

Reminder: Read about the interaction picture ... (references and comments will be provided during the tutorial)

1 Response to a perturbation

We consider a system described by the Hamiltonian $H'$. This Hamiltonian is the sum of a time independent term $H$, with ground state $\Phi_0$, and a perturbation explicitly time-dependent $\delta H(t)$.

$$H' = H + \delta H(t)$$

We want to calculate the mean value of an observable $A$. To do so, we’d better work within the interaction picture (described above). The mean value of $A$ reads:

$$\langle A \rangle(t) = \langle \Phi_I(t) | A_I(t) | \Phi_I(t) \rangle.$$

We suppose that the perturbation is switched on adiabatically at $t = -\infty$ ($\delta H(-\infty) = 0$) then $\Phi_I(t) = U_I(t, -\infty)\Phi_I(-\infty)$ and $\Phi_I(-\infty) = \Phi_0$. In interaction picture the evolution operator $U_I$ is:

$$U_I(t, -\infty) = T \left[ \exp \left( -i \int_{-\infty}^{t} dt_1 \delta H_I(t_1) \right) \right]$$

1.1 Prove that, to first order in $\delta H$, the mean value of $A$ is given by

$$\langle A \rangle(t) = A_0 + i \int_{-\infty}^{t} \langle \Phi_0 | [\delta H_I(t_1), A_I(t)] | \Phi_0 \rangle$$

$A_0$ has to be interpreted ...

1.2 Suppose that $\delta H$ is of the form

$$\delta H(t) = BF(t) \Rightarrow \delta H_I(t) = B_I F(t)$$

with $B$ being an operator, $F$ a function of time. We then define the susceptibility $\chi$ by:

$$\langle A \rangle(t) - A_0 = \int_{-\infty}^{\infty} F(t') \chi_{BA}(t, t')$$

with $\chi_{BA}(t, t') = i \langle \Phi_0 | [B_I(t'), A_I(t)] | \Phi_0 \rangle \theta(t - t')$.

If $H$ is independent of $t$, prove that $\chi_{BA}$ depends only on the difference $t - t'$. Write also the last relation in Fourier representation.

2 Application: The dielectric function

We consider a system of electrons. If there is a little charge fluctuation in the system, the associated potential will polarized the electrons, creating a response of the system. If
the fluctuation is small enough, the linear response theory is appropriate. The induced
density of charge created by the fluctuation $\delta \rho$ is:

$$\rho_{\text{ind}}(x, t) = \int dt' \int dx \phi_{\rho, \rho}(x - x', t - t') \delta \rho(x', t').$$

it is useful to introduce the so-called dielectric function defined by

$$\delta \rho(x, t) = \int dt' \int dx \epsilon(x - x', t - t') \rho_{\text{tot}}(x', t')$$

or in Fourier representation

$$\delta \rho(k, \omega) = \epsilon(k, \omega) \rho_{\text{tot}}(k, \omega)$$

2.1 Justify qualitatively the form of the perturbation

$$\delta H(t) = \int dr \int dr' \rho(r) \frac{e^2}{|r - r'|} \delta \rho(r', t)$$

2.2 Rewrite $\delta H(t)$ by introducing the momentum representation of $\rho$ and $\delta \rho$ to find

$$\delta H(t) = \frac{4\pi e^2}{\Omega} \sum_{q'} \rho(-q') \frac{1}{q'^2} \delta \rho(q', t).$$

We see from the above relation that this perturbation is the sum over $q'$ of elementary
perturbations having the usual form (cf part 1): $B = \rho(-q')$ and $F(t) = \frac{4\pi e^2}{\Omega} \delta(q', t)$.

2.3 Write the total response $\langle \rho(q, t) \rangle$ with the results of linear response theory.

2.4 Express, using the preceding results, the susceptibility $\chi_{\rho_{-q}, \rho_q}$ first as a function of
$\phi_{\rho, \rho}$ and then with the dielectric function making use of:

$$e^{-1} = \frac{\rho_{\text{ind}} + \delta \rho}{\delta \rho}$$