HomeWork 3 : Free Fermions

Nota Bene : in the present problem, it is assumed to work out the computations within the grand-canonical ensemble.

1 Preliminaries : the degenerate Fermi gas

In this exercise, we study the basic properties of a fermionic gas close to zero temperature (fully degenerate Fermi gas). The fermions composing the gas will be distributed between the various quantum states such that the total energy of the gas becomes minimum. Because of Pauli principle, the fermions will occupy all the states starting from energy zero up to an energy limit determined by the density of fermions in the gas. The upper value of the one-particle orbital is called the Fermi energy.

Reminder : the occupation number of a fermion at energy ε is given by the Fermi-Dirac distribution :

$$n(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/T} + 1}$$

where μ is the chemical potential, T the temperature.

1.1 Count the number of quantum states available to a particle which momentum takes value into the elementary volume p and $p + dp$.
NB : don't forget the degeneracy of spin that we will denote g.

1.2 Using the preceding result, express the Fermi momentum as a function of the density of fermions, and the corresponding Fermi energy also.

1.3 Discuss the physical content of the Fermi energy, analysing its $T \to 0$ limit.

1.4 Calculate the total energy of the gas and its pressure (state equation).

1.5 The above formulas are valid when the temperature is sufficiently low. Make this assumption quantitative by comparing the temperature of the Fermi gas to the only characteristic temperature of the system.

2 Free fermions' Green's function

For $t > 0$, and at $T = 0$, we define the Green's function as follows :

$$G_0(k,t) = -i\langle\phi_0|c_k(t)c_k^\dagger(0)|\phi_0\rangle$$

where $c_k(t)$ is the field operator that annihilates a fermion of momentum k. The time-dependence of the operators is taken into account in the Heisenberg representation.
NB : in the grand-canonical ensemble (unfixed number of particles), replace the Hamiltonian $\hat{H}$ by $\hat{H} - \mu\hat{N}$ (μ chemical potential) in the usual Heisenberg representation of the fields. $|\phi_0\rangle$ is the ground state of the system with N particles.

2.1 In the case of the "Fermi sea", calculate the Green's function as introduced so far.

2.2 Perform a Fourier transformation of $G_0(k,t)$ to obtain an expression for $G_0(k,\omega)$ in the frequency-space. Comment.

2.3 What does the Green's function measure?

To answer this question consider this little mind experiment.

a) At $t = 0$, we inject one fermion of momentum k in the system and then let it evolve. Write formally the wave function $|\psi(t)\rangle$ of the system at time t.

b) Let's now consider the wave function $|\psi_e(t)\rangle$ that describes, at time t, a system composed of a Fermi sea plus one electron of momentum k being included in the setup. Write formally $|\psi_e(t)\rangle$.

c) Finally, express the overlap $\langle\psi(t)|\psi_e(t)\rangle$ in terms of the Green's function. Comment.

2.4 In real space, we use this definition of the Green's function, for $t' > t$:

$$G(x,t|x',t') = -i\langle\phi_0|c(x,t)c^\dagger(x',t')|\phi_0\rangle$$

Calculate the spatial Green's function just introduced in the case of free fermions. Plot its modulus at $t = 0$ as a function of $r = |x - x'|$.