HomeWork 7 : The weakly interacting Bose gas

The theory of the weakly interacting Bose gas was originally developed by Bogoliubov in the late 1940s. It was meant to be a theory of superfluid helium, although for $^4$He the interatomic interactions are very strong. In this case the theory has some qualitative features which agree with experimental properties of $^4$He, most notably the linear phonon like quasiparticle excitation spectrum, $\epsilon_k = ck$, at small wave vectors. But it fails to reproduce other important phenomena, such as the roton minimum in the spectrum. On the other hand, the theory is believed to give a good description of atomic BEC, since the conditions under which it is derived are close to the experimental ones.

1. Mean-field approach, the Gross-Pitaevskii equation

Suppose that the many particle state is a coherent state (ie : eigenstate of the annihilation operator):

$$\hat{\psi}(r)|\psi\rangle = \psi_0(r)|\psi\rangle$$

The Hamiltonian of the interacting Bose gas is:

$$H = \int \hat{\psi}^\dagger(r) \left( -\frac{\hbar^2\nabla^2}{2m} + V_1(r) \right) \hat{\psi}(r) d^3r + \frac{1}{2} \int V(r - r') \hat{\psi}^\dagger(r) \hat{\psi}^\dagger(r') \hat{\psi}(r') \hat{\psi}(r') d^3rd^3r'$$

here $V_1(r)$ is an external potential.

Using $|\psi\rangle$ as a trial function, find that the dynamics of the system is ruled by the following equation:

$$\left( -\frac{\hbar^2\nabla^2}{2m} + V_1(r) + V_{eff}(r) - \mu \right) \psi_0(r) = 0$$

Hint : apply a variational principle like we did in Homework 6 ... and do not forget the normalization of $\psi_0$ that will be chosen as $N_0 = \int |\psi_0(r)|^2 d^3r$;

$$V_{eff}(r) = \int V(r - r')|\psi_0(r')|^2 d^3r'$$

The equation so derived is called the Gross-Pitaevskii equation.

2. Beyond mean-field, the Bogoliubov transformation

Up to second order in perturbation theory and in the simple case of no external potential ($V_1(r) = 0$), the Hamiltonian of the Bose liquid becomes:

$$H_2 = \sum_k \left( -\frac{\hbar^2k^2}{2m} - \mu \right) a_k^\dagger a_k + \frac{n_0g}{2} (a_k^\dagger a_{-k}^\dagger + 4a_k^\dagger a_k + a_{-k}a_k)$$

where $\mu = n_0g$

We introduce the Bogoliubov transformation:

$$a_k = u_k \alpha_k + v_k \alpha_{-k}^\dagger$$
$$a_k^\dagger = u_k \alpha_k^\dagger + v_k \alpha_{-k}$$
where \( u_k \) and \( v_k \) are real and obey \( u_k^2 - v_k^2 = 1 \).

The idea is to rewrite the Hamiltonian in terms of these new operators and then to vary the parameters \( u_k \) and \( v_k \) to make it diagonal.

a) Writing the Bogoliubov transformation in the matrix form,

\[
\begin{pmatrix}
    b_k \\
    b_{-k}^\dagger
\end{pmatrix} = \begin{pmatrix} u_k & v_k \\
                          v_k & u_k \end{pmatrix} \begin{pmatrix} a_k \\
                          a_{-k}^\dagger \end{pmatrix}
\]

show that the pair of equations can be inverted to yield

\[
\begin{pmatrix}
    a_k \\
    a_{-k}^\dagger
\end{pmatrix} = \begin{pmatrix} u_k & -v_k \\
                          -v_k & u_k \end{pmatrix} \begin{pmatrix} b_k \\
                          b_{-k}^\dagger \end{pmatrix}
\]

b) Rewrite the Hamiltonian \( H_2 \) in the matrix form

\[
H_2 = \sum_k \left( \begin{array}{cc} a_k^\dagger & a_{-k} \end{array} \right) \left( \begin{array}{cc} \epsilon_k + n_0 g & n_0 g/2 \\
                          n_0 g/2 & 0 \end{array} \right) \left( \begin{array}{c} a_k \\
                          a_{-k}^\dagger \end{array} \right)
\]

where \( \epsilon_k = \hbar^2 k^2 / 2m \).

c) Use the inverse of the Bogoliubov transformation to express \( H_2 \) in terms of the \( b' \)s operators, namely you should find :

\[
H_2 = \sum_k \left( \begin{array}{cc} b_k^\dagger & b_{-k} \end{array} \right) \left( \begin{array}{cc} M_{11} & M_{12} \\
                          M_{21} & M_{22} \end{array} \right) \left( \begin{array}{c} b_k \\
                          b_{-k}^\dagger \end{array} \right)
\]

where the coefficients \( M_{ij} \) have to be computed explicitly.

d) Show that the condition for the transformed matrix to be diagonal is that

\[
\frac{2 u_k v_k}{u_k^2 + v_k^2} = \frac{n_0 g}{\epsilon_k + n_0 g}
\]

e) Show that the trace of the \( M \) matrix is

\[
E = (\epsilon_k + n_0 g)(u_k^2 + v_k^2) - 2 n_0 g u_k v_k.
\]

Using the representation \( u_k = \cosh(\theta) \) and \( v_k = \sinh(\theta) \), show from d) that

\[
\tanh(2\theta) = \frac{n_0 g}{\epsilon_k + n_0 g}
\]

and hence prove that

\[
E = \sqrt{\epsilon_k (\epsilon_k + 2 n_0 g)}
\]

consistent with the Bogoliubov quasiparticle dispersion given in the lecture.