HomeWork 8 : The Bose-Einstien condensation

1. Bose condensate critical temperature

a) Using the Bose-Einstein distribution, the total number of particles in a box is

\[ N = \sum_k \frac{1}{e^{\beta (\epsilon_k - \mu)} - 1}. \]

In the continuum limit, we replace \( \sum_k \) by \( \int \frac{V}{(2\pi)^3} d^3k \), and introducing the density of states \( g(\varepsilon) \) we get the density :

\[ n = \int_0^\infty \frac{1}{e^{\beta (\varepsilon - \mu)} - 1} g(\varepsilon) d\varepsilon \]

we can see this equation as an implicit definition of the chemical potential as a function of \( T \) and \( n \). Rewrite (evaluate \( g(\varepsilon) \) explicitly !) \( n \) in terms of dimensionless variables \( z = e^{\beta \mu} \) (called the fugacity) and \( x = \beta \varepsilon \). You should find that

\[ n \propto \int_0^\infty \frac{ze^{-x}}{1 - ze^{-x} x^{1/2}} dx \]

To calculate this integral we can expand

\[ \frac{ze^{-x}}{1 - ze^{-x}} = \sum_{p=1}^\infty z^p e^{-px} \]

Prove that

\[ n = \left( \frac{mk_B T}{2\pi \hbar^2} \right)^{3/2} g_{3/2}(z) \]

For this you will certainly need the following results :

\[ \Gamma(t) = \int_0^\infty z^{-1} e^{-y} dy \]

with the value \( \Gamma(3/2) = \sqrt{\pi}/2 \), and

\[ g_{3/2} = \sum_{p=1}^\infty \frac{z^p}{p^{3/2}} \]

Invert this equation and calculate the critical temperature \( T_c \) where \( \mu = 0 \). Comment.

2. Number of particles in the condensate

NB : In this exercice, the notations will be taken from the lecture.

The goal is to determine the temperature dependence of the number of particles in the condensate \( k = 0 \), for a contact potential \( V_k = \lambda \).

a) Proceed by first calculating the thermodynamic expectation value of the particle number operator \( N = \sum_k a_k^\dagger a_k \). Rewrite it in terms of the Bogoliubov operators \( \alpha_k \) and switch to the continuum limit.
Result:

\[ N = N_0(T) + 2 \frac{(mn\lambda)^{3/2}}{\pi^2} \left( \frac{1}{6} + U_1(\gamma) \right), \]

where \( \gamma = \frac{\beta k_0^2}{2m}, \frac{k_0^2}{4mn\lambda}, \beta = \frac{1}{k_BT}, \) and

\[ U_n(\gamma) = \int_0^\infty dy \frac{x^n}{e^{\gamma y} - 1} \]

with \( y = x\sqrt{x^2 + 1} \)

\[ b) \quad \text{Show that for low temperature the depletion of the condensate increases quadratically with the temperature} \]

\[ \frac{N_0(T)}{V} = \frac{N_0(0)}{V} - \frac{m}{12c} (k_BT)^2 \]

where \( c = \sqrt{\frac{\mu \lambda}{m}}. \)