TOPOLOGICAL SPACES

One starts with sets $S$ of points. If a point $s_i$ is an element of $S$, one writes $s_i \in S$. Let $S_1$ and $S_2$ be two sets of points. One says that $S_1$ is a subset of $S_2$, if every point of $S_1$ is also contained in $S_2$, and denotes this as $S_1 \subset S_2$. If there exist points in $S_2$, which are not contained in $S_1$, then $S_1$ is called a proper subset of $S_2$. The set which does contain no points is called empty set, denoted by the symbol $\emptyset$. The union $S_1 \cup S_2$ of two sets $S_1$ and $S_2$ contains all points which are either in $S_1$ or in $S_2$. The intersection $S_1 \cap S_2$ of two sets $S_1$ and $S_2$ contains all points which are contained in both set, $S_1$ and $S_2$.

Topological space. A topological space $T$ is a set of points which is equipped with a topology $T$. A topology $T$ is a selection (i.e. a set) of subsets $S_1, S_2, \ldots$ of $T$, i.e. $S_1 \subset T, S_2 \subset T$, which satisfies the following three axioms:

[T1] The empty set $\emptyset$ and the whole space $T$ belong to $T$, i.e. $\emptyset \in T$ and $T \in T$.

[T2] Finite intersections of elements of $T$ are again contained in $T$, i.e. $\bigcap_{i \in I} S_i \in T$ for $|I| < \infty$.

[T3] Arbitrary unions of elements of $T$ are again contained in $T$, i.e. $\bigcup_{i \in I} S_i \in T$.

The elements $S_i$ of the topology $T$ are called open sets. This concept is too general for the definition of manifolds. One needs one further axiom, which expresses the concept of separability. This means that one can always find for two different points $p, q$ in $T$ two open subsets $S_p$ and $S_q$, which contain $p$ and $q$ respectively, but which do not overlap. More precisely, this reads:

[T4] If $p \in T$ and $q \in T$ with $p \neq q$ are two different point, then there exist $S_p \in T$ and $S_q \in T$ with the properties $p \in S_p, q \in S_q, S_p \cap S_q = \emptyset$.

A topological space, which additionally satisfies [T4], is called Hausdorff space. An open set $S_p$, which contains the point $p$, is also called neighborhood of $p$. To emphasize that $T$ is a topological space, one typically gives the tuple $(T, T)$ instead.

MANIFOLDS

The topological concept of continuity is very easy to define: A map $\phi : (T, T) \to (U, U)$ of a topological space $T$ to another topological space $U$ is called continuous, if the pre-image of every open set $U$ is an open set in $T$. Now we have everything in order to define manifolds.

Differentiable manifolds. A differentiable manifold $\mathcal{M}$ is a Hausdorff space $(T, T)$ together with a set $\Phi$ of maps $\phi_p \in \Phi, \phi_p : T \to \mathbb{R}^n, p \in T$, with the following four properties:

[M1] $\phi_p$ is a one-to-one map of an open set $T_p$ with $p \in T_p$ onto an open set in $\mathbb{R}^n$.

[M2] The union $\bigcup_p T_p = T$.

[M3] If $T_p \cap T_q$ is not empty, then $\phi_p(T_p \cap T_q)$ is an open set in $\mathbb{R}^n$, and $\phi_q(T_p \cap T_q)$ is an open set in $\mathbb{R}^n$ different from $\phi_q(T_p \cap T_q)$. The map $\phi_p \circ \phi_q^{-1}$ must be continuous and differentiable.

[M4] The maps $\phi_p \circ \phi_q^{-1}$ and $\phi_q \circ \phi_p^{-1}$ from axiom [M3] are maps in $\Phi$. This property is also called maximality.

The property [M1] admits to construct a coordinate system for the neighborhood of any point $p$. The point $p$ is mapped to the origin of $\mathbb{R}^n$ with the help of $\phi_p$. Each point $q$ near $p$ is thus mapped to a point $\phi_p(q)$ near $0 = \phi_p(p)$. In this way, one can associate the coordinates $\phi_{qi}^p(q), i = 1, \ldots, n$ of $\phi_p(q) \in \mathbb{R}^n$ with the original point $q \in T$. This association yields a (local) coordinate system for the whole space $T$.

Axiom [M2] ensures that such a local coordinate system can indeed be constructed for each point in $T$.

Axiom [M3] makes use of the map $\phi_p \circ \phi_q^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, which can be studied with standard methods from differential calculus. If here, and in the following, a map $\phi$ is called differentiable, we always mean that $\phi \in \mathcal{C}^\infty$, i.e. that $\phi$ is smooth.

Manifolds are a very important concept. They are just sufficiently benign to look everywhere locally like the Euclidean space $\mathbb{R}^n$. Therefore, we can use all methods and concepts, which we know for the study of Euclidean spaces $\mathbb{R}^n$, since we can transfer them via the axioms [M1] to [M3] to manifolds. In particular, the dimension of a manifold $\mathcal{M}$ is just the dimension of the space $\mathbb{R}^n$, namely $n$.

Complex manifolds. One can define other sorts of manifolds in the same manner. The property of smoothness is then replaced by other properties in the same consistent way. For example, we obtain a complex manifold, if we replace in the above axioms everywhere $\mathbb{R}^n$ by $\mathbb{C}^n$, and at the end of [M3] the word differentiable by complex analytic.