SYMMETRIC GROUP AND YOUNG TABLEAUX

As we learned, any element g of a symmetric group S_n on n elements can be written as a product of cycles. This resulted in a classification of conjugacy classes. A given conjugacy class consists of permutations of the form of k_1 1-cycles, k_2 2-cycles, and so on, such that the total number of elements in it is

$$C_{(k_1,k_2,\ldots k_n)} = n! \prod_j (j^{k_j}\, k_j!)^{-1}\,.$$

It is useful to represent each j-cycle by a column of boxes of length j. We arrage these topjustified in decreasing order in j from left to right. Thus, the trivial conjugacy class of S_n, which consists of n 1-cycles, is given by one row of n boxes. Each such tableau corresponds to a different conjugacy class (in fact, to a different partition of the number n into a sum of positive integers), and thus to an irrep. For example, the conjugacy class for a 4-cycle, a 3-cycle and a 1-cycle in S_8 is represented by

$$\ydiagram{3,2,2,1}$$

Conjugacy classes. In this way, we can easily obtain all conjugacy classes graphically. For example, for S_3 we obtain

$$\begin{array}{ccc} \ydiagram{3} & \ydiagram{2,1} & \ydiagram{1,1,1} \\ C_{(3)} = \frac{3!}{3!} = 1 & C_{(2,1)} = \frac{3!}{2} = 3 & C_{(1,1,1)} = \frac{3!}{3} = 2 \end{array}\,.$$

For S_4 we obtain in the same way the five possibilities

$$\begin{array}{ccccc} \ydiagram{4} & \ydiagram{3,1} & \ydiagram{2,2} & \ydiagram{2,1,1} & \ydiagram{1,1,1,1} \\ C_{(4)} = \frac{4!}{4!} = 1 & C_{(3,1)} = \frac{4!}{4} = 6 & C_{(2,2)} = \frac{4!}{8} = 3 & C_{(2,1,1)} = \frac{4!}{3} = 8 & C_{(1,1,1,1)} = \frac{4!}{4} = 6 \end{array}\,.$$

Irreps. The Young tableaux are very helpful to explicitly construct the irreps by identifying an appropriate subspace of the regular representation of S_n. Each Young tableau for S_n has n boxes, into which we can fill the numbers $1, 2, \ldots, n$ in many different ways. In fact, there are $n!$ different possibilities to do so. Such a labeling of a Young tableaux corresponds one-to-one to a state of the regular representation. For example, we can simply read from left to right, row by row from top to bottom, to obtain a state, e.g.

$$\begin{ytableau} 6 & 5 & 3 & 2 \\ 1 & 7 \\ 4 \end{ytableau} \longrightarrow |6532174\rangle\,,$$

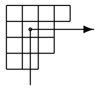
where the state $|6532174\rangle$ corresponds to the permutation $(123456) \to (6532174)$. To find the invariant subspace of the regular representation corresponding to a given Young tableau, one uses the rule that a Young tableau is completely symmetric in each row, and completely antisymmetric in each column. For instance,

$$\begin{ytableau} 1 & 2 \end{ytableau} = |12\rangle + |21\rangle\,, \quad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} = |123\rangle + |213\rangle - |321\rangle - |231\rangle\,.$$

Consider S_3 as an example. The Young tableau $\ydiagram{3}$ corresponds to a completely symmetric representation, which is the one-dimensional subsapce of the regular representation wich transforms under the trivial representation. The other one-dimensional representation is the alternating representation, which is completely antisymmetric and therefore is associated to the tableau $\ydiagram{1,1,1}$ where interchanges are represented by a sign-change. Finally, the remaining tableau yields a two-dimensional representation, since the $3! = 6$ different labelings give only two different states, since

$$\begin{aligned}
\begin{ytableau} 1 & 2 \\ 3 \end{ytableau} &= |123\rangle + |213\rangle - |321\rangle - |231\rangle\,, & \begin{ytableau} 2 & 3 \\ 1 \end{ytableau} &= |231\rangle + |321\rangle - |132\rangle - |312\rangle\,, \\
\begin{ytableau} 2 & 1 \\ 3 \end{ytableau} &= |213\rangle + |123\rangle - |312\rangle - |132\rangle\,, & \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} &= |132\rangle + |312\rangle - |231\rangle - |321\rangle\,, \\
\begin{ytableau} 3 & 2 \\ 1 \end{ytableau} &= |321\rangle + |231\rangle - |123\rangle - |213\rangle\,, & \begin{ytableau} 3 & 1 \\ 2 \end{ytableau} &= |312\rangle + |132\rangle - |213\rangle - |123\rangle\,.
\end{aligned}$$

Hook rule. We can determine the dimension of the irreps by the *hook number* H, where the irrep $\rho_{(j_1,j_2,\ldots,j_n)}$ corresponding to the Young tableau with columns $(j_1, j_2, \ldots, j_n)$ has dimension dim $\rho = \frac{n!}{H}$. The hook number is computed in the following way: A hook comes from the bottom passing through one of the columns to some box within the tableau, then turns right and leaves the tableau through one of its rows. Of course, each box has a corresponding hook. The hook to each box i passes on its way through a certain number h_i of boxes, where i is an arbitrary labeling of the boxes. Then, $H = \prod_i h_i$. Here is one example for a hook:



$$h = 4\,.$$

YOUNG TABLEAUX AND $\mathfrak{su}(3)$

We learned in the lecture that $\mathfrak{su}(3)$ has two fundamental representations, both of dimension three. Since one is the complex conjugate of the other, they are often simply denoted by $\rho_{\omega^1} = \rho_{(1,0)} = \mathbf{3}$ and $\rho_{\omega^2} = \rho_{(0,1)} = \overline{\mathbf{3}}$. We also know that any irrep with weight $\mu = n\omega^1 + m\omega^2$ is contained in the tensor product of the fundamental irreps as $\rho_{(n,m)} \subset \mathbf{3}^{\otimes n} \otimes \overline{\mathbf{3}}^{\otimes m}$. However, since $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \overline{\mathbf{3}}$, we only need one of the fundamental irreps to build all the irreps.

Now, irreps of $\mathfrak{su}(3)$, or more generally $\mathfrak{su}(n)$, have the property that they transform irreducibly under permutation of the labels of their tensor indices. Thus, they can be classified in a very similar fashion with the help of Young tableaux, as the irreps of the symmetric groups! In fact, a general irrep (n, m) is essentially a tensor with components $A^{i_1\cdots i_n}_{j_1\cdots j_m}$, seperately symmetric in upper and lower indices, and traceless. Lower indices can be raised with the help of the completely antisymmetric ϵ-tensors,

$$Y^{i_1\ldots i_n\, k_1\ell_1\ldots k_m\ell_m} = \epsilon^{j_1k_1\ell_1}\cdots\epsilon^{j_mk_m\ell_m}A^{i_1\cdots i_n}_{j_1\cdots j_m}\,.$$

The new tensor Y is antisymmetric in each pair $k_i \leftrightarrow \ell_i$, and it is symmetric in the exchange of pairs $k_i, \ell_i \leftrightarrow k_{i'}, \ell_{i'}$. To each such tensor Y we can associate a Young tableau of the form

$$\begin{array}{|c|c|c|c|c|c|}\hline k_1 & \cdots & k_m & i_1 & \cdots & i_n \\ \hline \ell_1 & \cdots & \ell_m \\ \cline{1-3}\end{array}$$

One can now show that the tensor Y has the right symmetry properties. First, one should keep in mind that the lowering operators, acting on the highest weight state, preserve the symmetry such that it is sufficient to study the symmetry of the highest weight, or the symmetry of the corresponding tensor components. The highest weight of the irrep (n, m) has tensor components of A with all $i_r = 1$ and all the $j_s = 2$, thus the components of the Y-tensor are $i_r = 1$ and all pairs $k_s, l_s = 1, 3$. Thus, the symmetry is as for the Young tableaux for the symmetric groups, symmetric in each of the rows, antisymmetric in each of the columns. The traceless condition of the A-tensor translates to the condition

$$\epsilon_{i_1k_1l_1}Y^{i_1\ldots i_n\, k_1\ell_1\ldots k_m\ell_m} = 0$$

of the Y-tensor. Now, this symmetrizing prescription can be used to symmetrize an arbitrary tensor in order to project out a specific irrep. For example, if $B^{j_1j_2k_1}$ is a general tensor with three upper indices, but no special symmetry property, the following Young diagram produces a symmetrized tensor,

$$\begin{array}{|c|c|}\hline j_1 & j_2 \\ \hline k_1 \\ \cline{1-1}\end{array} \;\to\; B^{j_1j_2k_1} + B^{j_2j_1k_1} - B^{k_1j_2j_1} - B^{j_2k_1j_1}\,,$$

which transforms according to the $(1, 1)$ irrep, i.e. the adjoint representation. The recipe can be generalized to Young tableaux with more than two rows. However, in the case of $\mathfrak{su}(3)$, no Young tableau can have more than three rows since there is no completely antisymmetric object with four or more indices which can take on only three values. Furthermore, any column with three boxes is irrelevant, since it is simply a completely antisymmetric object in three indices, thus it corresponds to an ϵ-tensor. In fact, for $\mathfrak{su}(3)$, we only need to consider Young tableaux with less than three rows. Any diagram which contain columns with three rows can be replaced by one, where these columns are simply erased.

Clebsh-Gordan decomposition. We can use all this to decompose tensor products with the help of Young tableaux. First, we explain in general the relation between the weight $\mu = \sum_{i=1}^{n-1} k_i\omega^i$ of an $\mathfrak{su}(n)$ irrep and its Young tableau Y which is given by building from left to right starting with k_{n-1} columns of $n-1$ boxes, adding to the left k_r columns of r boxes for $r = n-2, \ldots, 2$ until at the end we have k_1 columns with just one box each. This is the immediate generalization of antisymmetrization in n instead of just 3 possible index values. Note that legal Young tableaux always are such that the number of boxes per row does not increase when going down and that the number of boxes per column does not increase when moving to the right.

Given now two representations with weights $\mu = \sum_i n_i\omega^i$ and $\mu' = \sum_i n_i'\omega^i$ with corresponding Young tableaux Y and Y', the decomposition of $\rho_\mu \otimes \rho_{\mu'}$ works then as follows: One has to distribute all the boxes of the tableau Y' to the tableau Y, but in a certain manner. Fill the boxes of Y' from top to bottom with symbols, say a in the first row, b in the second row and so on. Now attach the boxes with as from the first row of Y' in all possible ways to the tableau Y which form legal tabelaux. Then go on with the boxes with bs from the second row of Y' and form again legal tabelaux, but with the additional rule that the number of bs accumulated by reading the rows from right to left from the top row to the bottom row must not be larger than the number of as to avoid double counting. Continue in the same way with the other rows of Y' always ensuring that the tableaux are not only legal, but that the number of the new symols accumulated from right to left top to bottom never exceeds the number of the preceeding symbol. Coming back to $\mathfrak{su}(3)$ we obtain in this way for example

$$\begin{array}{|c|}\hline \ \\ \hline\end{array} \otimes \begin{array}{|c|}\hline a \\ \hline\end{array} = \begin{array}{|c|c|}\hline \ & a \\ \hline\end{array} \oplus \begin{array}{|c|}\hline \ \\ \hline a \\ \hline\end{array}$$

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \overline{\mathbf{3}} \quad ,$$

$$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} \otimes \begin{array}{|c|}\hline a \\ \hline\end{array} = \begin{array}{|c|c|}\hline \ & a \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline a \\ \hline\end{array}$$

$$\overline{\mathbf{3}} \otimes \mathbf{3} = \mathbf{8} \oplus \mathbf{1} \quad .$$

The dimensions of the involved irreps will be explained further below. Let us now do a slightly more complicated example, namely the decomposition $\overline{\mathbf{3}} \otimes \overline{\mathbf{3}} = \overline{\mathbf{6}} \oplus \mathbf{3}$ which we can immediately guess from the things we did above, since this is simply the complex conjugate representation of $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \overline{\mathbf{3}}$. However, with the Young tableau recipe this goes as follows:

$$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} \otimes \begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array} = \xcancel{\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline a \\ \hline b \\ \hline\end{array}} \oplus \xcancel{\begin{array}{|c|c|}\hline \ & b \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1} a & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}} \oplus \xcancel{\begin{array}{|c|c|c|}\hline \ & a & b \\ \hline \ & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}} \oplus \begin{array}{|c|c|}\hline \ & a \\ \hline \ & b \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline \ & a \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1} b & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \;,$$

where we crossed out tableaux which either violate the rule that they cannot have more than $n = 3$ rows, or which violate the rule that the number of bs must not be larger than the number of as when counted from right to left and top to bottom to any of the bs (here, there is just one).

It is a very useful exercise to do the decomposition of $\mathbf{8} \otimes \mathbf{8} = \mathbf{27} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$. However, to check that you did everything right, you need a rule to compute the dimension of an irrep associated to a Young tableau. So, here it comes:

Dimension of irreps. It is convenient to introduce one more notation. For the Lie algebras $\mathfrak{su}(n)$, the irrep with highest weight $\mu = \sum_{i=1}^{n-1} k_i\omega^i$ has a Young tableau with columns $[\ell_1, \ell_2, \ldots]$ where ℓ_j is the length of the j-th column. As discussed above, the ℓ_j are easily obtained from the k_i because there are k_i columns of length i. The sequences $[\ell_1, \ell_2, \ldots]$ are sequences of non-increasing integers. For instance, the j-th fundamental representation is in this notation denoted by $[j]$, i.e. by a Young tableau with just one column of length j. In this notation, the exercise reads to verify that $[2,1] \otimes [2,1] = [2,2,1,1] \oplus [3,1,1,1] \oplus [2,2,2] \oplus [3,2,1] \oplus [3,2,1] \oplus [3,3]$.

Now, how do we compute the dimension of an irrep associated to the tableau $[\ell_1, \ell_2, \ldots]$? This goes much in the same way as the computation of the dimensions of the irreps of the symmteric groups with the help of the hook number H. What changes it the numerator. Thus, instead of $n!/H$ for the case of irreps of S_n, we now get the rule

$$\dim\,[\ell_1, \ell_2, \ldots, \ell_k] = \frac{F}{H}\,, \quad F = \frac{n!}{(n-\ell_1)!}\frac{(n+1)!}{(n+1-\ell_2)!}\cdots\frac{(n+k-1)!}{(n+k-1-\ell_k)!}\,.$$

This is the *factor over hooks rule*. You obtain the factor F by labeling the Young tableau in a certain way and then compute the product over all the lables. Start in the top left corner with n. Going along a row from left to right, increase the label one by one. Going down a column one by one, decrease the label one by one. Here a hopefully helpful picture explaining how to get F:

$$\begin{array}{|c|c|c|c|}\hline n & n+1 & n+2 & n+3 \\ \hline n-1 & n & n+1 & \multicolumn{1}{c}{} \\ \cline{1-3} n-2 & n-1 & \multicolumn{2}{c}{} \\ \cline{1-2} n-3 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}$$

$$F = [n(n-1)(n-2)(n-3)][(n+1)n(n-1)][(n+2)(n+1)][(n+3)]\,.$$

We note for completeness that the complex conjugate of an irrep $[\ell_1, \ell_2, \ldots, \ell_k]$ is given by the $\overline{[\ell_1, \ell_2, \ldots, \ell_k]} = [n - \ell_k, \ldots, n - \ell_2, n - \ell_1]$. You may convince yourself that this leads to the same dimension. The tableau Y and its complex conjugate $\overline{Y}$ obviously add up to a rectangle of height n and width k.

More on $\mathfrak{su}(n)$. The group $SU(n)$ is the group of special unitary $n \times n$ matrices, generated by hermitean traceless $n \times n$ matrices. We normalize the Killing form for the adjoint representations such that $\mathrm{tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$. For the raising and lowering operators in the Cartan-Weyl basis we choose the matrices, which have just one non-zero off-diagonal element, which takes the value $\frac{1}{\sqrt{2}}$. The group has rank $n-1$ because there can only be $n-1$ independent traceless diagonal matrices with real entries. We choose these $n-1$ Cartan generators as

$$(H_m)_{ij} = \frac{1}{\sqrt{2m(m+1)}} \left(\sum_{k=1}^{m} \delta_{ik}\delta_{jk} - m\delta_{i,m+1}\delta_{j,m+1} \right) .$$

Thus, $(H_m)_{ii} = 1$ for $1 \leq i \leq m$ and $(H_m)_{m+1,m+1} = -m$ upto the normalization factor $1/\sqrt{2m(m+1)}$. In total there are $n^2 - 1$ independent traceless matrices which generate the n-dimensional defining representation of $\mathfrak{su}(n)$, often denoted by its dimension **n**. The weights are $(n-1)$-dimensional vectors, obviously given by

$$(\mu^j)_m = (H_m)_{jj} = \frac{1}{\sqrt{2m(m+1)}} \left(\sum_{k=1}^{m} \delta_{jk} - m\delta_{j,m+1} \right) .$$

One can check easily that $\mu^j \cdot \mu^j = \frac{n-1}{2n}$ and $\mu^i \cdot \mu^j = -\frac{1}{2n}$ for $i < j$. Thus, they have all the same length and two weights have always the same angle between them. In fact, they form the $(n-1)$-simplex in $(n-1)$-dimensional space.

It is convenient to choose the ordering prescription for positivity in a different way than we have done so far. A weight is called positive if the *last* non-zero component is positive. Then the weights satisfy $\mu^1 > \mu^2 > \ldots > \mu^n$. As usual, the raising and lowering operators move from one weight to another, so the roots are the differences between the weights, $\alpha^i = \mu^i - \mu^{i+1}$ for $i = 1, \ldots, n-1$. The roots all have length one, and we have $\alpha^i \cdot \alpha^j = \delta_{ij} - \frac{1}{2}\delta_{i,j\pm 1}$. The fundamental weights can easily be found from the inverse of the Cartan matrix, but by simply looking at the formulæ we have so far, one can infer that they read $\omega^i = \sum_{j=1}^{i} \mu^j$.

Now, as in $\mathfrak{su}(3)$, we can associate states with tensors. Thus, $A^{[i_1 \cdots i_m]}|i_1 \ldots i_m\rangle$, where A is completely antisymmetric in all its (upper) indices, are the states of an antisymmetric combination of m defining representations. Due to its complete antisymmetry, this forms an irrep. Furthermore, no two indices can take the same value. Thus, the highest weight of this irrep corresponds to the state, where one index is 1, another is 2, and so on. So, the highest weight is the fumdamental weight $\omega^m = \sum_{j=1}^{m} \mu^j$ and the irrep is the fundamental rep $[m]$. The highest weight of any irrep can be expanded as $\mu = \sum_i k_i \omega^i$ with non-negative integers k_i. The tensor associated with this representation has, for each $i = 1, \ldots, n-1$, precisely k_i indices that are antisymmetric within each set among themselves. The evident generalization of the argument for $\mathfrak{su}(3)$ shows that the symmetry porperties of this tensor can be obtained from the Young tableau

$$[\underbrace{n-1, \ldots, n-1}_{k_{n-1}}, \underbrace{n-2, \ldots, n-2}_{k_{n-2}}, \ldots, \underbrace{2, \ldots, 2}_{k_2}, \underbrace{1, \ldots, 1}_{k_1}] .$$

This gives a tensor of the right form. Again, we have to check the symmetry properties only for the highest weight state, which has a term in which the top row of the Young tableau has only entries 1, the second only entries 2, the third only entries 3, etc. Thus, the desired tensor is obtained by first symmetrizing in the indices in each of the rows, and then antisymmetrizing in the indices in each of the columns. This is eaxtly the symmetry condition used to construct the irreps of the permutation groups! Therefore, the irreps of $\mathfrak{su}(n)$ with m indices are associated with the irreps of S_m. Tensors for Young tableaux with columns of length greater than n vanish identically, columns of length equal to n correspond to the unique totally antisymmetric rank n tensor (an ϵ-tensor with n indices), and can thus we omitted. So, Tableaux which differ only by a number of columns of lenght n belong to the same irrep.

Finally, the Clebsh-Gordan decomposition works for $\mathfrak{su}(n)$ in exactly the same way as for $\mathfrak{su}(3)$. The decomposition can be done via Young tableaux without even specifying n. Fixing n later then might result in eliminating certain tableau which then have too long columns, or in changing tableaux by omitting columns of length n. Note that boxes can only dissapear from a tableau when such a column of length n is removed. Therefore, the tensor product $Y \otimes Y'$ of tableaux with j and k boxes, respectively, will have a number of boxes which is equal to $j + k$ mod n. Final exercise: Decompose $[4] \otimes [4] = [4, 4] \oplus [5, 3] \oplus [6, 2] \oplus [7, 1] \oplus [8]$.