GROUPS

When a physicist describes some aspects of Nature by a theory, she will use two concepts: First of all, there is a set of configurations of the physical system. The possible configurations can be given in various ways, e.g. as tuples of quantum numbers in quantum mechanics, or as coordinates and velocities, or as coordinates in phase space, respectively, in classical mechanics. Secondly, there is the dynamics of the system, which again can be encoded in various way. Often, there is a Lagrangian or Hamiltonian which together with the principle of the least action will generate the equations of motion from which the time evolution of our physical system can be inferred.

In general, a symmetry of such a physical system is a map of one configuration to another such that the dynamics of the system is not changed. What this precisely means, depends on the system. However, the general statement immediately implies that the set of symmetries of a physical system forms a group in a very natural manner. We will use the notation $|\psi\rangle$ to denote a given configuration of a physical system, although we do not imply that we are dealing with quantum mechanics. Thus, $|\psi\rangle$ could, for instance, mean $\{q_1, \ldots, q_N, \dot{q}_1, \ldots \dot{q}_N\}$, i.e. a set of generalized coordinates and velocities.

Group Axioms. If I can map the configuration of a system to another without changing the dynamics, I can do the same thing again. Let us denote such a symmetry map by $g : |\psi\rangle \mapsto |\psi'\rangle$. Let us denote the set of all the symmetries by G. Finally, applying one symmetry operation after another is called composition. Then we have:

[G1] $\quad \forall\, g, g' \in G : g'' = g' \circ g \in G.$

If we repeat this procedure once more, we may ask ourselves whether it matters in which order the composition is done. Since, according to our definition of a symmetry transformation, the system is always in a definite state, the order should not matter. This is called *associativity*, and reads:

[G2] $\quad \forall\, g, g', g'' \in G : (g'' \circ g') \circ g = g'' \circ (g' \circ g).$

Of course, there exists always the symmetry transformation of doing nothing at all, i.e. a map $e : |\psi\rangle \mapsto |\psi\rangle$ for all configurations. This trivial symmetry operation is called the *identity*, and it satisfies:

[G3] $\quad \exists\, e \in G : \forall\, g \in G : g \circ e = e \circ g = e.$

Finally, when we can do a symmetry operation, we also can undo it. Thus, we expect that there exists an *inverse* operation g^{-1} to each operation g, i.e.

[G4] $\quad \forall g \in G : \exists\, g^{-1} \in G : g \circ g^{-1} = g^{-1} \circ g = e.$

As one sees, symmetry operations naturally satisfy the four axioms which define the algebraic structure of a *group*. Note that we have said nothing about the specific nature of the symmetry operations so far.

One Parameter Groups. Many symmetry transformations can be described with the help of parameters, and very often, these parameters are continuous variables. Examples are the angles of rotations or the shifts of translations. Thus, our symmetry operations g depend on some set of parameters u, i.e. $g = g(u)$, $u \in \mathcal{M}$. The parameters form a manifold $\mathcal{M}$, often called the *group manifold*. The group composition is then encoded in some complicated function c of the parameters, $g(u') \circ g(u) = g(c(u', u))$.

Since most descriptions of the dynamics of a system involve (partial) differential equations, we very often encounter the situation where the group elements g depend in a differentiable way on the parameters. Then $\mathcal{M}$ forms a differentiable manifold, and the composition function $c(u', u)$ is differentiable, too. Groups with this property are called *Lie groups*. There are also discrete symmetries such as parity, but these lectures will mostly be concerned with the continuous differentiable symmetries.

The group manifold $\mathcal{M}$ may have a high dimension, but we may restrict ourselves to one-dimensional sub-manifolds parametrized by a single real variable s. It is customary to put $g(u = 0) = e$. Let us consider one-parameter sub-manifolds which run through the point $u = 0$. The implicit function theorem then implies that we always can choose the parametrization in such a way that $g(s') \circ g(s) = g(s' + s)$ in some small neighborhood $s \in U_0 = (-\epsilon, \epsilon)$ around zero. Of course, $g(s = 0) = g(u = 0) = e$. A consequence of this parametrization is that these one-parameter subgroups are Abelian, i.e. $g(s') \circ g(s) = g(s) \circ g(s')$ commutes for all s, s' in the open neighborhood U_0.

The advantage of the one-parameter subgroups is that they allow us to define an infinitesimal transformation X. Such an infinitesimal transformation is a linearized version of the group elements very close to the identity. The infinitesimal transformation X is often called *generator* of the one-parameter subgroup, and it is defined by

$$X = \frac{\mathrm{d}}{\mathrm{d}s} g(s)\bigg|_{s=0} \in T\mathcal{M}_0\,.$$

It constitutes an element of the tangent space to the group manifold at the point zero. The full group G, or at least the component of it connected to the identity, can be recovered from the set of all possible one-parameter subgroups which run through the identity element. In fact, it will turn out that almost all information about G can be recovered from the infinitesimal generators of the one-parameter subgroups.

Noether Theorem. As an example, how groups enter physics when symmetries are around, we consider the Noether theorem which states that to every continuous symmetry there exists a conserved quantity in the physical system restricting the dynamics of it. To keep things simple, we consider a system of only one generalized coordinate q with Lagrangian $L(q, \dot{q})$. First, we define what we exactly mean by the statement that a symmetry operation must leave the dynamics of the system invariant. The principle of the least action, i.e. that $S = \int \mathrm{d}t\, L(t)$ must be extremal, implies the equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0\,. \tag{$*$}$$

Now let us suppose that we have a one-parameter family of symmetry maps, i.e. for any real number s in some small interval around 0, $s \in (-\epsilon, \epsilon)$, we have maps $g_s : q \mapsto q' = g_s(q)$, and induced maps $\tilde{g}_s : \dot{q} \mapsto \dot{q}' = \tilde{g}_s(\dot{q}) = \frac{\partial g_s(q)}{\partial q}\dot{q}$. The Lagrangian L is then *invariant* under the transformations given by $(g_s, \tilde{g}_s)$, if for any fixed s we can find a function $F_s(q, \dot{q}, t)$ such that

$$L(g_s(q), \tilde{g}_s(\dot{q})) = L(q, \dot{q}) + \frac{\mathrm{d}}{\mathrm{d}t}F_s\,. \tag{$**$}$$

Note that invariance holds only up to a total time derivative. Such terms are allowed since they do not change the equations of motions due to the fixed boundary conditions. Such total time derivatives give rise to so-called *surface terms* in the action integral. They play an important role in several applications, e.g. supersymmetry, where they cannot be neglected.

The invariance property $(**)$ of the Lagrangian implies a *conservation law*. To see this, we consider a trajectory q parametrized by a map ϕ in time, $\phi : t \mapsto q = \phi(t)$. Such a map is called a *path*. Obviously, we can concatenate the path map with the symmetry map, $\phi_s = g_s \circ \phi : t \mapsto q = g_s(\phi(t))$, to obtain a whole family of paths, one path for each value of the parameter s. The invariance property $(**)$ now implies that the action integral does not depend on the particular member ϕ_s of this family, i.e. it is independent of the parameter s. If the path ϕ solves the equations of motion $(*)$, so does ϕ_s for each value of s.

This can be recast in a more mathematical form by putting the variable t, which parametrizes a single path, and the parameter s, which labels different paths, on equal footing. We simply define $\Phi(s,t) = g_s \circ \phi(t)$. Plugging this into $(**)$, we obtain

$$\begin{array}{rcl} L(\phi(t), \partial_t\phi(t)) \;=\; L(\Phi(0,t), \partial_t\Phi(0,t)) & = & L(\Phi(s,t), \partial_t\Phi(s,t) - \dfrac{\mathrm{d}}{\mathrm{d}t}F_s \quad\Longrightarrow \\ 0 & = & \dfrac{\partial}{\partial s}\left[L(\Phi, \partial_t\Phi) - \dfrac{\mathrm{d}}{\mathrm{d}t}F_s\right] \;=\; \dfrac{\partial L}{\partial q}\dfrac{\partial \Phi}{\partial s} + \dfrac{\partial L}{\partial \dot{q}}\dfrac{\partial^2\Phi}{\partial s\partial t} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial F_s}{\partial s} \\ & = & \left(\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial \dot{q}}\right)\dfrac{\partial \Phi}{\partial s} + \dfrac{\partial L}{\partial \dot{q}}\dfrac{\partial^2\Phi}{\partial s\partial t} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial F_s}{\partial s}\,. \end{array}$$

In the last step, we have used the equations of motion. The last expression is a total time derivative. Since it vanishes, the quantity under this total time derivative is conserved in time. This quantity is called the *Noether charge*

$$Q = \frac{\partial L}{\partial \dot{q}}\frac{\partial \Phi}{\partial s} - \frac{\partial F_s}{\partial s}\,.$$

Note that Q is only conserved for solutions of the equations of motion! The Noether theorem shows that there exists to any differentiable family of symmetry transformations a conserved charge. For example, symmetry under translations implies conservation of momentum, symmetry under rotations conservation of angular momentum. Although it applies to any continuous differentiable group of symmetries, its proof only involves the existence of a locally defined one-parameter (sub)group of infinitesimal transformations.

Algebra. So far, we considered the Lagrangian approach to classical mechanics. There is an alternative theoretical approach, the Hamilton formalism. In this approach, the algebraic structure of infinitesimal symmetries becomes apparent. To keep things again simple, we consider a Hamiltonian $H(q,p)$ of just one mass point, i.e. of just one coordinate (q,p) in phase space. We note in passing that for a mechanical system which can be treated in both formalisms, the relation between the two approaches is given by the Legendre transformation $H(q,p) = p\dot{q} - L$, where $\dot{q}$ is implicitly defined via $p = \frac{\partial L}{\partial \dot{q}}$.

The equations of motion in the Hamilton formalism are given by

$$\dot{p} = \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial q}\,, \quad \dot{q} = \frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{\partial H}{\partial p}\,.$$

In the Hamilton formalism, one considers differentiable functions on phase space, $\mathfrak{g} = C^\infty(\mathcal{M}_{n,n})$, where $\mathcal{M}_{n,n}$ is a symplectic manifold. For two such functions, $f(q,p)$ and $g(q,p)$, one defines the *Poisson bracket*

$$\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}\,.$$

The Poisson bracket belongs to the class of so-called *Lie brackets*, which are characterized by the following three properties: First, it is bilinear, which says that the bracket is essentially a linear map $V \times V \mapsto V$ for a vector space V, i.e.

[A1] $\forall\, f,g,h \in \mathfrak{g}\,,\ \forall\, \mu,\lambda \in \mathbb{R} : \{\lambda f + \mu g, h\} = \lambda\{f,h\} + \mu\{g,h\}$ and $\{f, \lambda g + \mu h\} = \lambda\{f,g\} + \mu\{f,h\}$.

The next property is crucial. As we will see later, it corresponds in a certain sense to the axiom of the existence of the inverse element for groups. The bracket is *antisymmetric*, i.e.

[A2] $\forall\, f,g \in \mathfrak{g} : \{f,g\} = -\{g,f\}$.

Finally, there is a property, which will correspond to the law of associativity for groups. The bracket has to satisfy a certain relation, called the *Jacobi identity*, which reads

[A3] $\forall\, f,g,h \in \mathfrak{g} : \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$.

Any vector space which admits the definition of a bracket with the properties **[A1]** to **[A3]** is called an *algebra*. If there is the additional property of differentiability, such an algebra is called a *Lie algebra*. There is a precise relationship between Lie algebras and Lie groups which we will encounter later.

The interesting point about the existence of such a Lie bracket on the space of functions on phase space is that the time evolution of an arbitrary function $f(q(t),p(t))$ defined on a trajectory $(q(t),p(t)$ in phase space is simply given by the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}f(q(t),p(t)) = \{H,f\}$$

if H does not explicitly depend on t. If f is a conserved quantity, it obviously satisfies $\{H,f\} = 0$. One says that f Poisson-commutes with H, or that f is in involution with H. The Jacobi identity now ensures that if f and g are conserved, so is $\{f,g\}$. Thus, the conserved quantities form a closed sub-algebra. This is a further hint that in order to learn something about conserved quantities and their relation to symmetries, one should first learn something about the structure of Lie algebras.

Quantum mechanics can be seen in the same light by performing canonical quantization in the Heisenberg picture. The phase space is replaced by a Hilbert space $\mathcal{H}$, the functions on the phase space by self-adjoint operators acting on $\mathcal{H}$ representing the observables of the system. The quantization prescription is than

$$\{\cdot,\cdot\} \longrightarrow \frac{\mathrm{i}}{\hbar}[\cdot,\cdot]\,,$$

such that we obtain, for example, $\{p,q\} \longrightarrow \frac{\mathrm{i}}{\hbar}[p,q] = 1$. The appearance of $\hbar$ is necessary for dimensional reasons, while the factor i is needed, since the Poisson bracket of two real functions is again real, but the commutator of two self-adjoint operators is anti-self-adjoint. One of the deep difficulties in quantum mechanics stems from the fact that typically we can define the self-adjoint operators not on the whole of Hilbert space, but only on a dense subspace of it. Thus, the product of two operators might not be defined at all, such that we cannot represent the commutator in its naive way, $[A,B] = AB - BA$. However, if the commutator is defined, it satisfies **[A1]** to **[A3]**. Note that in the Heisenberg picture, states $|\psi\rangle$ do not change in time, while the operators have a time evolution defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = \frac{\mathrm{i}}{\hbar}[H,A]\,.$$

Conserved observables obviously have to commute with H, and they again form a closed sub-algebra. With the help of $A_{(\mathrm{H})}(t) = \exp(\mathrm{i}Ht/\hbar)A_{(\mathrm{S})}\exp(-\mathrm{i}Ht/\hbar)$, we can translate our statements to the Schrödinger picture.

The power of conserved quantities in quantum mechanics is that the induce *selection rules*, encoded in Wigner-Eckart theorems, as the reader might already know. Let us suppose that a self-adjoint operator A is conserved, i.e. $[H,A] = 0$. Then we can defined an operator

$$U_A(t) = \mathrm{e}^{\mathrm{i}tA} = 1\!\!1 + \mathrm{i}tA + \frac{1}{2!}(\mathrm{i}tA)^2 + \ldots$$

for real t in (at least) a small neighborhood around $t = 0$. These operators are unitary and satisfy

$$U_A(t')U_A(t) = U_A(t' + t)\,,$$

i.e. the operators form a one-parameter group! The symmetry group G is then generated by all operators $U_A(t)$ for all A which commute with H, thus

$$G = \Big\langle U_A(t) : A \text{ self-adjoint}, \quad [H, A] = 0 \Big\rangle\,.$$

Exercises

This is a small collection of exercises which partially are done in the tutorials. If the reader wishes, she might try the remaining ones for herself at home. The author believes that the exercises not marked with a star are not hard to solve. The last three exercises deal with finite groups. If you don't know what to do with your time, try them. Some of the words used in these exercises might not be known to you, but they will be explained in the lecture course in case you are so patient to wait until then to find out.

Selection rules. What can you say about matrix elements $\langle n', \ell', m'|H|n, \ell, m\rangle$ if angular momentum is conserved, i.e. $[H, L_z] = [H, \boldsymbol{L}^2] = 0$?

Noether theorem. Use Noether's theorem to calculate the conserved quantity associated to a rotation in a three dimensional space $\mathbb{R}^3$ around the axis given by the vector $\boldsymbol{a}$.
Hint: In this case we have $\frac{\mathrm{d}g_s}{\mathrm{d}s}(\boldsymbol{x}) = \boldsymbol{a} \times \boldsymbol{x}$, where $\times$ denotes the cross product.

Poisson bracket. Check the three characteristic properties **[A1]** to **[A3]** of the Poisson bracket

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \right)$$

of smooth functions f, g on the phase space for n mass points. Deduce the relation $\frac{\mathrm{d}}{\mathrm{d}t}f = \{H, f\}$ between time derivatives and the Poisson brackets with H. Show that, if f and g Poisson-commute with H, so does $\{f, g\}$.

Symplectic manifolds*. Consider an alternative definition of a bracket, given by

$$\{\{f, g\}\} = \sum_{i,j=1}^{n} \left(\Omega^{ij}\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_j} \right)\,.$$

Which property must the matrix Ω possess in order that this bracket satisfies the Jacobi identity. Can it depend on p and q ?

Translation in quantum mechanis. Check that a finite translation in one dimension can be represented by an operator of the form $\exp(a\frac{\mathrm{d}}{\mathrm{d}x})$.
Hint: Expand in a series.

Unitary operators. Verify that the operators $U_A(t)$ defined above are unitary, i.e. obey $(U_A)^\dagger = (U_A)^{-1}$, and that they indeed form a unitary one-paramter group under the multiplication law. Show that unitary operators preserve the inner product of Hilbert space vectors, $\langle U_A\psi'|U_A\psi\rangle = \langle\psi'|\psi\rangle$.

Small finite groups. Find the multiplication table for a group with three elements and prove that its unique. Find all essentially different possible multiplication tables for groups with four elements, i.e. tables which cannot be related by renaming elements.

Equivalent representations. Suppose that ρ and ρ' are equivalent irreducible representations of a finite group G, such that there is a matrix S with $\rho'(g) = S\rho(g)S^{-1}$ for all $g \in G$. What can you say about a matrix A that satisfies $A\rho(g) = \rho'(g)A$ for all $g \in G$?

The tetrahedron*. Find the group of all the discrete rotations which leave a regular tetrahedron invariant by labeling the four vertices and considering the rotations as permutations on the four indices. This defines a four-dimensional representation of a group. Find the conjugacy classes and the characters of the irreducible representations of this group.