KILLING FORM

After defining the adjoint representation of a Lie algebra $\mathfrak{g}$, we can act with the generators T_a, $(T_a)_b^{\ c} = -\mathrm{i}f_{ab}^{\ \ c}$ on the Lie algebra itself as a vector space, which is spanned by vectors $|T_a\rangle = e_a$. We may ask ourselves now, whether we can define a suitable inner product on this vector space. A natural choice is $\langle T_a, T_b\rangle = \mathrm{tr}(T_aT_b) = g_{ab}$, which yields a real, symmetric matrix. (Remember, that the structure constants have been chosen to be real.) This form is called the Killing form of the Lie algebra $\mathfrak{g}$.

Canonical form. Let X_a denote a basis of generators of the Lie algebra $\mathfrak{g}$ such that $[X_a, X_b] = \mathrm{i}f_{ab}^{\ \ c}X_c$. We now perform a linear transformation on the basis $\{X_a\}$, $X'_a = L_a^{\ b}X_b$. One finds that this implies

$$[X'_a, X'_b] = \mathrm{i}L_a^{\ d}L_b^{\ e}f_{de}^{\ \ g}(L^{-1})_g^{\ c}X'_c\,.$$

Hence, the transformed structure constants are $(f_{de}^{\ \ g})' = f'^{\ \ g}_{de} = L_a^{\ d}L_b^{\ e}f_{de}^{\ \ g}(L^{-1})_g^{\ c}$. From this, we can easily read off the matrix elements of the transformed generators in the adjoint representation, $(T'_a)_b^{\ c} = L_a^{\ d}L_b^{\ e}(T_d)_e^{\ g}(L^{-1})_g^{\ c}$, or in short $T'_a = L_a^{\ d}(LT_dL^{-1})$. The important point is that the similarity transformation, which shows up in the transformed adjoint generators, does not have any effect in the trace: $\mathrm{tr}(T'_aT'_b) = L_a^{\ d}L_b^{\ e}\mathrm{tr}(T_dT_e)$. We can choose L appropriately such that the trace gets diagonalized. (Remember, g_{ab} is real and symmetric.) In this new base, which we denote again by $\{T_a\}$, we have $\mathrm{tr}(T_aT_b) = k_a\delta_{ab}$ with $k_a \in \mathbb{R}$. Finally, as we can rescale all the X_a and thus the T_a arbitrarily by a further, diagonal, transformation L, we can always manage that $|k_a| = 1$ for all $k_a \neq 0$. Note, however, that we cannot change the sign of the k_a, as L appears twice in the transformation of the trace.

Compact Lie algebras. A particular nice case is that all $k_a > 0$. Such Lie algebras are called *compact*, because the group manifold of the corresponding Lie group is compact in this case. The Killing form g_{ab} defines a norm $\parallel T_a \parallel^2 = g_{aa}$. A generic element $T = c^aT_a$ of the adjoint representation has norm $\parallel T \parallel^2 = c^ag_{ab}c^b = \sum_a |c^a|^2$ for the choice $g_{ab} = \delta_{ab}$. Thu, in the compact case, the norm is positive definite.

Remark: Lie algebras with a Killing form, where some of the $k_a < 0$, have no non-trivial finite dimensional unitary representations. They are called *non-compact* Lie algebras. An important example is the Poincaré algebra of the Lorentz group.

Complete antisymmetry. In the compact case, it is often more convenient to choose $\mathrm{tr}(T_aT_b) = \lambda\delta_{ab}$ for some $\lambda > 0$. We can then use the Killing form to raise and lower indices. In particular,

$$f_{abc} = -\mathrm{i}\frac{1}{\lambda}\mathrm{tr}([T_a, T_b]T_c) = -\mathrm{i}\frac{1}{\lambda}\mathrm{i}f_{ab}^{\ \ d}\lambda\delta_{dc}$$

By considering

$$\begin{aligned}
\mathrm{tr}([T_a, T_b]Tc) &= \mathrm{tr}(T_aT_bT_c - T_bT_aT_c) \\
&= \mathrm{tr}(T_aT_bT_c) - \mathrm{tr}(T_bT_aT_c) \\
&= \mathrm{tr}(T_bT_cT_a) - \mathrm{tr}(T_cT_bT_a) \\
&= \mathrm{tr}([T_b, T_c]T_a)
\end{aligned}$$

one easily concludes that $f_{abc} = f_{bca}$ is symmetric under cyclic permutations. Anti-symmetry in the first two indices yields then the complete anti-symmetry of the structure constants with all indices down: $f_{abc} = f_{bca} = f_{cab} = -f_{bac} = -f_{cba} = -f_{acb}$.

In this basis, the adjoint representation is unitary, because the T_a are imaginary and anti-symmetric, thus Hermitean. As already mentioned, the Killing form g_{ab} can be used ti raise and lower indices. In the compact case, this is trivial since $g_{ab} \propto \delta_{ab}$.

SIMPLE LIE ALGEBRAS

In order to classify all possible Lie algebras, a good first step is to clarify, whether Lie algebras can be composed out of smaller Lie algebras, and whether there are "elementary" Lie algebras, which cannot be further decomposed into smaller ones.

Invariant sub-algebra. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie sub-algebra. It is called an *invariant sub-algebra*, if for all $X \in \mathfrak{h}$ and all $Y \in \mathfrak{g}$, always $[X, Y] \in \mathfrak{h}$.

Via the exponential map, a similar statement holds for the corresponding Lie groups. With $h = \mathrm{e}^{\mathrm{i}X}$ and $g = \mathrm{e}^{\mathrm{i}Y}$, we have that $h' = g^{-1}hg = \mathrm{e}^{\mathrm{i}X'}$ with $X' \in \mathfrak{h}$. To see this, we observe that

$$X' = \mathrm{e}^{-\mathrm{i}Y} X \mathrm{e}^{\mathrm{i}Y} = X - \mathrm{i}[Y, X] - \frac{1}{2!}[Y, [Y, X]] + \dots$$

which one obtains by expanding $X'(\varepsilon) = \mathrm{e}^{-\mathrm{i}\varepsilon Y} X \mathrm{e}^{\mathrm{i}\varepsilon Y}$ around $\varepsilon = 0$ and putting $\varepsilon = 1$ at the end. As each (nested) commutator maps back to $\mathfrak{h}$, it follows that $X' \in \mathfrak{h}$.

We remark that $\{0\}$ and $\mathfrak{g}$ are trivial invariant sub-algebras of $\mathfrak{g}$.

Simple Lie algebras. A Lie algebra $\mathfrak{g}$, which does not possess non-trivial invariant sub-algebras, is called *simple*. Simple Lie algebras generate simple Lie groups.

The adjoint representation of a simple compact Lie algebra $\mathfrak{g}$ is irreducible.

Proof: Assume the contrary. Thus, there exists an invariant subspace $\mathfrak{h} \subset \rho_{\mathrm{ad}}(\mathfrak{g})$. However, the vectors of the adjoint representation vector space, which is $\mathfrak{g}$, correspond to generators. Let us choose a basis of $\mathfrak{g}$ such that $\mathrm{span}\{T_r : r = 1, \dots, k\} = \mathfrak{h}$, $\mathrm{span}\{T_s : s = k+1, \dots, n\} = \mathfrak{h}^\perp$. So, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ as vector space. But this implies that the matrix elements $(T_a)_r^s = -\mathrm{i}f_{ar}{}^s = 0$ for all $a = 1, \dots, n$, $r = 1, \dots, k$, $s = k+1, \dots, n$. This must be so, since $[X_a, X_r] = -\mathrm{i}f_{ar}{}^b X_b$. Then, if $X_r \in \mathfrak{h}$, we have $f_{ar}{}^b = 0$ for all b such that $X_b \notin \mathfrak{h}$. As we can lower indices trivially, we find $-\mathrm{i}f_{airs} = 0$. Now, we use the complete anti-symmetry of the structure constants with all indices down to conclude that $f_{rr's} = f_{r'sr} = f_{srr'} = 0$ and $f_{rss'} = f_{srs'} = f_{rs's} = 0$. This means that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ decomposes into a direct sum as algebras, i.e. it decomposes into two invariant sub-algebras. The only non-vanishing structure constants are hence $f_{rr'r''}$ and $f_{ss's''}$. Therefore, $\mathfrak{g}$ is not simple which is a contradiction. It follows that the adjoint representation of a simple compact Lie algebra must be irreducible. ∎

Abelian sub-groups. Let G be a Lie group. Let us define a one-parameter sub-group by the group elements $g(\lambda) = \mathrm{e}^{\mathrm{i}\lambda X}$. If this sub-group has the property that for all $h \in G$ and all λ we have $g(\lambda)h = hg(\lambda)$, we call this sub-group an *Abelian sub-group*. It is generated by a singe generator X, and we thus have $[X, Y] = 0$ for all $Y \in \mathfrak{g}$.

Such Abelian sub-groups are called $U(1)$ factors, because $\mathrm{e}^{\mathrm{i}\lambda X}\mathrm{e}^{\mathrm{i}\lambda' X} = \mathrm{e}^{\mathrm{i}(\lambda+\lambda')X}$. The group $U(1)$ is the group of unitary transformations of a one-dimensional complex vector space. Hence, $g \in U(1)$ if for all $z \in \mathbb{C}$ we have $|gz| = |z|$. Thus, $g = \mathrm{e}^{\mathrm{i}\varphi}$ is just a phase. So, $U(1)$ is the group of phase transformations.

Let us now choose a basis in $\mathfrak{g}$ such that the above X coincides with the basis element X_a for a certain a. It is then clear that $f_{ab}{}^c = 0$ for all b, c. Therefore, $\mathrm{tr}(T_a T_b) = \mathrm{tr}(f_{ac}{}^{c'} f_{bc'}{}^d) = f_{ac}{}^{c'} f_{bc'}{}^c = 0$. On the other hand, this is $\mathrm{tr}(T_a T_b) = k_a \delta_{ab}$ so we must have $k_a = 0$. Abelian sub-groups lead to zero eigen values in the Killing form. If there are $U(1)$ factors, the Killing form does not induce a norm on the space $\mathfrak{g}$. The $U(1)$ factors of the Lie group G correspond to Abelian invariant sub-algebras of the Lie algebra $\mathfrak{g}$.

Semi-simple Lie algebra. A Lie algebra $\mathfrak{g}$ without any Abelian invariant sub-algebra is called *semi-simple*. This means the following: $\nexists Y \in \mathfrak{g} : \forall X \in \mathfrak{g} : [X, Y] = 0$.

We will see later that for $\mathfrak{g}$ semi-simple, we have $\mathfrak{g} = \bigoplus_r \mathfrak{g}_r$ with $\mathfrak{g}_r$ simple Lie algebras. Furthermore, we have $\forall X \in \mathfrak{g} : \exists Y \in \mathfrak{g} : [X, Y] \neq 0$.