SUBGROUPS

Nature has the habit to very often realize symmetries not perfectly. Probably, this is why the universe is so beautiful and why it would be unbearable boring otherwise. Thus, having found a large symmetry group, a good question in physics is to look for its subgroups. One of the prime examples for this is the search for GUTs (*grand unified theories*). These theories have a large gauge group G nut are valid only at very high energies, e.g. at the first few moments of the universe. Later, when the average available energy in a cooling universe decreased, this large gauge symmetry which unifies all known fundamental forces, somehow gets broken to the direct product $U(1) \times SU(2) \times SU(3) \subset G$ of the gauge groups we know today. The first factor stands for the gauge theory of electromagnetism, which is an Abelian gauge theory of one gauge boson, the photon. The second factor describes weak interactions via the intermediate vector bosons $W^{\pm}$ and Z. Finally, the last factor represents the gauge group of quantum chromodynamics, the strong interaction of gluons acting between the quarks. But how can we find out, which subgroups a given Lie group contains?

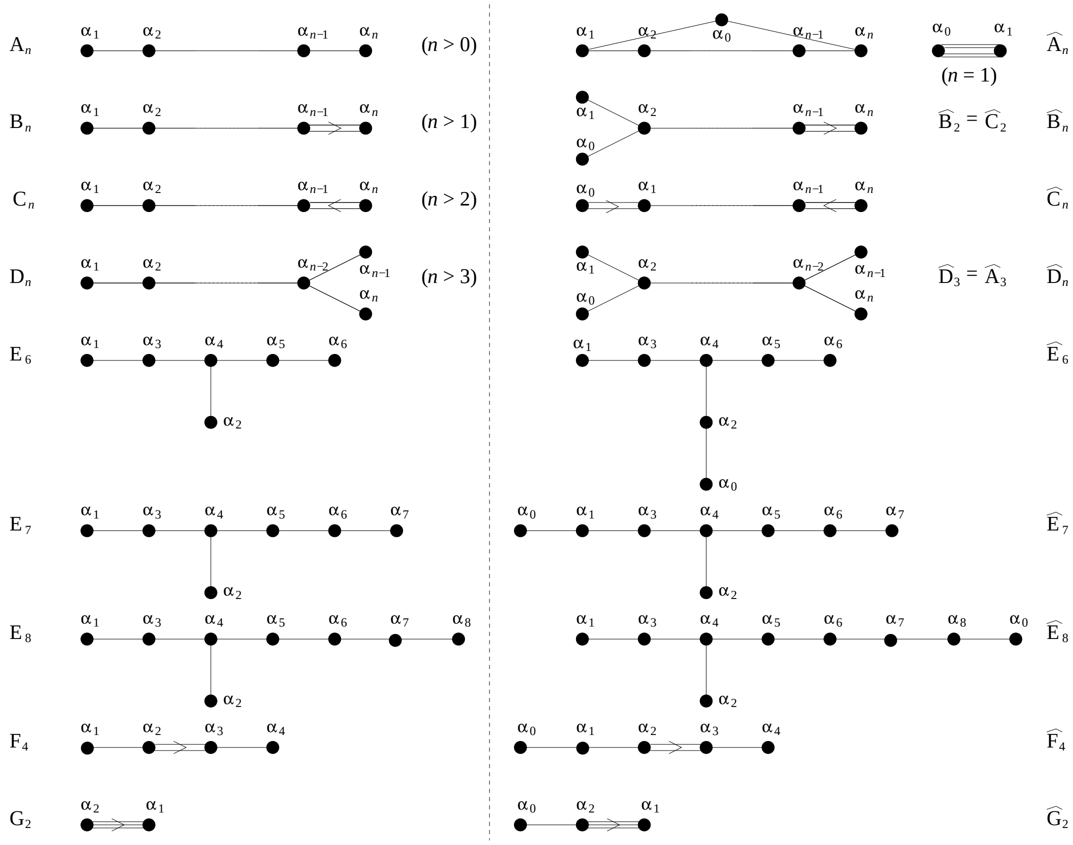
Regular subalgebra. Given a simple Lie algebra $\mathfrak{g}$, a *regular subalgebra* $\mathfrak{p}$ is a subalgebra such that the roots α of $\mathfrak{p}$ are a subset of the roots of $\mathfrak{g}$ and the generators of the Cartan subalgebra of $\mathfrak{p}$ are linear combinations of the Cartan generators of $\mathfrak{g}$. A regular subalgebra is called *maximal*, if $\operatorname{rank}\mathfrak{p} = \operatorname{rank}\mathfrak{g}$. Of course, the Cartan subalgebras are identical in this case which means that the maximal set of simultaneously commuting observables remains the same.

Subalgebras from Dynkin diagrams. Given a Dynkin diagram for a simple Lie algebra $\mathfrak{g}$, we can leave out a node together with the lines connected to it. This inevitably will split the Dynkin diagram into two new diagrams. These are then associated with a regular subalgebra of the original algebra $\mathfrak{g}$ which, however, is not semi-simple. The subalgebra has a subset of the roots of the original algebra, but we also lost one generator from the Cartan algebra. By removing a node, the rank of the subalgebra is reduced by one, and the simple roots are a subset of the original simple roots. On the level of the groups, we thus find $G = G_1 \times G_2 \times U(1)$, where the additional $U(1)$ factor comes from the left out Cartan generator. For example, $SU(n+m)$ can be reduced in this way into $SU(n) \times SU(m) \times U(1)$. This is the classical ansatz for a GUT: $SU(5)$ gets broken into $SU(3) \times SU(2) \times U(1)$.

There are other regular subalgebras, which cannot be obtained by leaving out a node. These can be found with the help of the merging procedure which we used in the lecture to prove the classification theorem. Thus, $SU(n)$ naturally contains $SU(k)$, $k < n$, as regular subalgebras. Another nice merging yields that $SO(2n)$ contains an $Sp(2n)$ subalgebra, by merging the branch of the D_n Dynkin diagram to a double line connecting to a (longer!) new root in the C_n diagram.

Extended Dynkin diagrams. As just explained, these subalgebras all have a smaller rank than the original algebra. There is, however, an elegant way to obtain all the semi-simple maximal regular subalgebras. Let us define the so-called *lowest root* α^0 by the property that $\alpha^0 - \alpha^j$ is not a root for all simple roots α^j, $j = 1, \ldots, r = \operatorname{rank}\mathfrak{g}$. That implies that $2(\alpha^0 \cdot \alpha^j)/(\alpha^0)^2$ and $2(\alpha^0 \cdot \alpha^j)/(\alpha^j)^2$ are non-positive integers for all simple roots α^j. Therefore, the system $\{\alpha^j : j = 1, \ldots r\} \cup \{\alpha^0\}$ of vectors satisfies all the conditions for a Π-system (root system) except that there is now one linear relation among the vectors. Such a Π-system is called an *extended* Π*-system*, to which belongs an *extended Dynkin diagram*. If we now remove a node from an extended Dynkin diagram, the resulting corresponding set of vectors will again be linearly independent. These roots still satisfy the master formula. However, the Dynkin diagram might be disconnected, so the root system might be decomposable. Thus, we will obtain the simple roots of a maximal regular subalgebra of the original algebra, but this subalgebra may be semi-simple instead of simple. It is maximal, since we now have as many nodes as the original algebra's Dynkin diagram had, so the ranks must be equal.

In the proof of the classification theorem, we already encountered all the root systems which satisfied all conditions but linear independence. Thus, we already know how the lowest root α^0 then looks, and in fact, the lowest root can be computed explicitly for all Dynkin diagrams. Thus, to each Dynkin diagram exists a unique extended Dynkin diagram. The following table lists all the Dynkin diagram to the left together with their extended version to the right. The extended Dynkin diagram to the Lie algebra X is denoted by $\hat{X}$. The additional node for the lowest root α^0 is explicitly indicated. Note a few exceptions: $\hat{A}_1 = \hat{B}_1 = \hat{C}_1$ cannot be extended without introducing an additional notation, since the lowest root for $\mathfrak{su}(2)$ with simple root α is simply $-\alpha$. Thus, the angle π between two roots, which so far could never appear, is denoted by a link out of four lines. Exercise: How should $\hat{D}_2$ look like? Remember that $D_2 = \mathfrak{so}(2) \oplus \mathfrak{so}(2)$ is not semi-simple. Finally, $\hat{B}_2$ and $\hat{D}_3$ do not have extension as indicated for $\hat{B}_n$ and $\hat{D}_n$, respectively, since the number of nodes is too small. If we would extend in the indicated manner, we would be led to forbidden diagrams such as a branch directly attached to a double line. The correct extended diagrams are then given by $\hat{B}_2 = \hat{C}_2$ and $\hat{D}_3 = \hat{A}_3$, respectively.



Maximal subalgebras. The recipe to obtain maximal semi-simple regular subalgebras is then simple: Take the *extended* Dynkin diagram and remove one node from it. The first thing one notices is that $A_n = \mathfrak{su}(n+1)$ does not have any non-trivial maximal semi-simple regular sublagebras, because removing any node from $\hat{A}_n$ just takes us back to A_n, so nothing interesting here.

The case $B_n = \mathfrak{so}(2n+1)$ is more interesting. Removing a node from the left end of $\hat{B}_b$ just gives back B_n, but removing the node at the right end yields D_n. So, this tells us that $SO(2n+1)$ contains an $SO(2n)$ subgroup as maximal regular subgroup. Finally, we can remove a node somewhere from the inside to obtain $SO(2k) \times SO(2n-2k+1)$. Of course, you can continue this procedure for the factors to break this group further down.

The case $D_n = \mathfrak{so}(2n)$ is less interesting, since removing nodes from either end of $\hat{D}_n$ just gives back D_n. Removing a node from the inside simply yields $D_k \oplus D_{n-k}$ corresponding to the subgroup $SO(2k) \times SO(2n-2k)$.

In a similar way, we obtain for $C_n = \mathfrak{sp}(2n)$, that removing a node from either end of $\hat{C}_n$ simple gives back C_n. Removing any node corresponding to a shorter root from the middle simply breaks C_n into $C_k \oplus C_{n-k}$. Removing the first or last of the shorter roots yields instead $A_1 \oplus C_{n-1}$ which is just the same since $A_1 = C_1 = \mathfrak{su}(2)$. Thus, the nontrivial subgroups are $Sp(2k) \times Sp(2n-2k)$.

Our special friend G_2 is so small, that we easily can list all its maximal regular subalgebras. Removing the node from the left end of $\hat{G}_2$ gives back G_2. Deleting the node from the right end gives us $A_2 = \mathfrak{su}(3)$. Finally, taking out the middle node gives $SU(2) \times SU(2)$.

The exceptional algebra F_4 possesses a B_4 subalgebra by removing the shorter root at the right end op $\hat{F}_4$. Removing the other shorter root instead gives $A_1 \oplus A_3$. Going further to the left, the next root we could delete is a longer root right in the middle of the diagram, which would yield $A_2 \oplus A_2$. Removing the penultimate node to the left we get $C_3 \oplus A_1$, and finally deleting the node at the left end just gives back F_4.

Grand Unified Theories

Some small remarks about GUTs. We have already seen that the broken symmetry group $SU(3) \times SU(2) \times U(1)$ is naturally contained in $SU(5)$ via $A_4 \to A_2 \oplus A_1$. However, if one investigates this in detail, one finds that parity symmetry gets lost in $SU(5)$ in a phenomenologically unsatisfactory manner. In the same way, we can take $B_4 = \mathfrak{so}(9)$ and cut out the penultimate node to the right, $B_4 \to A_2 \oplus A_1$. However, the relative lengths of the roots from A_2 then differ from the ones in A_1. There are other reasons why $SO(9)$ does not make a good gauge group for a unified theory. What happens is that the representations of the larger group do not contain the representations of the broken direct product group in a useful manner to easily yield the low energy particle spectrum we actually observe. This rules out $C_4 = \mathfrak{sp}(8)$ as well. However, we can take $SO(10)$. The Lie algebra is D_5, which can be broken into A_4 by removing one node of the branched end of its Dynkin diagram. However, removing a node from the middle of its extended Dynkin diagram gives the phenomenological interesting maximal subgroup $SU(4) \times SU(2) \times SU(2) \cong SO(6) \times SO(4)$. Thus, we get back our $SU(5)$ as a subalgebra of $SO(10)$. Note, that $SU(5)$ is a maximal subgroup of $SO(10)$. In comparison, the maximal subgroups of $SO(9)$ are $SO(8)$ or $SO(9-2k) \times SO(2k)$. Unification with $SO(10)$ works quite nicely, since the $SU(4)$ contains a color $SU(3)$ subgroup. The weak interaction is given by one of the $SU(2)$ subgroups. It turns out that together with the other $SU(2)$ factor, the particle spectrum becomes completely symmetric with respect to chirality and the weak interaction (which we know does not conserve parity!). In particular, it contains the as yet unobserved right-handed neutrino. The problem with all such unifying theories is to find an explicit description, how the symmetry breaking works in such a way that all the unobserved particles become extremely heavy. One can go on in this manner and look for even larger unifying algebras. It is interesting that the chain $E_6 \to D_5 = E_5 = \mathfrak{so}(10) \to A_4 = E_4 = \mathfrak{su}(5)$ works, but cannot be continued to contain E_7 and E_8 as well, since the latter algebras do not yield any sensible unified theories. The interesting point about E_6 is that it contains a maximal subgroup $SU(3) \times SU(3) \times SU(3)$. This is very attractive to get a hierarchical way of symmetry breaking. First, at very high energies, we break down to the three $SU(3)$ factors. At lower energies, two of the factors are further broken down to $SU(2) \times U(1)$ which at our every-day-energies gets broken down to pure electromagnetism $U(1)$.

Electroweak interaction $SU(2) \times U(1)$. One major motivation to search for unifying gauge groups is the problem of charge quantization. Electromagnetism is a $U(1)$ gauge theory. The problem is that $U(1)$ does not yield a discrete spectrum of quantum numbers. In principle, any value q for the charge of a particle is possible. Furthermore, the $U(1)$ factor in the standard model commutes with the other gauge groups, the color $SU(3)$ and the $SU(2)$. If this direct product of gauge groups were a subgroup of a simple Lie group, then all its representations had to fit into the representations of this larger group. But simple Lie groups have the wonderful property that the weights of all the states of any representation are quantized according to the discrete points on the weight lattice. And these weights are the quantum numbers of the states with respect to the maximal set of commuting observables, the Cartan algebra. So, unifying with a simple Lie algebra would enforce that charge had to be quantized as well. This would then solve the puzzle that the charges of the leptons and the charges of the quarks are so closely related to each other, although they belong to completely different representations with respect to $SU(2) \times U(1)$. We conclude this handout with a very brief tour through the concept of unification, where we take $SU(5)$ as an example.

The Glashow-Salam-Weinberg model of electroweak interaction uses $SU(2) \times U(1)$ as unifying gauge theory. If we restrict ourselves to one generation of particles, this theory contains the following seven right-handed particles (note that all particles are assumed to be massless so that helicity is a relativistically conserved quantity, the handedness): $u, d, e^-, \bar{u}, \bar{d}, e^+, \bar{\nu}_e$. There is no right-handed neutrino due to the parity-violating nature of the weak interaction. To ease notation, we will denote the electron by e, the positron by $\bar{e}$ and the electron anti-neutrino by $\bar{\nu}$. Since color commutes with the electroweak interaction, we don't have to bother with color indices for the quarks yet. Under the electroweak interaction, $(\bar{e}, \bar{\nu})$ transform as a doublet, i.e. in a spin $1/2$ representation. Thus, the corresponding creation operators for these particles, denotes $p^\dagger$ for particle p, can be arranged as the components of an irreducible tensor with respect to $SU(2)$, such as $\bar{\ell}_1^\dagger = \bar{e}^\dagger$ and $\bar{\ell}_2^\dagger = \bar{\nu}^\dagger$. The same is true for the $(\bar{d}, \bar{u})$ anti-quarks, which we can collect as $\bar{\psi}_1^\dagger = \bar{d}^\dagger$ and $\bar{\psi}_2^\dagger = \bar{u}^\dagger$. Let us denote the generators of $SU(2)$ by X^a, and the generator of the $U(1)$ by S. One finds the following commutation relations between the gauge group generators and the particle creation operators:

$$\begin{aligned}
[X^a, u^\dagger] &= 0, & [X^a, d^\dagger] &= 0, & [X^a, e^\dagger] &= 0, & [X^a, \bar{\psi}_j^\dagger] &= +\tfrac{1}{2}\bar{\psi}_k^\dagger(\sigma^a)_{kj}, & [X^a, \bar{\ell}_j^\dagger] &= +\tfrac{1}{2}\bar{\ell}_k^\dagger(\sigma^a)_{kj}, \\
[S, u^\dagger] &= +\tfrac{2}{3}u^\dagger, & [S, d^\dagger] &= -\tfrac{1}{3}d^\dagger, & [S, e^\dagger] &= -e^\dagger, & [S, \bar{\psi}_j^\dagger] &= -\tfrac{1}{6}\bar{\psi}_j^\dagger, & [S, \bar{\ell}_j^\dagger] &= +\tfrac{1}{2}\bar{\ell}_j^\dagger.
\end{aligned}$$

Thus, all the particles transform as tensors, either as singlets or as doublets. The annihilation operators for the right-handed particles are given by the adjoints of their creation operators. Therefore, they transform in the complex conjugate representation, such that, in particular, all S eigenvalues change sign. The creation operators of the left-handed particles transform exactly as the annihilation operators of their right-handed anti-particles. Furthermore, the operator for the electric charge is given by $Q = X^3 + S$ such that

$$[Q, u^\dagger] = +\frac{2}{3}u^\dagger, \ \ [Q, d^\dagger] = -\frac{1}{3}d^\dagger, \ \ [Q, e^\dagger] = -e^\dagger, \ \ [Q, \bar{u}^\dagger] = -\frac{2}{3}\bar{u}^\dagger, \ \ [Q, \bar{d}^\dagger] = +\frac{1}{3}\bar{d}^\dagger, \ \ [Q, \bar{\nu}^\dagger] = 0.$$

The idea of any gauge theory is now that the generators (i.e. the particles from the adjoint representation of the gauge group's Lie algebra) are associated with the force particles. Thus, the three X^a generators are the three intermediate vector bosons W^a, and the generator S is often denoted X in the electroweak model. Only one linear combination, namely $W^3 + X = Q$, remains a massless force particle, namely the photon, the other particles, the $W^{\pm} = W^1 \pm \mathrm{i}W^2$ and $Z = W^3 - X$ are responsible for the weak interaction. These latter three acquire heavy masses due to the Higgs mechanism which also breaks the $SU(2) \times U(1)$ model down to the embedded $U(1)$ symmetry of electromagnetism. This mechanism then also manages to make the weak interaction short ranged due to the masses of its gauge bosons. The mass for the gauge bosons is not the only thing the Higgs mechanism is needed for. Without it, the electron and the quarks would be massless as well, since only for massless particles is a parity violating interaction consistent with relativity. What the Higgs mechanism does is essentially to create a vacuum state of the theory which is non-trivial, i.e. not just a singlet of $SU(2) \times U(1)$. Only the $U(1)$ symmetry via Q is left by this *spontaneous symmetry breaking*, under which the vacuum state is a singlet.

Higgs mechanism. We will sketch the Higgs mechanism very briefly here. Suppose that there exists an additional *scalar* and Lorentz-invariant field. Such a field can have a non-zero expectation value in the vacuum state without breaking Lorentz-invariance. If this field now transforms non-trivially under $SU(2) \times U(1)$, a non-zero vacuum expectation value leads to spontaneous symmetry breaking. In fact, a Higgs field ϕ transforming as a doublet under $SU(2)$ and with $S = \frac{1}{2}$ does the trick,

$$[X^a, \phi_j^\dagger] = +\frac{1}{2}\phi_k^\dagger(\sigma^a)_{kj}\,, \;\; [S, \phi_j^\dagger] = +\frac{1}{2}\phi_j^\dagger.$$

If such a field exists, it may interact with itself. This interaction can be described by a potential $V(\phi)$, which simply is the energy stored in a constant ϕ field. The potential should be invariant under $SU(2) \times U(1)$ in order to construct a physical vacuum state, so $V(\phi)$ is actually a function of $\phi^\dagger\phi$ only. The lowest energy state corresponds to the minimum value of $V(\phi)$. It can now happen, e.g. for $V(\phi) = \lambda(\phi^\dagger\phi - v)^2$, that the minimum of V is exhibited not for $\phi = 0$, but for $\langle\phi^\dagger\phi\rangle = v^2$ for $\lambda > 0$. We could thus take the vacuum expectation value of ϕ as $\langle\phi_1\rangle = 0$, $\langle\phi_2\rangle = v$. This choice implies that $[Q, \phi]|_{\langle\phi^\dagger\phi\rangle = v^2} = 0$ such that the particular subgroup of $SU(2) \times U(1)$ associated to electromagnetism (generated by $Q = X^2 + S$) is not broken by the Higgs field. However, any other linear combination of generators of $SU(2) \times U(1)$, acting on ϕ, yields a non-zero result. This means that these generators are all spontaneously broken and correspond to transformation of the physical vacuum state to an unphysical one. The precise form of the unbroken part of the symmetry depends on our choice for ϕ, but any other choice satisfying $\langle\phi^\dagger\phi\rangle = v^2$ yields the same physics, since it is related via an $SU(2) \times U(1)$ transformation to our initial choice. The matter particles (electron and quarks) get mass from the Higgs as well. The rule of thumb is that a Higgs field can produce mass for a spin $1/2$ particle if the tensor product of the representation of the right-handed particle with the representation of the corresponding anti-particle contains the representation of the Higgs field (or its complex conjugate). In quantum field theory, this rule of thumb implies that an $SU(3) \times SU(2) \times U(1)$ invariant action can be written down, which involves the Higgs field as well as all particle creation and annihilation operators, that becomes a pure mass term when the Higgs field is replaced by its vacuum expectation value.

Example: Unifying with $SU(5)$. Let us finally turn to the question of unification. The $SU(2) \times U(1)$ symmetry for the electroweak interaction is a partial unification of the weak and electromagnetic interactions. A search for full unification of this theory within a simple Lie group failed until the strong interaction of color was incorporated as well via an $SU(3)$ gauge theory – quantum chromodynamics – whose generators we denote by T^a. A particle creation operator $a_{rj}^\dagger$ transforms according to a representation $(\rho, \varrho)_s$ of $SU(3) \times SU(2) \times U(1)$, if it satisfies

$$[T^a, a_{rj}^\dagger] = a_{sj}^\dagger(\rho(T^a))_{sr}\,, \;\; [X^a, a_{rj}^\dagger] = a_{rk}^\dagger(\varrho(X^a))_{kj}\,, \;\; [S, a_{rj}^\dagger] = s\, a_{rj}^\dagger.$$

Thus, r is a color $SU(3)$ index associated with the $SU(3)$ representation ρ, and j is an $SU(2)$ index associated with the $SU(2)$ representation ϱ. Our choice $Q = X^3 + S$ implies that s is simply the averaged electromagnetic charge of the full representation, since $\mathrm{tr}\, Q = \mathrm{tr}\, X^3 + \mathrm{tr}\, S = \mathrm{tr}\, S$, because the trace of each $SU(2)$ generator vanishes (the quantum numbers of $SU(2)$ are symmetric with respect to the origin). Denoting the representations ρ and ϱ by their respective dimensions, we identify the particle creation operators for the right-handed ones as members of the representations

$$u^\dagger : (\mathbf{3}, \mathbf{1})_{+\frac{2}{3}}\,, \;\; d^\dagger : (\mathbf{3}, \mathbf{1})_{-\frac{1}{3}}\,, \;\; e^\dagger : (\mathbf{1}, \mathbf{1})_{-1}\,, \;\; \bar{\psi}^\dagger : (\bar{\mathbf{3}}, \mathbf{2})_{-\frac{1}{6}}\,, \;\; \bar{\ell}^\dagger : (\mathbf{1}, \mathbf{2})_{+\frac{1}{2}}.$$

Thus, the full $SU(3) \times SU(2) \times U(1)$ representation, in which the right-handed particle's creation operators reside, together with the one for the left-handed fields are thus

$$(\mathbf{3}, \mathbf{1})_{+\frac{2}{3}} \oplus (\mathbf{3}, \mathbf{1})_{-\frac{1}{3}} \oplus (\mathbf{1}, \mathbf{1})_{-1} \oplus (\bar{\mathbf{3}}, \mathbf{2})_{-\frac{1}{6}} \oplus (\mathbf{1}, \mathbf{2})_{+\frac{1}{2}}\,, \;\; (\bar{\mathbf{3}}, \mathbf{1})_{+\frac{2}{3}} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{-\frac{1}{3}} \oplus (\mathbf{1}, \mathbf{1})_{-1} \oplus (\mathbf{3}, \mathbf{2})_{-\frac{1}{6}} \oplus (\mathbf{1}, \mathbf{2})_{+\frac{1}{2}}\,.$$

The latter is the complex conjugate of the former, where we made use of the fact that the singlets are always real representations, as is the $SU(2)$ double representation, $\bar{\mathbf{2}} = \mathbf{2}$. The two representations above are not the same, so the representation is complex, which stems from the parity violating nature of the electroweak interaction.

To find a unifying theory, we have to find a gauge group G which contains $SU(3) \times SU(2) \times U(1)$ as a subgroup, and which possesses a representation which transforms under this subgroup precisely as given above. The rank of G must be at least four such that it can contain T^3, T^8, X^3 and S in its Cartan algebra. The simplest possibility is indeed the rank four group $SU(5)$. In fact, the other simple rank four groups do not work, because they do not have complex representations. The algebra $\mathfrak{su}(5)$ has two five-dimensional representations $\mathbf{5}$ and $\bar{\mathbf{5}}$, which are the fundamental representations $(1,0,0,0) = [1]$ and $(0,0,0,1) = [4]$, respectively (for this notation, see the seminar on Young tableaux). Since $\mathbf{5}$ is complex, these two are not equivalent. There exists an $SU(2) \times U(1)$ subgroup of $SU(5)$ such that the $\mathbf{5}$ transforms as a five dimensional subset of the creation operators, namely $(\mathbf{3},\mathbf{1})_{-\frac{1}{3}} \oplus (\mathbf{1},\mathbf{2})_{+\frac{1}{2}}$. The other five-dimensional subset $(\mathbf{3},\mathbf{1})_{+\frac{2}{3}} \oplus (\mathbf{1},\mathbf{2})_{+\frac{1}{2}}$ cannot work, since the generator S is not realized traceless on it. This implies that S cannot be a generator of $SU(5)$ at all. It is indeed possible to embed $SU(3) \times SU(2) \times U(1)$ in $SU(5)$ to obtain the above five-dimensional representation, namely by taking the $SU(3)$ generators to be traceless matrices acting on the first three indices in the $\mathbf{5}$, $\left(\begin{smallmatrix} T^a & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and by taking the $SU(2)$ generators to be traceless matrices acting on the last two indices, $\left(\begin{smallmatrix} 0 & 0 \\ 0 & X^a \end{smallmatrix}\right)$. Then S is the generator that commutes with both of these, given by $\mathrm{diag}(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2})$. Thus, we can collect the $d^\dagger$ and $\bar{\ell}^\dagger$ creation operators into an $SU(5)$ $\mathbf{5}$ representation $\lambda_j^\dagger$ as follows:

$$\lambda_j^\dagger = d_j^\dagger \text{ for } j = 1,2,3\,, \;\; \lambda_4^\dagger = \bar{\ell}_1^\dagger = \bar{e}^\dagger\,, \;\; \lambda_5^\dagger = \bar{\ell}_2^\dagger = \bar{\nu}^\dagger\,.$$

We are left with $(\mathbf{3},\mathbf{1})_{+\frac{2}{3}} \oplus (\mathbf{1},\mathbf{1})_{-1} \oplus (\bar{\mathbf{3}},\mathbf{2})_{-\frac{1}{6}}$. This representation is ten-dimensional. Fortunately, the other two fundamental representations of $SU(5)$ are ten-dimensional, $\mathbf{10} = (0,1,0,0) = [2]$ and $\overline{\mathbf{10}} = (0,0,1,0) = [3]$. In fact, $\mathbf{10} = \mathbf{5} \wedge \mathbf{5}$ is an anti-symmetric tensor product of two $\mathbf{5}$ representations, which we could use to identify how it transforms under $SU(3) \times SU(2) \times U(1)$. The $SU(3)$ and $SU(2)$ representations decompose as discussed in the seminar, the S quantum numbers simply add:

$$\left[(\mathbf{3},\mathbf{1})_{-\frac{1}{3}} \oplus (\mathbf{1},\mathbf{2})_{+\frac{1}{2}}\right] \wedge \left[(\mathbf{3},\mathbf{1})_{-\frac{1}{3}} \oplus (\mathbf{1},\mathbf{2})_{+\frac{1}{2}}\right] = \left[(\bar{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}} \oplus (\mathbf{1},\mathbf{1})_{+1} \oplus (\mathbf{3},\mathbf{2})_{+\frac{1}{6}}\right]\,.$$

This is the complex conjugate of our proposal, so we actually want the $\overline{\mathbf{10}}$ of $SU(5)$. We can fill in the remaining right-handed fermion creation operators in this $SU(5)$ representation, anti-symmetric in two upper indices (since we used lower indices so far),

$$\xi^{ab\dagger} = \epsilon^{abc} u_c^\dagger\,, \;\; \xi^{a4\dagger} = \bar{\psi}_2^{a\dagger} = \bar{u}^{a\dagger}\,, \;\; \xi^{a5\dagger} = \bar{\psi}_1^{a\dagger} = \bar{d}^{a\dagger}\,, \;\; \xi^{45\dagger} = e^\dagger\,.$$

where the indices $a,b,c \in \{1,2,3\}$ and where $\xi^{jk\dagger} = -\xi^{kj\dagger}$. This is the standard $SU(5)$ unified model, where the creation operators for all right-handed particles transform in the representation $\mathbf{5} \oplus \overline{\mathbf{10}}$. Of course, the creation operators for the left-handed particles transform then in the complex conjugate representation $\overline{\mathbf{5}} \oplus \mathbf{10}$. As one sees, it is quite a non-trivial thing that the content of the representations fits so nicely with the observed particles.

Consequences. Having identified a potential candidate unifying theory, the next step is to explain how it is actually broken down to the symmetry we observe nowadays. The Higgs mechanism can do this for us, and there is one particular simple solution for this. One can show that in the adjoint representation, the $\mathbf{24}$, the S generator has just the properties of the vacuum expectation value of the Higgs field. That is similar to the so-called hypercharge generator in the adjoint representation of $SU(3)$, which commutes with isospin. The $U(1)$ generators S in $SU(5)$ commutes with all the generators of the $SU(3) \times SU(2) \times U(1)$ subgroup. Thus, taking the adjoint representation $\mathbf{24}$ with a vacuum expectation value for the Higgs in the direction of S (there are physically inequivalent directions for the vacuum expectation value of the Higgs field in $SU(5)$, and one can show that the direction of S is an admissible value) indeed leads to the desired symmetry breaking. Next, the Higgs field should also be responsible to give the leptons and the quarks their masses. This can happen, if the Higgs couples to the fermions. In turn, this can be the case when its representation (or its complex conjugate) appears in the tensor product of the $SU(5)$ representations of the fermion in question and its anti-particle, respectively. Furthermore, the Higgs representation must have a component that transforms under $SU(3) \times SU(2) \times U(1)$ like the Higgs field of this model (or its complex conjugate).

Now, the right-handed positron $\bar{e}$ and the d quark reside in the $\mathbf{5}$, while their anti-particles, the electron e and the $\bar{d}$ quark, live in the $\overline{\mathbf{10}}$. As an exercise, perform the Clebsh-Gordan decomposition of $\mathbf{5} \otimes \overline{\mathbf{10}}$ in $SU(5)$, which gives $\overline{\mathbf{5}} \oplus \overline{\mathbf{45}}$. Using the Young notation, this is $[1] \times [3] = [4] \oplus [3,1]$. The hard work is to show that the two irreps on the right hand side do indeed contain components with respect to $SU(3) \times SU(2) \times U(1)$ with the correct properties to represent the Higgs field. The $\mathbf{5}$, for example, contains the $(\mathbf{1},\mathbf{2})_{+\frac{1}{2}}$. The theory has now a chance to produce a mass term for the electron and the d quark, since the action can contain a term involving the Higgs field and the particle creation and annihilation operators. This argument is in complete analogy to our study of the dipole matrix elements for electrons in crystals with octahedral symmetry, where we asked for the irreps which couple via the dipole operator. These were just these with non-vanishing matrix elements. A necessary condition for a matrix element to be non-vanishing is that the involved three irreps (for the bra- and ket-state as well as the operator) are linked together by a tensor product of two of them yielding the third. Here, the mass for the electron and d quark can arise from both, the $\mathbf{5}$ or the $\mathbf{45}$. The right-handed u quark and its anti-particle $\bar{u}$ both reside in the $\overline{\mathbf{10}}$. Now, $[3] \otimes [3] = [1] \oplus [4, 2] \oplus [3, 3] = \mathbf{5} \oplus \mathbf{45} \oplus \mathbf{50}$. It turns out that the last irrep, the $\mathbf{50}$ does not give rise to a mass term from the Higgs field, since it does not contain a component transforming like $(\mathbf{1}, \mathbf{2})_{\pm\frac{1}{2}}$ under $SU(3) \times SU(2) \times U(1)$.

Another consequence worth mentioning of unified theories is that certain particles can decay which are stable in the broken theory. The prime example of this is *proton decay*. The point is that in the $SU(5)$ theory, all the quarks, anti-quarks and the electron appear in the same irrep. Thus, $SU(5)$ admits interactions which do not conserve baryon number. It can happen, when two quarks in a proton interact with each other via the Higgs field. If the vacuum expectation value of the Higgs is very large (as it presumably is, since the Higgs seems to be a very heavy particle not yet identified in accelerators), the interaction is extremely short ranged and the decay probability is very small. One can actually predict on theoretical grounds and the experimentally observed differences between the color $SU(3)$, the electroweak $SU(2)$ and the $U(1)$ forces, how large the vacuum expectation value of the Higgs field should be, and derive from it an average proton live time. Experiments are conducted to look for decaying protons. None were observed so far which pushed the current value of the average proton live time beyond anything one could realize with a standard $SU(5)$ unified theory, which therefore has to be considered as ruled out. Thus, theorists are looking for other Lie group candidates for unifying theories, or try completely new concepts such as supersymmetry. In fact, supersymmetric $SU(5)$, $SO(10)$ or E_6 models are still very hot candidates for unifying theories. One beautiful side effect of supersymmetry is that the energy scales, where unification actually takes place, becomes the same for strong and electroweak interaction. However, this alone is no proof of the existence of supersymmetry, although it is highly suggestive.

There are many interesting questions to ask, and there are many things left unexplained here, since they would require a detailed study of the fundamental and adjoint representations (at least!) of the mentioned algebras in order to find the particle spectrum. Most difficult is always the question how the symmetry breaking actually takes place, and where the particles do get their mass from, as sketched for the $SU(5)$ case. The latter question can only partially be answered by the Higgs mechanism. Unification would solve other problems which need explanation, such as the origin of charge quantization. This is, by the way, one of the reasons why $SO(10)$ looks promising, since its maximal subgroup $SU(4) \times SU(2) \times SU(2)$ is the smallest semi-simple Lie group containing $SU(3) \times SU(2) \times U(1)$. Breaking symmetry this way would yields charge quantization for free. The reader might now have reached a point where she can appreciate the beauty and the power of symmetries (and their imperfect realization!) in physics. Next term, a seminar on Lie groups in elementary particle physics, supervised by me, should put these symmetries to use in order to explain from which elementary particles and fundamental forces our world is made of. So, please stay tuned.