**Classification of Coxeter Graphs**

In the lecture, we did not have time to prove the theorem on admissible Coxeter graphs. These are precisely the graphs, out of which root systems can be constructed in a unique way, which belong to Lie algebras. A root system to a graph with \( r \) nodes essentially is a set of \( r \) linearly independent vectors which possess certain properties as explained in the lecture, such that they can represent the \( r \) simple roots of a Lie algebra of rank \( r \). The proof argues in a nice way with properties of graphs and the corresponding properties of systems of unit vectors. A few steps of this proof shall be performed in this tutorial. We hope that this may enhance the feeling for beauty and elegance of this mathematical subject.

Let \( R_p^+ \) be the set of simple roots of a Lie algebra. We know that the elements of this set have the following properties:

- **[A]** The \( \alpha \in R_p^+ \) are linearly independent vectors.
- **[B]** If \( \alpha, \beta \in R_p^+ \) with \( \alpha \neq \beta \), then \( 2g(\alpha, \beta)/g(\alpha, \alpha) \in \mathbb{Z} \) is a non-positive integer.
- **[C]** The root system \( R_p^+ \) is indecomposable.

**Π-Systems.** A set of vectors satisfying **[A]**, **[B]** and **[C]** is called a Π-system.

The last condition **[C]** is necessary to guarantee that a root system, which satisfies **[A]** and **[B]**, belong to a simple Lie algebra. A root system \( R_p^+ \) is called decomposable, if we can split it into to mutually orthogonal sub-systems. Otherwise, it is called indecomposable. Decomposable root systems belong to Lie algebras, which are direct sums of simple Lie algebras, which therefore are semi-simple. The Classification of the simple (and hence of the semi-simple) Lie algebras is therefore achieved when one succeeds to classify all Π-systems. To any Π-system there belongs in a unique way a Coxeter graph. Condition **[C]** tells that the Coxeter graph cannot be divided into parts without cutting (at least) one line.

**Lemma 1.** Show that only the two following systems with three nodes are Π-systems:

Show then that the following three systems with three nodes violate condition **[A]**, since the three vectors lie in a plane:

**Sub-graphs.** Argue that any connected sub-graph of a Coxeter graph is again a Coxeter graph. Therefore, any indecomposable sub-system of a Π-system is again a Π-system. Any three connected vectors of an arbitrary Π-system must hence yield one of the two allowed forms of Lemma 1. Draw with this the only one Π-system which contains a triple line.

**Lemma 2.** Show the following: If a Π-system contains two vectors which are connected by a single line, then one obtains a new admissible Π-system by replacing this line including its two end nodes by one single node. Thus, the graph is reduced by one node and one line.

**Corollaries.** Lemma 2 has two important corollaries. Argue that the shrinking method described in Lemma 2 excludes the following cases: No Π-system can have more than one double line, and no Π-system contains a closed loop.

**Lemma 3.** Let \( A \) be an admissible sub-graph. Show that, if the first of the following graphs is a Π-system, then so is the second:

**More Corollaries.** Argue with Lemma 3 and Lemma 1, which of the following (sub-)graphs represent allowed branchings of an admissible Coxeter graph.

Conclude then that no Π-system can contain two or more branchings.
**Exceptional Cases.** So far we have analyzed general properties of \( \Pi \)-systems, considered as graphs. In order to conclude the classification, one has to consider a few exceptional cases, which cannot be excluded in this simple manner. These are:

![Graphs](image)

Each node \( j \) stands for a vector \( \alpha_j \). Assume now that all vectors \( \alpha_j \) have the same length (except in the last graph, there the vectors right from the double line have a length differing by a factor of \( \sqrt{2} \) or \( 1/\sqrt{2} \)). Find now positive integers \( \mu_j \in \mathbb{Z}_+ \), such that

\[
\left( \sum \mu_j \alpha_j \right)^2 = 0. 
\]

Hint: It is helpful to enumerate the nodes. In the last case there are two solutions, depending on whether the vectors right from the double line are the longer or the shorter ones.

**The Classification.** With our results we easily can draw a list of all still admissible graphs. Convince yourself that the following list is complete and that indeed any of these graphs is a \( \Pi \)-system. In order to also display the information that vectors may have different lengths in the graphs, one proceeds as follows: Our considerations so far show that vectors can only be of differing lengths, if they are connected by a multiple line. Such multiple lines now get an arrow which shows from the longer to the shorter vector. (Crib: The arrow can be interpreted as the symbol for “greater than” or “less than” in the resulting inequality on the lengths of the roots.)

- \( A_n \)
  - \( a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_n \) (\( n > 0 \))
- \( B_n \)
  - \( a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_n \) (\( n > 1 \))
- \( C_n \)
  - \( a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_n \) (\( n > 2 \))
- \( D_n \)
  - \( a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_n \) (\( n > 3 \))
- \( E_6 \)
- \( E_7 \)
- \( E_8 \)
- \( F_4 \)
- \( G_2 \)