GOOD AND BAD PARAMETRIZATIONS

Let G be a Lie group with parametrization u such that $g(u=0)=1\!\!1$. The dimension of the Lie group is n. The parametrization u can then, for small $|u|\ll 1$, be chosen out of an open subset of $\mathbb{R}^n$.

Example 1. Let us consider rotations in $\mathbb{R}^2$. These form a matrix Lie group, more precisely the matrices $g\in\mathrm{Mat}(2,\mathbb{R})$, which satisfy $g^{-1}=g^t$ and $\det\, g=1$. This matrix Lie group is called $SO(2)$. What is the dimension of this group? We chose the parametrization

$$g(u)=\begin{pmatrix}\cos(u) & \sin(u)\\ -\sin(u) & \cos(u)\end{pmatrix}\,,\quad u\in(-\pi,\pi)\subset\mathbb{R}\,.$$

Show that $g(0)=1\!\!1$. Find the group law, i.e. the function $w(u,v)$, such that $g(w(u,v))=g(v)\circ g(u)$. Is there an interesting observation you can make?

Example 2. We once more consider $SO(2)$. Now chose the following parametrization:

$$g(u)=\begin{pmatrix}1-u & \sqrt{1-(1-u)^2}\\ -\sqrt{1-(1-u)^2} & 1-u\end{pmatrix}\,,\quad u\in(0,2)\subset\mathbb{R}\,.$$

Check that $g(0)=1\!\!1$, $\det\, g(u)=1$ and $(g(u))^{-1}=(g(u))^t$. Derive again the group law $w(u,v)$ for the group multiplication $g(w(u,v))=g(v)\circ g(u)$. Consider first $(g)_{11}$. Check whether this is compatible in a simple way with the result which you get for $(g)_{12}$. Obviously, this is not a particularly suitable parametrization of this group.

GENERATORS

Next, we consider the group $SU(2)$, i.e. the matrix Lie group of matrices $g\in\mathrm{Mat}(2,\mathbb{C})$ with $\det\, g=1$ and $g^{-1}=g^\dagger$. We already know from Tutorial I, that $\dim SU(2)=3$.

Parametrization. Show. that with $u=(\phi,\alpha,\beta)$ and $\gamma=\sqrt{1-\alpha^2-\beta^2}$ as abbreviations

$$g(u)=\begin{pmatrix}\gamma\cos\frac{\phi}{2}-\mathrm{i}\gamma\sin\frac{\phi}{2} & -\mathrm{i}(\alpha-\mathrm{i}\beta)\\ -\mathrm{i}(\alpha+\mathrm{i}\beta) & \gamma\cos\frac{\phi}{2}+\mathrm{i}\gamma\sin\frac{\phi}{2}\end{pmatrix}$$

yields a good parametrization. We now wish to find the group law $w^k=w^k(u,v)$, at least rudimental.

Generators. Determine the three generators

$$X_1=\frac{\partial}{\partial\phi}g(u)\bigg|_{u=0}\,,\quad X_2=\frac{\partial}{\partial\alpha}g(u)\bigg|_{u=0}\,,\quad X_3=\frac{\partial}{\partial\beta}g(u)\bigg|_{u=0}\,.$$

Conversely one has $\exp(-\mathrm{i}\frac{\phi}{2}\vec{n}\cdot\vec{\sigma})$ with

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\,,\quad\sigma_2=\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix}\,,\quad\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

and $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ for all ϕ and $\vec{n}=(\alpha,\beta,\gamma)$ an element of $SU(2)$. Moreover, we have

$$\exp\left(-\mathrm{i}\frac{\phi}{2}\vec{n}\cdot\vec{\sigma}\right)=\cos\frac{\phi}{2}1\!\!1-\mathrm{i}\sin\frac{\phi}{2}\vec{n}\cdot\vec{\sigma}\,.$$

Algebra. Determine the structure constants of the Lie algebra of the generators σ_k of $SU(2)$. To do so, you have to compute the commutators $[\sigma_j,\sigma_k]=\mathrm{i}f_{jk}^{\ \ l}\sigma_l$. Result: $f_{jk}^{\ \ l}=2\varepsilon_{jk}^{\ \ l}$. With this you can write down the group law in a first order approximation: $w^l(u,v)=u^l+v^l-\frac{1}{2}u^jv^kf_{jk}^{\ \ l}$.