**Adjoint Representation of the Lie Algebra**

In the lecture, we introduced the adjoint representation $\text{ad}$ of a Lie algebra $\mathfrak{g}$ on itself as a vector space. This representation has some remarkable and useful properties. We wish to study the adjoint representation in the example of the Lie algebra $\mathfrak{su}(2)$ in some detail. Remember the last tutorial, where the generators for $\mathfrak{SU}(2)$, the Pauli matrices, were introduced. We note here in advance, that the Pauli matrices are, more precisely, the generators of $\mathfrak{SU}(2)$ in the fundamental representation on a two-dimensional vector space.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

**Adjoint generators.** The structure constants of $\mathfrak{su}(2)$ with respect to the above given generators are $f_{jk}^l = 2e_{jk}^l$. Write down the generators $T_j$ in the adjoint representation, for which we must have $(T_j)_k^l = -if_{jk}^l$. Convince yourself that the $T_j$ do indeed satisfy the correct algebra by computing $[T_j, T_k]$.

**Algebra as vector space.** The generators $T_j$ span in a natural way the vector space of the algebra. Define a basis $\{T_j\} = e_j$ associated with the generators, such that $e_j$ is the $j$-th basis vector of a standard basis. Thus, $e_j$ is the column vector, whose components are given by $(e_j)_k = \delta_{jk}$. Derive the action of the algebra on itself as vector space by computing $T_j|T_k\rangle = T_j \cdot e_k$. Express the result again in the basis $\{T_j\}$, i.e. compute the coefficients $a^j$ in $T_j|T_k\rangle = a^j |T_i\rangle$. Compare your result with the following definition of the action of the algebra on itself as vector space:

$$T_j|T_k\rangle = [T_j, T_k].$$

**Adjoint representation of the group.** With the adjoint representation of the algebra $\mathfrak{g}$ on itself as vector space at hand, it seems natural to also introduce a representation of the group $G$ on the vector space of its algebra. To do so, we first introduce the operation of conjugation, $\Psi_g(h) = ghg^{-1}$ for $g, h \in G$. Obviously, $\Psi_g$ is an automorphism of the group $G$ for each $g \in G$. Choose for $h$ an element close to the identity, i.e. $h = 1 + \varepsilon X$. Then you can easily read off, how the derivative of any $\Psi_g$ looks like at the identity, i.e. at the point $h = 1$. This derivative at the identity defines the adjoint representation of the group $G$ on its algebra $\mathfrak{g}$. The derivative of $\Psi_g$ is usually denoted $\text{Ad}_g$ and is an automorphism of the algebra $\mathfrak{g}$ for each $g \in G$.

**Our example again.** With the notation from the last exercise sheet we have that $g = \exp(-i \frac{\vec{n}}{2} \cdot \vec{\sigma})$ is a generic element of the group $\mathfrak{SU}(2)$. Compute $\text{Ad}_g(\sigma_j)$.

**From the group to the algebra.** It is now possible to go from the adjoint representation of the group to the adjoint representation of the algebra. All you have to do is to take the derivative of $\text{Ad}_g$ at the point $g = 1$. This defines the operation $\text{ad}_X$, where the generator $X$ is precisely the one, which yields the group element $g$ in the from $g(\lambda) = \exp(\lambda X)$. Remark: Obviously, this is not unique. This fixes $X$ only up to a multiplicative constant. Show that with $\text{Ad}_g(Y) = gYg^{-1}$ and $g = 1 + \varepsilon X$, $g^{-1} = 1 - \varepsilon X$, we have $\text{ad}_X(Y) = [X, Y]$.

**Killing form.** In the lecture, we will introduce the Killing form $g_{ab} = \text{tr}(T_a T_b)$. Compute the Killing form for $\mathfrak{su}(2)$ with the generators computed above in the adjoint representation. Then, diagonalize $g_{ab}$, i.e. bring the Killing form into the form $g_{ab} = k_a \delta_{ab}$. Finally, compute now the form $g'_{ab} = \text{tr}(\sigma_a \sigma_b)$ associated to the fundamental representation of the algebra $\mathfrak{su}(2)$. Compare your result with what you obtained for the Killing form $g_{ab}$. Diagonalize $g'_{ab}$ as well.