EXPONENTIALS OF GENERATORS

In the lecture, we use the physics notation in which the generators are chosen to be Hermitean. Let X be such an generator. Then, $U_X(\lambda) = \exp(\mathrm{i}\lambda X)$ defines a so called one-parameter sub-goup of the Lie group. In the lecture, we motivated that each group element $g \in G$, which lies in the connection component of the one, can be written in such a way as the exponential of a generator. In the following exercises, we once more assume, taht we have choses a representation ρ for both, the group G, as well as the algebra $\mathfrak{g}$. The dimension of the representation is d. Then, we have that $\rho(G) \subset \mathrm{GL}_d(\mathbb{R})$ and $\mathrm{d}\rho(\mathfrak{g}) \subset \mathrm{End}(\mathbb{R}^d)$. This assumption is useful because then $u^a X_a \in \mathrm{d}\rho(\mathfrak{g})$ is a matrix. Hence, the exponential

$$\exp(\mathrm{i}u^a X_a) = \sum_{n=0}^{\infty} \frac{(\mathrm{i}u^a X_a)^n}{n!}$$

is defined. We now wish to compute a few useful identities, in particular for derivations of exponentials.

Preparation. We wish to compute what $\frac{\partial}{\partial u^b}\exp(\mathrm{i}u^a X_a)$ yields. First, consider the family $U(\lambda) = \exp(\mathrm{i}\lambda u^a X_a)$. Show, that

$$\frac{\partial}{\partial\lambda}U(\lambda) = \mathrm{i}(u^b X_b)U(\lambda) = \mathrm{i}U(\lambda)(u^b X_b)\,.$$

Generators again. Show now by Taylor expansion, that

$$\left.\frac{\partial}{\partial u^b}\exp(\mathrm{i}u^a X_a)\right|_{u=0} = \mathrm{i}X_b\,.$$

Getting serious. Compute with the Leibniz rule, and by keeping in mind that the commutator $[u^a X_a, X_b] \neq 0$ in general, the Taylor expansion of $\frac{\partial}{\partial u^b}\exp(\mathrm{i}u^a X_a)$.

Auxiliary calculation. Show the identity

$$\int_0^1 \mathrm{d}\lambda\, \lambda^k (1-\lambda)^m = \frac{k!m!}{(k+m+1)!}\,.$$

Hint: The easiest way is by induction over m, starting with $m = 0$.

Getting very serious. Compute now the expression

$$\int_0^1 \mathrm{d}\lambda\, \exp(\mathrm{i}\lambda u^a X_a)(\mathrm{i}X_b)\exp(\mathrm{i}(1-\lambda)u^c X_c)\,.$$

To do so, expand the exponentials in Taylor series and make use if the result of the above auxiliary calculation.

Finally. Compare now the coefficients of the last computation with the ones from "getting serious", and prove in this way the beautiful identity

$$\frac{\partial}{\partial u^b}\exp(\mathrm{i}u^a X_a) = \int_0^1 \mathrm{d}\lambda\, \exp(\mathrm{i}\lambda u^a X_a)(\mathrm{i}X_b)\exp(\mathrm{i}(1-\lambda)u^c X_c)\,.$$

With the notations from above, this can be written in a shorter way: $\frac{\partial}{\partial u^b}U(1) = \int_0^1 \mathrm{d}\lambda\, U(\lambda)\mathrm{i}X_b U(1-\lambda)$

On the adjoint representation. Assume that $[A, B] = B$. Show that then

$$\exp(\mathrm{i}\alpha A)B\exp(-\mathrm{i}\alpha A) = \exp(\mathrm{i}\alpha)B\,.$$

The rationale behind this formula is the following: We have $\mathrm{ad}(A)(B) = B$. Realize that this means nothing else than that B is an eigen vector of $\mathrm{ad}(A)$ with eigen value 1. From this follows without long calculation that $\mathrm{Ad}(\exp(\mathrm{i}\alpha A))(B) = \exp(\mathrm{i}\alpha A)B\exp(-\mathrm{i}\alpha A)$ has also B as eigen vector, with eigen value $\exp(\mathrm{i}\alpha)$. If you like, you may ponder on the identity

$$\mathrm{Ad}(\mathrm{Exp}(X)) = \exp(\mathrm{ad}(X))\,.$$

Mathematically, the exponential function Exp, which yields the connection between Lie algebra and Lie group, is defined precisely as that functional for which one has the above relation, for which therefore the diagram

$$\begin{array}{ccccc} & & \mathrm{ad} & & \\ & X \in \mathfrak{g} & \longrightarrow & \mathrm{ad}(X) & \\ \mathrm{Exp} & \downarrow & & \downarrow & \exp \\ & g \in G & \longrightarrow & \mathrm{Ad}(g) & \\ & & \mathrm{Ad} & & \end{array}$$

commutes. Note that Exp, with the help of the adjoint representation, is defined on the right hand side by the ordinary exponential function, understood as its Taylor series, acting on matrices.