ROOTS

The weights of the adjoint representation of a Lie algebra $\mathfrak{g}$ are called roots. One can reconstruct the entire Lie algebra out of them.

Multiples. In the lecture, it was stated that the only multiples of a root α, which also are roots, are α and $-\alpha$. Prove this.

Further properties. Roots (as also weights) are foremost the multiple eigen values of simultaneous eigen vectors of the Cartan sub-algebra. Just as well, we can consider them as linear functionals, via $\mathrm{ad}(H)(X) = \alpha(H)\,X$, when X lies in the eigen space $\mathfrak{g}_\alpha$ to α. If α and β are roots, then we can form the quantity $\eta_{\beta\alpha} = \beta(H_\alpha)$. This quantity can be expressed in terms of the Killing form, $\eta_{\beta\alpha} = 2g(\beta,\alpha)/g(\alpha,\alpha)$. Note that $\eta_{\beta\alpha}$ is not symmetric! To begin with, discuss why the Killing form $g(X,Y) \equiv \mathrm{tr}(\mathrm{ad}(X),\mathrm{ad}(Y))$ is obviously also defined for roots. For α,β roots, we have $\eta_{\beta\alpha} \in \mathbb{Z}$. (It might be helpful to recapitulate from the lecture, why this is so.)

Angles. Contemplate how you can express the cosine of the angle between two roots in terms of the Killing form. Use this in order to express $\eta_{\beta\alpha}$ only in terms of the norms $g(X,X)$ and this angle. What do you therefore find for $\eta_{\beta\alpha}\eta_{\alpha\beta}$? This must be an integer, as well! Which angles between two roots are hence the only possible ones? Write down a table with all possibilities for the cosine of the angle between two roots α,β, the angle itself, as well as the values of $\eta_{\alpha\beta}$ and $\eta_{\beta\alpha}$. Since there is an asymmetry, assume for your convenience $g(\beta,\beta) \geq g(\alpha,\alpha)$.

Long and short roots. Disucss, which ratios of lengths of two roots are possible according to your above results.

Examples. Construct all root systems of dimensions one and two. It is helpful to take the following property to heart: A root system must be symmetric with respect to reflections in the planes $\alpha^\perp$ for all roots α.

WEIGHTS

The weights do characterize an irrep V uniquely. We wish to discuss a few basic properties of weights as well. We denote with $W(V)$ the set of all weights of V.

Representation from the weights. Ponder that obviously $V = \bigoplus_{\alpha\in W(V)} V_\alpha$ with V_α the weight spaces, i.e. the eigen spaces to the weights α, such that for all $v \in V_\alpha$ and for all H from the Cartan sub-algebra $H(v) = \alpha(H)\,v$.

Action of the roots. Let $\mathfrak{g}_\beta$ be the root space to the root β. This space operates on the weight space in an obvious manner, $\mathfrak{g}_\beta : V_\alpha \to V_{\alpha+\beta}$. Argue that this implies for all $\alpha,\alpha' \in W(V)$, that $\alpha - \alpha'$ must be an integer linear combination of roots, i.e. that it must be an element of the root lattice.