Four-Point Functions in LCFT

Surprises from SL(2,\mathbb{C}) covariance

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Motivation

- LCFT important for many applications such as
  - abelian sandpiles,
  - percolation,
  - Haldane-Rezayi fractional quantum Hall state,
  - disorder etc.

- Presumably LCFT will play a role in string theory, e.g.
  - D-brane recoil,
  - world-sheet formulation on $AdS_3$,
  - or, more generally, when non-compact CFTs arise.

- Subtleties in non-compact CFTs, e.g. Liouville theory:
  - non-uniqueness of fusion matrices,
  - non-trivial factorization properties of correlators,
  - difficulties in definition of consistent OPEs,
  - additional constraints for unitarity and locality.

- These subtleties are typical for LCFT!
Correlation functions have to satisfy the global conformal Ward identities, i.e. for \( m = -1, 0, 1 \) we must have

\[
0 = L_m \langle \Psi_1(z_1) \cdots \Psi_n(z_n) \rangle \\
= \sum_{i=1}^{n} z_i^m \left[ z_i \partial_i + (m + 1)(h_i + \hat{\delta}_h_i) \right] \langle \Psi_1(z_1) \cdots \Psi_n(z_n) \rangle .
\]

In case of rank \( r > 1 \) Jordan cells of indecomposable representations with respect to \( \text{Vir} \), we have

\[
\hat{\delta}_{h_i} \Psi_{(h_j;k_j)} = \begin{cases} 
\delta_{i,j} \Psi_{(h_j;k_j-1)} & \text{if } 1 \leq k_j \leq r - 1 , \\
0 & \text{if } k_j = 0 .
\end{cases}
\]

Equivalently, \( L_0|h;k\rangle = h|h;k\rangle + (1 - \delta_{k,0})|h;k - 1\rangle .\)
Ward identities become *inhomogeneous* in LCFT. The inhomogeneities are given by correlation functions with total Jordan-level \( K = \sum_{i=1}^{n} k_i \) decreased by one,

\[
\langle \Psi_{(h_1;k_1)}(z_1) \cdots \Psi_{(h_n;k_n)}(z_n) \rangle \equiv \langle k_1 k_2 \cdots k_n \rangle ,
\]

\[
\frac{1}{(m + 1)L'_m} \langle k_1 k_2 \cdots k_n \rangle = - z_1^m \langle k_1 - 1, k_2 \cdots k_n \rangle
\]

\[
- z_2^m \langle k_1, k_2 - 1, k_3 \cdots k_n \rangle
\]

\[
- \cdots
\]

\[
- z_n^m \langle k_1 \cdots k_{n-1}, k_n - 1 \rangle .
\]

We obtain a hierarchical scheme of solutions, starting with correlators of total Jordan-level \( K = r - 1 \).
Generic form of 1-, 2- and 3-pt functions for fields forming Jordan cells, pre-logarithmic fields and fermionic fields in arbitrary rank $r$ LCFT known:

$$\langle \Psi_{(h;k)} \rangle = \delta_{h,0} \delta_{k,r-1} ,$$

$$\langle \Psi_{(h;k)}(z) \Psi_{(h';k')}(0) \rangle = \delta_{hh'} \sum_{j=r-1}^{k+k'} D_{(h;j)} \sum_{0 \leq i \leq k, 0 \leq i' \leq k'} \frac{(\partial h)^i}{i!} \frac{(\partial h')^{i'}}{i'!} z^{-h-h'} ,$$

$$\langle \Psi_{(h_1;k_1)}(z_1) \Psi_{(h_2;k_2)}(z_2) \Psi_{(h_3;k_3)}(z_3) \rangle = \sum_{j=r-1}^{k_1+k_2+k_3} C_{(h_1h_2h_3;j)}$$

$$\times \sum_{0 \leq i_1 \leq k_1, l=1,2,3} \frac{(\partial h_1)^{i_1}}{i_1!} \frac{(\partial h_2)^{i_2}}{i_2!} \frac{(\partial h_3)^{i_3}}{i_3!} \prod_{\sigma \in S_3, \sigma(1) < \sigma(2)} (z_{\sigma(1)}z_{\sigma(2)})^{h_{\sigma(3)} - h_{\sigma(1)} - h_{\sigma(2)}} .$$
Foundations: OPE

\[ \Psi_{(h_1;k_1)}(z_1) \Psi_{(h_2;k_2)}(z_2) = \sum_{(h;k)} \Psi_{(h;k)}(z_2) \lim_{z_1 \to z_2} \sum_{k'} \]

\[ \langle \Psi_{(h_1;k_1)}(z_1) \Psi_{(h_2;k_2)}(z_2) \Psi_{(h;k')} (z_3) \rangle \left( \langle \Psi_{(h';\cdot)}(z_2) \Psi_{(h';\cdot)}(z_3) \rangle^{-1} \right)_{k',k} . \]

• Crucial role of zero modes worked out: all known LCFTs have realizations which include fermionic fields.

• Maximal power of logs bounded by zero mode content:

\[ Z_*(\Psi_{(h;k)}) \leq Z_*(\Psi_{(h_1;k_1)}) + Z_*(\Psi_{(h_2;k_2)}) . \]

• Non-quasi-primary members of Jordan-cells: zero mode content yields BRST structure for correlators under action of Virasoro algebra.
To find a useful algorithm to fix the generic form of 4pt-functions, visualize a logarithmic field $\Psi_{(h;k)}$ by a vertex with $k$ outgoing lines.

Contractions of logarithmic fields give rise to logarithms in the correlators. The possible powers with which $\log(z_{ij})$ may occur, can be determined by graph combinatorics.
Terms of generic form of $n$-pt function given by sum over all admissible graphs subject to the rules:

△ Each $k_{\text{out}}$-vertex may receive $k_{\text{in}}' \leq (r - 1)$ lines.

△ Vertices with $k_{\text{out}} = 0$ (primary fields) do not receive any legs.

△ Vertex $i$ can receive legs from vertex $j$ only for $j \neq i$.

△ Precisely $r - 1$ lines in correlator remain open.

**Example:** 4pt function for $r = 2$ and all fields logarithmic yields, up to permutations, the graphs
Linking numbers $A_{ij}(g)$ of given graph $g$ yield *upper bounds* for power with which logarithms occur.

**Recursive procedure:** start with all ways $f_i$ to choose $r - 1$ free legs, find at each level $K'$ and for each configuration $f_i$ all graphs, which connect the remaining $K - K' - (r - 1)$ legs to vertices.

Write down corresponding monomial in $\log(z_{ij})$, multiplied with an as yet undetermined constant $C(g)$ for each graph $g$.

Determine some constants by imposing global conformal invariance.

Fix further constants by imposing admissible permutation symmetries.
Generic form of the LCFT 4pt functions $\langle k_1 k_2 k_3 k_4 \rangle \equiv \langle \Psi(h_1,k_1)(z_1) \ldots \Psi(h_4,k_4)(z_4) \rangle$ is

$$\langle k_1 k_2 k_3 k_4 \rangle = \prod_{i<j} (z_{ij})^{\mu_{ij}} \sum_{(k'_1,k'_2,k'_3,k'_4)} \sum_{g \in G_{K-K'}} C(g) \left( \prod_{i<j} \log^{A_{ij}(g)} (z_{ij}) \right) F_{k'_1 k'_2 k'_3 k'_4}(x) ,$$

where

- $G_{K-K'}$ is set of graphs for $(k_1 - k'_1, \ldots, k_4 - k'_4)$,
- $A_{ij}(g)$ is linking number of vertices $i, j$ of graph $g$,
- $x$ is the crossing ratio $x = \frac{z_{12} z_{34}}{z_{14} z_{23}}$,
- $\mu_{ij}$ is typically $\mu_{ij} = \frac{1}{3} (\sum_k h_k) - h_i - h_j$. 


4pt Functions: \( r = 2 \)

- The only direct dependence on the conformal weights is through the \( \mu_{ij} \). Put \( h_1 = \ldots = h_4 = 0 \) for simplicity.

- The generic form obeys some symmetry under permutations. Put \( \ell_{ij} \equiv \log(z_{ij}) \) and assume \( i < j \) throughout.

\[
\begin{align*}
\langle 1000 \rangle &= F_0, \\
\langle 1100 \rangle &= F_{1100} - 2\ell_{12}F_0, \\
\langle 1110 \rangle &= F_{1110} \\
&\quad + (\ell_{12} - \ell_{13} - \ell_{23})F_{1100} + (\ell_{13} - \ell_{12} - \ell_{23})F_{1010} \\
&\quad - (\ell_{23} - \ell_{12} - \ell_{13})F_{0110} \\
&\quad + (-\ell_{12}^2 - \ell_{13}^2 - \ell_{23}^2 + 2\ell_{12}\ell_{23} + 2\ell_{12}\ell_{13} + 2\ell_{23}\ell_{13})F_0 \\
&= F_{1110} + \mathcal{P}_{(123)} \left\{ (\ell_{12} - \ell_{23} - \ell_{13})F_{1100} \right\} \\
&\quad + \mathcal{P}_{(123)} \left\{ \ell_{12}(\ell_{12} - \ell_{23} - \ell_{13})F_0 \right\}.
\end{align*}
\]
Symmetry under permutations allows to write formulæ in more compact form.

The permutation operators $P$ run over all inequivalent permutations such that $i < j$ in all the $z_{ij}$ and $\ell_{ij}$ involved.

In the last example, $P_{(123)} = (123) + (231) + (312)$ subject to the above rule.

The ordering rule for $\ell_{ij}$ may be neglected, since in the full correlators, combined out of holomorphic and anti-holomorphic part in a single-valued way, only $\log |z_{ij}|^2$ will appear.
Interestingly, there remain *free constants*, when all fields are logarithmic!

\[
\langle 1111 \rangle = F_{1111} + \mathcal{P}_{(1234)} \left\{ \left[ (-\ell_{12} - \ell_{34} + \ell_{23} + \ell_{14}) C_1 + (\ell_{13} + \ell_{24} - \ell_{12} - \ell_{34}) C_2 \\
- \ell_{14} + \ell_{34} - \ell_{13} \right] F_{0111} \right\}
+ \mathcal{P}_{(12)(34)} \left\{ \left[ (\ell_{13}^2 + \ell_{24}^2 - \ell_{14}^2 - \ell_{23}^2 + 2(-\ell_{34}\ell_{24} - \ell_{12}\ell_{24} + \ell_{34}\ell_{14} + \ell_{13}\ell_{24} \\
- \ell_{13}\ell_{34} + \ell_{23}\ell_{34} + \ell_{12}\ell_{23} - \ell_{12}\ell_{13} - \ell_{23}\ell_{14} + \ell_{12}\ell_{14}) C_3 \\
+ (-(\ell_{23} + \ell_{14})^2 + \ell_{23}\ell_{34} + \ell_{12}\ell_{14} - \ell_{13}\ell_{34} + \ell_{34}\ell_{14} + \ell_{13}\ell_{14} \\
- \ell_{34}\ell_{24} - \ell_{12}\ell_{13} - \ell_{12}\ell_{24} + \ell_{23}\ell_{24} + \ell_{23}\ell_{13} + \ell_{12}\ell_{23} + \ell_{24}\ell_{14}) C_4 \\
- \ell_{34}^2 - \ell_{23}^2 - \ell_{14}^2 + 2\ell_{23}\ell_{34} + 2\ell_{34}\ell_{14} - 2\ell_{12}\ell_{34} - \ell_{23}\ell_{14} + \ell_{23}\ell_{24} \\
- \ell_{12}\ell_{13} + \ell_{12}\ell_{14} + \ell_{12}\ell_{23} - \ell_{12}\ell_{24} + \ell_{13}\ell_{14} + \ell_{13}\ell_{24} \right] F_{1100} \right\}
+ \left[ 2(\ell_{12}\ell_{24}\ell_{14} - \ell_{23}\ell_{13}\ell_{14} + \ell_{23}\ell_{34}\ell_{24} - \ell_{24}\ell_{13}\ell_{34} - \ell_{23}\ell_{34}\ell_{14} \\
- \ell_{12}\ell_{23}\ell_{34} - \ell_{12}\ell_{34}\ell_{24} - \ell_{23}\ell_{13}\ell_{24} + \ell_{12}\ell_{23}\ell_{13} + \ell_{13}\ell_{34}\ell_{14} \\
- \ell_{13}\ell_{14}\ell_{24} - \ell_{23}\ell_{24}\ell_{14} - \ell_{12}\ell_{13}\ell_{24} - \ell_{12}\ell_{23}\ell_{14} - \ell_{12}\ell_{13}\ell_{34} \\
- \ell_{12}\ell_{34}\ell_{14} \right]
+ 2(\ell_{13}\ell_{24} + \ell_{12}\ell_{34} + \ell_{14}\ell_{23} + \ell_{23}\ell_{14} + \ell_{34}\ell_{12} + \ell_{24}\ell_{13}) \right] F_0
\]
Next trivial case: Jordan cells of rank $r = 3$. Each Jordan level $0 \leq k_i \leq 2$.

Graph combinatorics gets more involved.

\[
\begin{align*}
\langle 2000 \rangle &= \langle 1100 \rangle = F_0 , \\
\langle 2100 \rangle &= F_{2100} - 2\ell_{12}F_0 , \\
\langle 1110 \rangle &= F_{1110} - (\ell_{12} + \ell_{23} + \ell_{13})F_0 \\
&= F_{1110} - \mathcal{P}_{(123)} \{\ell_{12}F_0\} ,
\end{align*}
\]
Again, if all \( k_i > 0 \), free constants remain:

\[
\langle 1111 \rangle = F_{1111} + P_{(1234)} \left\{ \left[ (\ell_{13} - \ell_{12} + \ell_{24} - \ell_{34})C_1 \right. \\
+ (\ell_{23} + \ell_{14} - \ell_{34} - \ell_{12})C_2 - \ell_{14} + \ell_{24} - \ell_{12} \right] F_0_{1111} \right\} + \\
\left[ (\ell^2_{12} + \ell^2_{24} + \ell^2_{34} + \ell^2_{13} + 2(\ell_{12}\ell_{13} + \ell_{13}\ell_{24} - \ell_{13}\ell_{34} \\
- \ell_{34}\ell_{24} - \ell_{12}\ell_{24} + \ell_{12}\ell_{34}))C_3 \right. \\
+ \left. (-2\ell_{13}\ell_{14} + \ell^2_{24} + 2\ell_{23}\ell_{14} - 2\ell_{23}\ell_{24} + \ell^2_{23} + 2\ell_{13}\ell_{24} \\
+ \ell^2_{13} - 2\ell_{23}\ell_{13} + \ell^2_{14} - 2\ell_{24}\ell_{14})C_4 \right. \\
+ \left. \left[ (\ell_{24} + \ell_{13})^2 \\
+ \ell_{12}\ell_{14} - \ell_{23}\ell_{24} - \ell_{12}\ell_{24} - \ell_{24}\ell_{14} - \ell_{23}\ell_{13} + \ell_{34}\ell_{14} \\
- \ell_{13}\ell_{34} - \ell_{13}\ell_{14} + \ell_{23}\ell_{34} - \ell_{34}\ell_{24} + \ell_{12}\ell_{23} - \ell_{12}\ell_{13} \right] C_5 \\
+ 2(\ell_{13}\ell_{24} + \ell_{23}\ell_{14} + \ell_{12}\ell_{34}) \right] F_0.
\]
Further examples . . . need bigger transparencies ; -(  

Problem: Computational complexity grows heavily with rank $r$ and total Jordan level $K$. Already the generic solution for $r = 2$ and $K_{\text{max}} = 4(r - 1) = 4$ needs a computer program.  

Solution: MAPLE package, written by Marco Krohn, almost finished. Need to make implementation of algorithm more efficient. So far, $K > 2(r - 1)$ for $r > 3$ still too complex.  

Permutation symmetry for the highest degree polynomial in the $\ell_{ij}$, appearing in front of $F_0(x)$, is not obvious and difficult to find.
Questions:

△ Need to understand origin of additional free constants.

△ Include explicit crossing symmetry. Should decrease number of different functions $F_{k_1',k_2',k_3',k_4'}(x)$, in particular for cases where several conformal weights are equal, $h_i = h_j$.

△ Need to generalize to $c = 0$ LCFTs important for percolation and for disorder. Problem: the naive vacuum representation is trivial.

△ Adapt algorithm to include pre-logarithmic fields: Skip the rule that primary vertices do not receive legs.
We found a method to fix the generic form of 4-pt and $n$-pt functions in arbitrary rank LCFT. Already the form of 4-pt functions, as determined by global conformal invariance, is much more complicated than in the ordinary case. There seems to exist additional degrees of freedom not present in ordinary CFT. We showed a few examples of non-trivial solutions. Already the solution for $r = 2$ and $K = 4$, is new and generalizes known expressions.