Logarithmic Conformal Field Theory

in a (Nut)Shell

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Identifies mathematical structures which describe reality.

- **Newton**: Everything is *matter* $\rightarrow$ Analysis.
  He assumed even light consists of particles.

- **Einstein**: Everything is *energy* $\rightarrow$ Geometry.
  We all know the famous $E = mc^2$.

- **Heisenberg**: Everything is *symmetry* $\rightarrow$ Algebra.
  Conservation laws, Noether theorem, selection rules, gauge groups, . . .

Symmetries govern many aspects of modern theoretical physics.

- Natural question: What possible symmetries are there?
- More fundamental questions: What does it *mean* that Nature can be described by mathematical structures? Why is Nature so “symmetric”?
Experience shows that the laws of Nature are fixed by symmetries to a sometimes miraculous extent.

- Look at the spectrum of atoms in crystals. The discrete finite group of rigid symmetries of the crystal predicts which degeneracies are lifted.

- The known fundamental forces (except gravity) are described by gauge field theories. Quantum numbers appear as weights of representations of the gauge groups, which are all Lie groups such as $U(1)$, $SU(2)$ and $SU(3)$.

- Extended or composite objects possess even larger symmetries such as infinite-dimensional Lie algebras.

One symmetry is particularly interesting: symmetry under scaling (think of dimensional analysis!).
Suppose, a theory is invariant under **local** scale transformations: \( g^{\mu \nu}(x) \mapsto \tilde{g}^{\mu \nu}(\tilde{x}) = \lambda(x)g^{\mu \nu}(x) \). Such maps locally conserve angles. That’s why they are called **conformal**.

There is something very special about conformal maps in **two** dimensions . . .
In two dimensions, we can work on the complex plane \( \mathbb{C} \):
\[ z = x + iy, \bar{z} = x - iy. \]
Any \textit{holomorphic} map \( z \mapsto z' = f(z) \),
\( \partial f(z) = 0 \), is conformal. Thus, two-dimensional conformally
invariant theories have an infinite number of symmetries.

△ My personal interest is in \textit{quantum field theories}.

\[
f(z) = z + \varepsilon(z) = z + \sum_{n \in \mathbb{Z}} \varepsilon_n z^n,
\]
\[
[L_\varepsilon, \Phi(z)] = \Phi(z + \varepsilon(z)) \quad \Rightarrow
\]
\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}.
\]

This is the \textbf{Virasoro algebra} of the generators of local
conformal transformations. It is an example of an infinite-
dimensional Lie algebra.
To classify conformally invariant theories means to study the representation theory of this algebra.

△ Universality classes of two-dimensional statistical systems at their points of criticality are classified by the value of the central extension $c$.

△ Critical exponents are given in terms of the scaling dimensions of primary fields, i.e. highest weights of irreps:

$$
\Phi_h(f(z)) = \left( \frac{\partial f(z)}{\partial z} \right)^{-h} \Phi_h(z), \quad \lim_{z \to 0} \Phi_h(z)|0\rangle \equiv |h, c\rangle.
$$

In particular $\Phi_h(\lambda z) = \lambda^{-h} \Phi_h(z)$.

△ The two-dim. Ising model possesses three basic observables, the identity $\Phi_0 = \mathbb{I}$, the energy operator $\Phi_{1/2} = \epsilon$, and the spin field or order parameter $\Phi_{1/16} = \sigma$ or $\mu$, respectively.
The simplest conformally invariant statistical field theories are classified very similar to spin in quantum mechanics: Put $L_z = L_0$, $L_\pm = L_{\pm1}$ and the $su(2)$ algebra takes the form $[L_n, L_m] = (n - m)L_{n+m}$. The role of the Casimir $C = \vec{L}^2$ is roughly given by $c$.

\begin{align*}
c_{p,q} &= 1 - 6 \frac{(p-q)^2}{pq}, \quad p, q \geq 1 \text{ coprime} \\
L_0|h, c\rangle &= h|h, c\rangle \\
L_n|h, c\rangle &= 0 \quad \text{for } n > 0 \\
h_{r,s}(c) &= \frac{(pr-qs)^2-(p-q)^2}{4pq}, \quad 1 \leq r < q, 1 \leq s < p \\
|h; \{n\}, c\rangle &= L_{-n_k} \ldots L_{-n_1}|h, c\rangle \\
0 &= \sum|\{n\}|=N \beta\{n\} L_{-\{n\}}|h, c\rangle
\end{align*}

\begin{align*}
\vec{L}^2|\ell, m\rangle &= \ell(\ell + 1)|\ell, m\rangle, \quad \ell \in \mathbb{Z}_+ \\
L_z|\ell, m\rangle &= m|\ell, m\rangle \\
L_+|\ell, \ell\rangle &= 0 \\
h(\ell) &= \ell \\
|\ell, \ell - m\rangle &= (L_-)^m|\ell, \ell\rangle \\
0 &= (L_-)^{2\ell+1}|\ell, \ell\rangle
\end{align*}
Ultimately, one wants to compute expectation values of observables, \( \langle 0 | \Phi_{h_1}(z_1) \Phi_{h_2}(z_2) \ldots \Phi_{h_n}(z_n) | 0 \rangle \).

\[
\langle \Phi_{h_1}(z_1) \Phi_{h_2}(z_2) \rangle = D_{h_1}(z_1 - z_2)^{-h_1-h_2} \delta_{h_1,h_2}, \\
\langle \Phi_{h_1}(z_1) \Phi_{h_2}(z_2) \Phi_{h_3}(z_3) \rangle = C_{h_1 h_2 h_3} (z_1 - z_2)^{h_3-h_1-h_2} \\
\times (z_1 - z_3)^{h_2-h_1-h_3} (z_2 - z_3)^{h_1-h_2-h_3}, \\
\langle \Phi_{h_1}(z_1) \Phi_{h_2}(z_2) \Phi_{h_3}(z_3) \Phi_{h_4}(z_4) \rangle = \prod_{i<j} (z_i - z_j)^{\mu_{ij}} F_{h_1 h_2 h_3 h_4}^{(p)}(x),
\]

where \( \sum_{j \neq i} \mu_{ij} = -2h_i \), \( x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \) is the crossing ratio, and \( p \) labels the conformal blocks.

During the last 20 years, a lot of technology has been developed to efficiently and exactly compute the \( F^{(p)}(x) \) and higher \( n \)-point functions.
We are used to expect that (tensor) representations can be completely reduced into irreps. Coupling angular momentum in quantum mechanics amounts to

\[ [\ell_1] \otimes [\ell_2] = \sum_{\ell = |\ell_1 - \ell_2|}^{\ell_1 + \ell_2} [\ell]. \]

In conformal field theory, coupling fields works much the same,

\[ [h_1, c] \ast [h_2, c] = \sum_h N_{h_1 h_2}^h [h, c], \quad N_{h_1 h_2}^h \in \mathbb{Z}_+. \]

Classifying these so-called fusion algebras is a very important problem in conformal field theory, but . . .
... but it may happen, that the fusion product of two irreps cannot again be decomposed into irreps!

△ There exists a conformal field theory with $c = c_{2,1} = -2$. It contains an innocent and admissible irrep corresponding to a primary field $\mu$ with $h = h_{1,2} = -1/8$. However,

$$\langle \mu(\infty)\mu(1)\mu(x)\mu(0) \rangle =$$

$$[x(1 - x)]^{1/4} \begin{cases} F^{(1)} = & 2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; x\right) \\ F^{(2)} = & \log(x) \ 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \\ & + \partial_\epsilon \ 3F_2\left(\frac{1}{2} + \epsilon, \frac{1}{2} + \epsilon, 1; 1 + \epsilon, 1 + \epsilon; x\right) \bigg|_{\epsilon=0} \end{cases}$$

One can show that this implies $[-\frac{1}{8}, -2] * [-\frac{1}{8}, -2] = [\tilde{0}, -2]$, where $L_0|\tilde{h}, c\rangle = h|\tilde{h}, c\rangle + |h, c\rangle$ spans a Jordan cell. Thus, the representation on the rhs is indecomposable.

$$\tilde{\Phi}_{\tilde{h}}(\lambda z) = \lambda^{-h} \left(\tilde{\Phi}_{\tilde{h}}(z) - \log(\lambda) \Phi_{\tilde{h}}(z)\right).$$
Indecomposable representations are at the heart of logarithmic conformal field theory.

Correlation functions have to satisfy the global conformal Ward identities, i.e. for \( m = -1, 0, 1 \) we must have

\[
0 = L_m \langle \Psi_1(z_1) \ldots \Psi_n(z_n) \rangle \\
= \sum_{i=1}^{n} z_i^m \left[ z_i \partial_i + (m + 1)(h_i + \hat{\delta}_h_i) \right] \langle \Psi_1(z_1) \ldots \Psi_n(z_n) \rangle.
\]

In case of rank \( r > 1 \) Jordan cells of indecomposable representations with respect to \( Vir \), we have

\[
\hat{\delta}_{h_i} \Psi_{(h_j;k_j)} = \begin{cases} 
\delta_{i,j} \Psi_{(h_j;k_j-1)} & \text{if } 1 \leq k_j \leq r - 1, \\
0 & \text{if } k_j = 0.
\end{cases}
\]
Although logarithms break scale invariance, correlators can still be invariant under global conformal maps.

Generic form of 1-, 2- and 3-pt functions for fields forming Jordan cells in arbitrary rank $r$ LCFT is known:

$$\langle \Psi(h;k) \rangle = \delta_{h,0} \delta_{k,r-1},$$

$$\langle \Psi(h;k)(z)\Psi(h';k')(0) \rangle = \delta_{h,h'} \sum_{j=r-1}^{k+k'} D(h;j) \sum_{0 \leq i \leq k, 0 \leq i' \leq k'} \frac{(\partial h)^i}{i!} \frac{(\partial h')^{i'}}{i'!} z^{-h-h'},$$

$$\langle \Psi(h_1;k_1)(z_1)\Psi(h_2;k_2)(z_2)\Psi(h_3;k_3)(z_3) \rangle = \sum_{j=r-1}^{k_1+k_2+k_3} C(h_1h_2h_3;j)$$

$$\times \sum_{0 \leq i_l \leq k_l, l=1,2,3} \frac{(\partial h_1)^i_1}{i_1!} \frac{(\partial h_2)^i_2}{i_2!} \frac{(\partial h_3)^i_3}{i_3!} \prod_{\sigma \in S_3, \sigma(1) < \sigma(2)} (z_{\sigma(1)}(1) \sigma(2)) h_{\sigma(3)} - h_{\sigma(1)} - h_{\sigma(2)}.$$
Some Achievements

- LCFT on a torus and other non-trivial Riemann surfaces imply modular invariants and characters of indecomposable representations $N_{ij}^k$.
- Null vectors in indecomposable representations imply exploiting local conformal symmetry to exactly compute correlators in LCFT $C_{ijk}$ and $F^{(p)}(x)$.
- Classification of LCFTs similar to the minimal models imply identifying theories of potential interest in physics implying LCFTs as limits of sequences of ordinary CFTs.
- LCFT on surfaces with boundaries, LCFT wrt extended chiral algebras, LCFT and vertex operator algebras, LCFT and modular differential eqn, ...
LCFT important for many applications such as
- abelian sandpiles,
- percolation and disorder,
- Haldane-Rezayi fractional quantum Hall state,
- mathematics (e.g. alternating sign matrices).

Presumably LCFT will play a role in string theory, e.g.
- \( D \)-brane recoil,
- world-sheet formulation on \( AdS_3 \),
- or, more generally, when non-compact CFTs arise.

Subtleties in non-compact CFTs, e.g. Liouville theory:
- non-uniqueness of fusion matrices \( N_{ij}^k \),
- non-trivial factorisation properties of correlators into \( F^{(p)} \),
- difficulties in definition of consistent OPEs via \( C_{ijk} \),
- additional constraints for unitarity and locality: \( h, c \leq 0 \).

These subtleties are typical for LCFT!