Fermionic Expressions for the Characters of $c_{p,1}$
Logarithmic Conformal Field Theories
Talk given at XIX Workshop Beyond the Standard Model, Bad Honnef

Michael Flohr  Carsten Grabow  Michael Koehn

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Motivation

Bosonic-Fermionic $q$-Series Identities

$c_{p,1}$ Logarithmic Conformal Field Theories

Classification of rational conformal field theories

RCFT has finitely many representations

All representations are irreducible

LCFTs as a limiting case of the classification

Recently: LCFT in the context of instantons

\[ \text{Frenkel, Losev, Nekrasov: hep-th/0610149 (2006)} \]

Logarithmic conformal field theories

have finitely many representations

reducible, but indecomposable representations are involved

are rational in a generalized sense

\[ \text{N. Carqueville, MF: math-ph/0508015 (2006)} \]

We show:

$c_{p,1}$ LCFTs behave just like RCFTs in one more important aspect!
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Bosonic-Fermionic $q$-Series Identities

Motivation

Bosonic and Fermionic Characters

Nahm's conjecture

$c_{p,1}$ Logarithmic Conformal Field Theories

Bosonic-Fermionic $q$-Series Identities
Rogers-Ramanujan identities \((a \in \{0, 1\})\):

\[
\sum_{n=0}^{\infty} \frac{q^n(n+a)}{(q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{5n-1-a})(1 - q^{5n-4+a})}
\]

\[
= \frac{1}{(q)_{\infty}} \sum_{n=-\infty}^{\infty} (q^{n(10n+1+2a)} - q^{(5n+2-a)(2n+1)})
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Rogers-Ramanujan identities \((a \in \{0, 1\})\):

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(q)_n = \prod_{i=1}^{n} (1 - q^n)
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\[
(q)_n = \prod_{i=1}^{n} (1 - q^n) \quad (q)_0 = 1
\]
Bosonic and Fermionic Character Expressions

Rogers-Ramanujan identities \((a \in \{0, 1\})\):

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(1)

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(2)

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\]

\[
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\]

bosonic-fermionic $q$-series identities

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- **bosonic-fermionic** \( q \)-series identities

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bosonic-fermionic \(q\)-series identities

\(\mathcal{M}(5, 2)\) minimal model

\[
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Generalized fermionic expression:
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\[ f_{A,\vec{b},c}(\tau) = \sum_{\vec{m} \in (\mathbb{Z}_{\geq 0})^r \text{ restrictions}} q^{\frac{1}{2} \vec{m}^t A \vec{m} + \vec{b}^t \vec{m} + c} \]  

(3)
Generalized fermionic expression:

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with \( A \in M_r(\mathbb{Q}) \)
Generalized fermionic expression:

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\]

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**Nahm’s Conjecture:**

- Given \( A \) obeying certain conditions, there exist \( \vec{b} \) and \( c \) such that \( f_{A, \vec{b}, c} \) is a modular function.
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Generalized fermionic expression:

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Nahm’s Conjecture:

- Given \( A \) obeying certain conditions, there exist \( \vec{b} \) and \( c \) such that \( f_{A, \vec{b}, c} \) is a modular function.
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- ADET classification: \( A = C_{X_r} \otimes C_{Y_s}^{-1} \) with \( X_r, Y_s \in \{ A_i, D_j, E_k, T_i \mid i \geq 1, j \geq 4, 6 \leq k \leq 8 \} \) and \( C_{X_r} \) its Cartan matrix.
Motivation

Bosonic-Fermionic \(q\)-Series Identities

Logarithmic Conformal Field Theories

Central Charges and Conformal Dimensions

Bosonic Character Expressions for \(c_{p,1}\)

Fermionic Character Expressions for \(c_{2,1} = -2\)

Fermionic Character Expressions for \(c = -2\)

Relation to Simple Lie Algebras

Ansatz for Generalization

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Fermionic Expressions for \(c_{p,1}\) Characters (\(p \geq 2\))

Summary and Outlook

\(c_{p,1}\) Logarithmic Conformal Field Theories
The $\mathcal{W}(2, 2p - 1, 2p - 1, 2p - 1)$ triplet algebra models have the central charges and conformal dimensions

$$c_{p,1} = 1 - 6\frac{(p - 1)^2}{p}$$

with $p \geq 2$
The $\mathcal{W}(2, 2p - 1, 2p - 1, 2p - 1)$ triplet algebra models have the central charges and conformal dimensions

$$c_{p,1} = 1 - 6\frac{(p - 1)^2}{p}$$

with $p \geq 2$ \hspace{1cm} (4)

and

$$h_{r,s}^{p,1} = \frac{(pr - s)^2 - (p - 1)^2}{4p}$$

with $1 \leq r \leq 2$ and $1 \leq s \leq 3p - 1$ \hspace{1cm} (5)
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- The $c_{p,1}$ are rational LCFTs in a generalized sense

Motivation

Bosonic-Fermionic q-Series Identities

c_{p,1} Logarithmic Conformal Field Theories

Central Charges and Conformal Dimensions

Bosonic Character Expressions for c_{p,1}

Fermionic Character Expressions for c_{2,1} = -2

Fermionic Character Expressions for c = -2

Relation to Simple Lie Algebras

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Example

Fermionic Expressions for c_{p,1} Characters (p \geq 2)

Summary and Outlook

The \mathcal{W}(2, 2p - 1, 2p - 1, 2p - 1) triplet algebra models have the central charges and conformal dimensions

\begin{align*}
c_{p,1} &= 1 - 6 \frac{(p - 1)^2}{p} \\
\text{with } p \geq 2
\end{align*}

and

\begin{align*}
h_{r,s}^{p,1} &= \frac{(pr - s)^2 - (p - 1)^2}{4p} \\
\text{with } 1 \leq r \leq 2 \text{ and } 1 \leq s \leq 3p - 1
\end{align*}

i.e. the operator content is that of the 'augmented minimal model' to c_{3p,3}.

The c_{p,1} are rational LCFTs in a generalized sense


(reducible, but indecomposable representations are involved)
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Fermionic Expressions for $c_{p,1}$ Characters ($p \geq 2$)

Summary and Outlook

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\[ \chi_{\lambda,p} = \frac{\Theta_{\lambda,p}}{\eta} \quad 0 \leq \lambda \leq p, \; \lambda = pr - s \]
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Bosonic Character Expressions for $c_{p,1}$

$$\chi_{\lambda,p} = \frac{\Theta_{\lambda,p}}{\eta}$$

$$0 \leq \lambda \leq p, \lambda = pr - s$$

$$\Theta_{\lambda,k}(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{(2kn+\lambda)^2}{4k}}$$

$$(q)_{\infty} = q^{-\frac{1}{24} \eta(\tau)}, \; q = e^{2\pi i \tau}$$
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Bosonic Character Expressions for $c_{p,1}$

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(q)_{\infty} = q^{-\frac{1}{24}} \eta(\tau), \quad q = e^{2\pi i \tau}
\]

\[
\chi^{+}_{\lambda',p} = \frac{(p - \lambda')\Theta_{\lambda',p} + (\partial \Theta)_{\lambda',p}}{p \eta} \quad 0 < \lambda' < p
\]
Bosonic Character Expressions for $c_{p,1}$

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Fermionic Expressions for $c_{p,1}$ Characters ($p \geq 2$)

Summary and Outlook

$\chi_{\lambda,p} = \frac{\Theta_{\lambda,p}}{\eta}$ \hspace{1cm} $0 \leq \lambda \leq p$, $\lambda = pr - s$

$\Theta_{\lambda,k}(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{(2kn+\lambda)^2}{4k}}$

$(q)_\infty = q^{-\frac{1}{24}} \eta(\tau)$, $q = e^{2\pi i \tau}$

$\chi_{\lambda',p}^+ = \frac{(p - \lambda') \Theta_{\lambda',p} + (\partial \Theta)_{\lambda',p}}{p \eta}$

$0 < \lambda' < p$

$(\partial \Theta)_{\lambda,k}(\tau) = \sum_{n \in \mathbb{Z}} (2kn + \lambda) q^{\frac{(2kn+\lambda)^2}{4k}}$
Bosonic Character Expressions for $c_{p,1}$

\begin{align*}
\chi_{\lambda,p} &= \frac{\Theta_{\lambda,p}}{\eta} & 0 \leq \lambda \leq p, \quad \lambda = pr - s \\
\Theta_{\lambda,k}(\tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{(2kn+\lambda)^2}{4k}} \\
(q)_{\infty} &= q^{-\frac{1}{24}} \eta(\tau), \quad q = e^{2\pi i \tau} \\
\chi_{\lambda',p}^+ &= \frac{(p - \lambda')\Theta_{\lambda',p} + (\partial \Theta)_{\lambda',p}}{p\eta} & 0 < \lambda' < p \\
(\partial \Theta)_{\lambda,k}(\tau) &= \sum_{n \in \mathbb{Z}} (2kn + \lambda)q^{\frac{(2kn+\lambda)^2}{4k}} \\
\chi_{\lambda',p}^- &= \frac{\lambda'\Theta_{\lambda',p} - (\partial \Theta)_{\lambda',p}}{p\eta} 
\end{align*}
Fermionic Character Expressions for $c_{2,1} = -2$

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Fermionic Expressions for $c_{p,1}$ Characters ($p \geq 2$)

Summary and Outlook

\[ \chi_{\lambda,2} = \frac{\Theta_{\lambda,2}}{\eta} \]

\[ 0 \leq \lambda \leq 2, \quad \lambda = 2r - s \]

\[ \Theta_{\lambda,k}(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{(2kn+\lambda)^2}{4k}} \]
Fermionic Character Expressions for $c_{2,1} = -2$

\[ \chi_{\lambda,2} = \frac{\Theta_{\lambda,2}}{\eta} \quad 0 \leq \lambda \leq 2, \quad \lambda = 2r - s \]

\[ \Theta_{\lambda,k}(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{(2kn+\lambda)^2}{4k}} \]

(6)

via Durfee rectangle identity

\[ \chi_{\lambda,2} = \sum_{\vec{m} \in (\mathbb{Z}_{\geq 0})^2} \frac{\frac{1}{2} \vec{m}^t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{m} + \frac{1}{2} \left( -\lambda \right)^t \vec{m} + c}{\left( q \right)^{\vec{m}}} \]

(7)

For $c_{p,1}$ Characters ($p \geq 2$)
Fermionic Character Expressions for $c = -2$

\[ \chi_{\lambda',2}^+ = \frac{(2 - \lambda') \Theta \lambda',2 + (\partial \Theta) \lambda',2}{2\eta} \]
\[ \chi_{\lambda',2}^- = \frac{\lambda' \Theta \lambda',2 - (\partial \Theta) \lambda',2}{2\eta} \]

$0 < \lambda' < 2$
Fermionic Character Expressions for $c = -2$

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Fermionic Expressions for $c_{p,1}$ Characters ($p \geq 2$)
Summary and Outlook

\[ \chi^+_{\lambda',2} = \frac{(2 - \lambda')\Theta_{\lambda',2} + (\partial\Theta)_{\lambda',2}}{2\eta}, \quad 0 < \lambda' < 2 \]

\[ \chi^-_{\lambda',2} = \frac{\lambda'\Theta_{\lambda',2} - (\partial\Theta)_{\lambda',2}}{2\eta} \]

\[ \chi^+_{1,2} = \sum_{\tilde{m} \in (\mathbb{Z}_{\geq 0})^2} \frac{q^{\frac{1}{2} \tilde{m}^t (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \tilde{m} + \frac{1}{2} (\begin{smallmatrix} 1 \\ 1 \end{smallmatrix})^t \tilde{m} + \frac{1}{12}}}{(q)^{\tilde{m}}} \] (8)

\[ \chi^-_{1,2} = \sum_{\tilde{m} \in (\mathbb{Z}_{\geq 0})^2} \frac{q^{\frac{1}{2} \tilde{m}^t (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \tilde{m} + \frac{1}{2} (\begin{smallmatrix} 1 \\ 1 \end{smallmatrix})^t \tilde{m} + \frac{1}{12}}}{(q)^{\tilde{m}}} \] (9)
Fermionic Character Expressions for $c = -2$

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**Bosonic-Fermionic \( q \)-Series Identities**

- \( c_{p,1} \) Logarithmic Conformal Field Theories
- Central Charges and Conformal Dimensions
- Bosonic Character Expressions for \( c_{p,1} \)
- Fermionic Character Expressions for \( c_{2,1} = -2 \)
- Fermionic Character Expressions for \( c = -2 \)
- Relation to Simple Lie Algebras
- Ansatz for Generalization
- Example
- Fermionic Expressions for \( c_{p,1} \) Characters (\( p \geq 2 \))
- Summary and Outlook

\[ \chi_{\lambda',2}^+ = \frac{(2 - \lambda')\Theta_{\lambda',2} + (\partial \Theta)_{\lambda',2}}{2\eta} \quad 0 < \lambda' < 2 \]

\[ \chi_{\lambda',2}^- = \frac{\lambda'\Theta_{\lambda',2} - (\partial \Theta)_{\lambda',2}}{2\eta} \]

\[ \chi_{1,2}^+ = \sum_{\vec{m} \in (\mathbb{Z}_{\geq 0})^2} q^{\frac{1}{2} \vec{m}^t \begin{pmatrix} 1 & 0 \end{pmatrix} \vec{m} + \frac{1}{2} \begin{pmatrix} 1 \end{pmatrix}^t \vec{m} + \frac{1}{12}} \frac{(q)\vec{m}}{(q)\vec{m}} \]  

\[ m_1 + m_2 \equiv 0 \pmod{2} \]  

\[ \chi_{1,2}^- = \sum_{\vec{m} \in (\mathbb{Z}_{\geq 0})^2} q^{\frac{1}{2} \vec{m}^t \begin{pmatrix} 1 & 0 \end{pmatrix} \vec{m} + \frac{1}{2} \begin{pmatrix} 1 \end{pmatrix}^t \vec{m} + \frac{1}{12}} \frac{(q)\vec{m}}{(q)\vec{m}} \]  

\[ m_1 + m_2 \equiv 1 \pmod{2} \]
Fermionic Character Expressions for $c = -2$

\[ 
\chi_{\lambda',2}^+ = \frac{(2 - \lambda')\Theta \lambda',2 + (\partial \Theta) \lambda',2}{2\eta} \\
\chi_{\lambda',2}^- = \frac{\lambda' \Theta \lambda',2 - (\partial \Theta) \lambda',2}{2\eta} \\
\chi_{1,2}^+ = \sum_{\bar{m} \in (\mathbb{Z}_{\geq 0})^2} q^{\frac{1}{2} \bar{m}^t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bar{m} + \frac{1}{2} \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right)^t \bar{m} + \frac{1}{12}} (q)^{\bar{m}} \\
\chi_{1,2}^- = \sum_{\bar{m} \in (\mathbb{Z}_{\geq 0})^2} q^{\frac{1}{2} \bar{m}^t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bar{m} + \frac{1}{2} \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right)^t \bar{m} + \frac{1}{12}} (q)^{\bar{m}} 
\]
Fermionic Character Expressions for $c = -2$

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Fermionic Character Expressions for $c = -2$

$$\chi^{+}_{\lambda',2} = \frac{(2 - \lambda')\Theta_{\lambda',2} + (\partial \Theta)_{\lambda',2}}{2\eta} \quad 0 < \lambda' < 2$$

$$\chi^{-}_{\lambda',2} = \frac{\lambda'\Theta_{\lambda',2} - (\partial \Theta)_{\lambda',2}}{2\eta}$$

$$\chi^{+}_{1,2} = \sum_{\tilde{m} \in (\mathbb{Z}_{\geq 0})^2 \atop m_1 + m_2 \equiv 0 \pmod{2}} q^{\frac{1}{2} \tilde{m}^t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{m} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tilde{m} + \frac{1}{12}} \left( q \right)^{\tilde{m}}$$

$$\chi^{-}_{1,2} = \sum_{\tilde{m} \in (\mathbb{Z}_{\geq 0})^2 \atop m_1 + m_2 \equiv 1 \pmod{2}} q^{\frac{1}{2} \tilde{m}^t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{m} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tilde{m} + \frac{1}{12}} \left( q \right)^{\tilde{m}}$$

$$C^{-1}_{D_2} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
Relation to Simple Lie Algebras

Figure 1: Dynkin diagram of $D_2 = su(2) \times su(2) = so(4)$
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Figure 2: Dynkin diagram of $D_p = so(2p)$
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$D_p = so(2p)$

Figure 2: Dynkin diagram of $D_p = so(2p)$

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CASE $p = 2$

\[ \overrightarrow{b} = \left( \pm \frac{\lambda}{2}, \frac{\lambda}{2} \right) \]

**Ansatz for generalization**

\[ \begin{array}{c}
1 \\
2 \\
p \end{array} \]

1 2 3 \ldots p-2 p
**Ansatz for Generalization**

**Case $p = 2$**

**Ansatz for generalization**

\[
\vec{b} = \left( \pm \frac{\lambda}{2} \right)
\]

\[
\vec{b} = \begin{pmatrix}
    b_1 \\
    \vdots \\
    b_{p-2} \\
    \pm \frac{\lambda}{2}
\end{pmatrix}
\]
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**Case $p = 2$**

\[ \vec{b} = \left( \pm \frac{\lambda}{2} \right) \]

**Ansatz for Generalization**

\[ \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_{p-2} \\ \pm \frac{\lambda}{2} \end{pmatrix} \]

Sum restrictions ($a \in \{0, 1\}$)

\[ m_1 + m_2 \equiv a \pmod{2} \]

\[ m_{p-1} + m_p \equiv a \pmod{2} \]
Example: vacuum representation of the $c_{5,1} = -\frac{91}{5}$ LCFT

\[ \chi_{4,5}^+ = \frac{\Theta_{4,5} + (\partial \Theta)_{4,5}}{5\eta} \] (10)
Example: vacuum representation of the $c_{5,1} = -\frac{91}{5}$ LCFT

$$\chi_{4,5}^+ = \frac{\Theta_{4,5} + (\partial \Theta)_{4,5}}{5\eta}$$  \hspace{1cm} (10)$$

$$= \sum_{\tilde{m} \in (\mathbb{Z}_{\geq 0})^5 \atop m_4 + m_5 \equiv 0 \text{ (mod 2)}} q^{\sum_{i=4}^{5} m_i} (q)^{\tilde{m}}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1/2 & 1/2 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 3 & 3/2 & 3/2 \\ 1/2 & 1 & 3/2 & 3 & 3 \\ 1/2 & 1 & 3/2 & 4/4 & 4/4 \end{pmatrix} \tilde{m} + \begin{pmatrix} 1 \\ 2/3 \\ 2 \\ 2 \\ 2 \end{pmatrix} \tilde{m} + \frac{91}{120}$$ \hspace{1cm} (11)
Example: vacuum representation of the $c_{5,1} = -\frac{91}{5}$ LCFT

$$\chi^{+}_{4,5} = \frac{\Theta_{4,5} + (\partial\Theta)_{4,5}}{5\eta}$$  \hspace{1cm}(10)$$

$$= \sum_{\vec{m} \in (\mathbb{Z}_{\geq 0})^5} \frac{q^{\vec{m}^t \left(\begin{array}{cccc} 1 & 1 & 1 & 1/2 & 1/2 \\ 1/2 & 1/2 & 2 & 1 & 1 \\ 1 & 2 & 3 & 3 & 3 \\ 1/2 & 1 & 3 & 5 & 3 \\ 1/2 & 1 & 3 & 3 & 5 \end{array}\right) \vec{m} + \left(\begin{array}{c} 1 \\ 2/3 \\ 2 \end{array}\right)^t \vec{m} + \frac{91}{120}}{(q)^{\vec{m}}}$$  \hspace{1cm}(11)$$

- Bosonic-fermionic $q$-series identities
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Fermionic Expressions for $c_{p,1}$ Characters ($p \geq 2$)

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\[
\chi_{\lambda,p} = \sum_{\tilde{m} \in (\mathbb{Z}_{\geq 0})^p} \frac{q \tilde{m}^t C_p^{-1} \tilde{m} + \tilde{b}^t_{\lambda,p} \tilde{m} + c^{\lambda}_{\lambda,p}}{(q)\tilde{m}}
\]  \hspace{1cm} (12)

with $0 \leq \lambda \leq p$

\[
\text{with } 0 \leq \lambda \leq p
\]
Fermionic Expressions for $c_{p,1}$ Characters ($p \geq 2$)

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\[ \chi_{\lambda,p} = \sum_{\vec{m} \in (\mathbb{Z} \geq 0)^p} \frac{\vec{m}^t C_{D_p}^{-1} \vec{m} + \vec{b}_{\lambda,p}^t \vec{m} + c_{\lambda,p}}{(q)^{\vec{m}}} \]  

(12)

with $0 \leq \lambda \leq p$, $\vec{b}_{\lambda,p}^t = (0, 0, \ldots, 0, \frac{\lambda}{2}, -\frac{\lambda}{2})$
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\[ \chi_{\lambda,p} = \sum_{\tilde{m} \in (\mathbb{Z}_{\geq 0})^p \atop m_{p-1} + m_p \equiv 0 \pmod{2}} q^{\tilde{m}^t C_{Dp}^{-1} \tilde{m} + b_{\lambda,p}^t \tilde{m} + c_{\lambda,p}} (q)^{\tilde{m}} \]  

(12)

\[ \chi_{\lambda',p} = \sum_{\tilde{m} \in (\mathbb{Z}_{\geq 0})^p \atop m_{p-1} + m_p \equiv 0 \pmod{2}} q^{\tilde{m}^t C_{Dp}^{-1} \tilde{m} + b_{\lambda',p}^t \tilde{m} + c_{\lambda',p}} (q)^{\tilde{m}} \]  

(13)

with $0 < \lambda' < p$
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\[ \chi_{\lambda,p} = \sum_{\vec{m} \in (\mathbb{Z}_{\geq 0})^p} \frac{q^{\vec{m}^t C_{Dp}^{-1} \vec{m} + \vec{b}_\lambda^t, p \vec{m} + c_{\lambda, p}^*}}{(q)_{\vec{m}}} \]  \hspace{1cm} (12)

\[ \chi_{\lambda',p}^+ = \sum_{\vec{m} \in (\mathbb{Z}_{\geq 0})^p} \frac{q^{\vec{m}^t C_{Dp}^{-1} \vec{m} + \vec{b}_{\lambda', p}^t + \vec{c}_{\lambda', p}^*}}{(q)_{\vec{m}}} \]  \hspace{1cm} (13)

with $0 < \lambda' < p$, $\vec{b}_{\lambda', p}^t = (0, \ldots, 0, 1, 2, 3, \ldots, k, \frac{\lambda}{2}, \frac{\lambda}{2})$
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Fermionic Expressions for $c_{p,1}$ Characters ($p \geq 2$)

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\[ \chi_{\lambda, p} = \sum_{\bar{m} \in (\mathbb{Z}_{\geq 0})^p} \frac{q^{\bar{m}^t C_{Dp}^{-1} \bar{m} + \bar{b}^t_{\lambda, p} \bar{m} + c^*_\lambda, p}}{(q) \bar{m}} \]

(12)

\[ \chi^+_{\lambda', p} = \sum_{\bar{m} \in (\mathbb{Z}_{\geq 0})^p} \frac{q^{\bar{m}^t C_{Dp}^{-1} \bar{m} + \bar{b}^{++t}_{\lambda', p} \bar{m} + c^*_p \lambda', p}}{(q) \bar{m}} \]

(13)

\[ \chi^-_{\lambda', p} = \sum_{\bar{m} \in (\mathbb{Z}_{\geq 0})^p} \frac{q^{\bar{m}^t C_{Dp}^{-1} \bar{m} + \bar{b}^{--t}_{\lambda', p} \bar{m} + c^*_p - \lambda', p}}{(q) \bar{m}} \]

(14)

with $0 < \lambda' < p$
Fermionic Expressions for $c_{p,1}$ Characters ($p \geq 2$)

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Fermionic Expressions for $c_{p,1}$ Characters ($p \geq 2$)

Summary and Outlook

\[ \chi_{\lambda,p} = \sum_{\tilde{m} \in \mathbb{Z}_{\geq 0}^p} \frac{q^{\tilde{m}^t C_{D_p}^{-1} \tilde{m} + b_{\lambda',p}^t \tilde{m} + c_{\lambda',p}}}{(q)^m} \] (12)

\[ \chi_{\lambda',p}^+ = \sum_{\tilde{m} \in \mathbb{Z}_{\geq 0}^p} \frac{q^{\tilde{m}^t C_{D_p}^{-1} \tilde{m} + b_{\lambda',p}^{t+} \tilde{m} + c_{\lambda',p}}}{(q)^m} \] (13)

\[ \chi_{\lambda',p}^- = \sum_{\tilde{m} \in \mathbb{Z}_{\geq 0}^p} \frac{q^{\tilde{m}^t C_{D_p}^{-1} \tilde{m} + b_{\lambda',p}^{-t} \tilde{m} + c_{\lambda',p}^{p-\lambda',p}}}{(q)^m} \] (14)

with $0 < \lambda' < p$, $(b_{\lambda',p}^-)_i = (b_{p-\lambda',p}^+)_i$
Further evidence for the well-definedness of LCFTs as being rational
Further evidence for the well-definedness of LCFTs as being rational
Fermionic character expressions admit quasi-particle interpretations
Further evidence for the well-definedness of LCFTs as being rational

Fermionic character expressions admit quasi-particle interpretations

In the case of the \(c_{p,1}\) LCFTs:
Further evidence for the well-definedness of LCFTs as being rational

Fermionic character expressions admit quasi-particle interpretations

In the case of the $c_{p,1}$ LCFTs:

$\Box \quad p = 2 \Rightarrow c = -2$: 

- Symplectic fermions
- $p > 2$: Realization in terms of $p - 2$ ordinary fermions and a pair of symplectic fermions
- Interesting direction for further research
### Summary and Outlook

- Further evidence for the well-definedness of LCFTs as being rational
- Fermionic character expressions admit quasi-particle interpretations
- In the case of the $c_{p,1}$ LCFTs:
  - $p = 2 \Rightarrow c = -2$: Symplectic fermions

### Bosonic-Fermionic \(q\)-Series Identities

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- Further evidence for the well-definedness of LCFTs as being rational
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  - $p > 2$: 

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Summary and Outlook

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    $\Rightarrow$ interesting direction for further research
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Fermionic Character
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Fermionic Expressions
for $c_{p,1}$ Characters
($p \geq 2$)

Summary and Outlook

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Generalized Character Functions:

\[
\chi_{0,p} = \frac{\Theta_{0,p}}{\eta}
\]

\[
\chi_{p,p} = \frac{\Theta_{p,p}}{\eta}
\]

where \(0 < \lambda < p\), \(\lambda = pr - p's = pr - s\).
Bosonic Character Expressions for $c_{p,1}$

Generalized Character Functions:

\[
\chi_{0,p} = \frac{\Theta_{0,p}}{\eta}
\]
\[
\chi_{p,p} = \frac{\Theta_{p,p}}{\eta}
\]

\[
\Theta_{\lambda,k}(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{(2kn+\lambda)^2}{4k}}
\]

where $0 < \lambda < p$, $\lambda = pr - p's = pr - s$. 
Bosonic Character Expressions for $c_{p,1}$

Generalized Character Functions:

\begin{align*}
\chi_{0,p} &= \frac{\Theta_{0,p}}{\eta} \\
\chi_{p,p} &= \frac{\Theta_{p,p}}{\eta} \\
\chi^+_{\lambda,p} &= \frac{(p - \lambda)\Theta_{\lambda,p} + (\partial \Theta)_{\lambda,p}}{p \eta} \\
\chi^-_{\lambda,p} &= \frac{\lambda \Theta_{\lambda,p} - (\partial \Theta)_{\lambda,p}}{p \eta}
\end{align*}

where $0 < \lambda < p$, $\lambda = pr - p's = pr - s$. 

\[ \Theta_{\lambda,k}(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{(2kn + \lambda)^2}{4k}} \]
Bosonic Character Expressions for $c_{p,1}$

Generalized Character Functions:

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\end{align*}
\]

where $0 < \lambda < p$, $\lambda = pr - p's = pr - s$. 

\[
\Theta_{\lambda,k}(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{(2kn+\lambda)^2}{4k}}
\]

\[
(\partial\Theta)_{\lambda,k}(\tau) = \sum_{n \in \mathbb{Z}} (2kn + \lambda)q^{\frac{(2kn+\lambda)^2}{4k}}
\]
Generalized Character Functions:

\[
\chi_{0,p} = \frac{\Theta_{0,p}}{\eta} \\
\chi_{p,p} = \frac{\Theta_{p,p}}{\eta} \\
\chi_{\lambda,p}^+ = \frac{(p - \lambda)\Theta_{\lambda,p} + (\partial \Theta)_{\lambda,p}}{p\eta} \\
\chi_{\lambda,p}^- = \frac{\lambda\Theta_{\lambda,p} - (\partial \Theta)_{\lambda,p}}{p\eta} \\
\tilde{\chi}_{\lambda,p}^+ = \frac{\Theta_{\lambda,p} + i\alpha \lambda (\nabla \Theta)_{\lambda,p}}{\eta} \\
\tilde{\chi}_{\lambda,p}^- = \frac{\Theta_{\lambda,p} - i\alpha (p - \lambda) (\nabla \Theta)_{\lambda,p}}{\eta}
\]

where \( 0 < \lambda < p \), \( \lambda = pr - p's = pr - s \).
Generalized Character Functions:

\[
\chi_{0,p} = \frac{\Theta_{0,p}}{\eta} \\
\chi_{p,p} = \frac{\Theta_{p,p}}{\eta} \\
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\chi_{\lambda,p}^- = \frac{\lambda\Theta_{\lambda,p} - (\partial\Theta)_{\lambda,p}}{p\eta} \\
\tilde{\chi}_{\lambda,p}^+ = \frac{\Theta_{\lambda,p} + i\alpha\lambda(\nabla\Theta)_{\lambda,p}}{\eta} \\
\tilde{\chi}_{\lambda,p}^- = \frac{\Theta_{\lambda,p} - i\alpha(p - \lambda)(\nabla\Theta)_{\lambda,p}}{\eta}
\]

where \(0 < \lambda < p\), \(\lambda = pr - p's = pr - s\).

\[
\Theta_{\lambda,k}(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{(2kn + \lambda)^2}{4k}}
\]

\[
(\partial\Theta)_{\lambda,k}(\tau) = \sum_{n \in \mathbb{Z}} (2kn + \lambda)q^{\frac{(2kn + \lambda)^2}{4k}}
\]

\[
(\nabla\Theta)_{\lambda,k}(\tau) = \frac{\ln q}{2\pi i} (\partial\Theta)_{\lambda,k}(\tau)
\]
Motivation

Bosonic-Ifermionic q-Series Identities

- **prototype**: free boson
  \[ \chi = q^{-\frac{1}{24}} \frac{1}{(q)_\infty} = \frac{1}{\eta(\tau)} \]

- obtained by BRST projection of a (typically bosonic) free-field construction

- subsequent subtraction of submodules corr. to singular vectors, e.g.
  \[ \frac{1}{(q)_\infty} \sum_{n=-\infty}^{+\infty} (q^{h_n} - q^{h_n+1}) \]

- easily expressible in terms of \( \Theta \)-functions
  \[ \frac{1}{(q)_\infty} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \ldots \] (15)

  is the generating function for additive partitions of an integer \( n \) into an arbitrary number of integers
Fermionic Expression

- **prototype:** free fermion Fock space with PBC

\[
q^{\frac{1}{48}} \chi = \prod_{n=0}^{\infty} (1 + q^n) = \sum_{n=0}^{\infty} \frac{q^{\frac{n^2-n}{2}}}{(q)_n}
\]

- \(q^{\frac{n^2-n}{2}} / (q)_n\) is generating function for partitions of natural numbers into \(n \in \mathbb{N}\) distinct parts

- admits interpretation of the theory in terms of quasi-particles
Theta Functions

- **Motivation**

  - Bosonic-Fermionic $q$-Series Identities

  - $c_{p,1}$ Logarithmic Conformal Field Theories

- **Central Charges and Conformal Dimensions**

  - Bosonic Character Expressions for $c_{p,1}$

  - Fermionic Character Expressions for $c_{2,1} = -2$

  - Fermionic Character Expressions for $c = -2$

- **Relation to Simple Lie Algebras**

- **Ansatz for Generalization**

- **Example**

  - Fermionic Expressions for $c_{p,1}$ Characters ($p \geq 2$)

- **Summary and Outlook**

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**Jacobi-Riemann $\Theta$-functions:**

\[
\Theta_{\lambda,k}(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{(2kn+\lambda)^2}{4k}}
\]  \hspace{1cm} (16)

**Affine $\Theta$-functions:**

\[
(\partial \Theta)_{\lambda,k}(\tau) = \sum_{n \in \mathbb{Z}} (2kn + \lambda)q^{\frac{(2kn+\lambda)^2}{4k}}
\]  \hspace{1cm} (17)

with $\lambda \in \mathbb{Z}/2$ and $k \in \mathbb{Z}_{\geq 1}/2$

\[
(\nabla \Theta)_{\lambda,k}(\tau) = \frac{\ln q}{2\pi i} \sum_{n \in \mathbb{Z}} (2kn + \lambda)q^{\frac{(2kn+\lambda)^2}{4k}}
\]  \hspace{1cm} (18)