Fermionic Expressions for the Characters of $c_{p,1}$ Logarithmic Conformal Field Theories

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Talk given at XIX Workshop Beyond the Standard Model, Bad Honnef

Michael Flohr Carsten Grabow Michael Koehn

March 14th, 2007

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- We show: $c_{p,1}$ LCFTs behave just like RCFTs in one more important aspect!

Bosonic-Fermionic q-Series Identities

Bosonic and Fermionic Character Expressions

Rogers-Ramanujan identities ($a \in \{0, 1\}$):

$$\sum_{n=0}^{\infty} \frac{q^{n(n+a)}}{(q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1-a})(1-q^{5n-4+a})} \tag{1}$$

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$c_{p,1}$ Logarithmic Conformal Field Theories

Central Charges and Conformal Dimensions

The $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$ triplet algebra models have the central charges and conformal dimensions

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$$C^{-1}_{D_2} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Relation to Simple Lie Algebras

Figure 1: Dynkin diagram of $D_2 = su(2) \times su(2) = so(4)$

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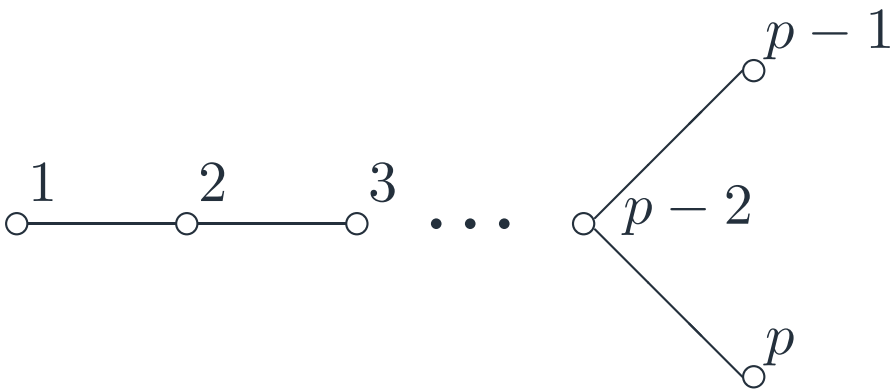
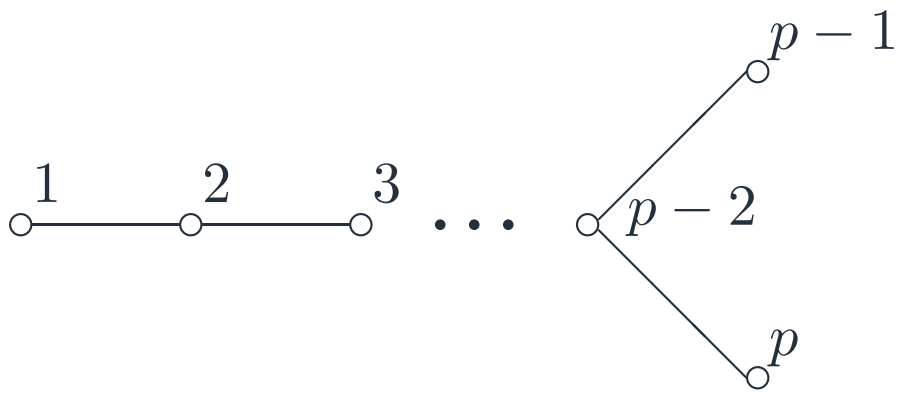


Figure 2: Dynkin diagram of $D_p = so(2p)$

Relation to Simple Lie Algebras



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$$C_{D_p}^{-1} = \frac{1}{4}\begin{pmatrix} 4 & 4 & \dots & 4 & 2 & 2 \\ 4 & 8 & \dots & 8 & 4 & 4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 4 & 8 & \dots & 4p-8 & 2p-4 & 2p-4 \\ 2 & 4 & \dots & 2p-4 & p & p-2 \\ 2 & 4 & \dots & 2p-4 & p-2 & p \end{pmatrix}$$

Case $p = 2$

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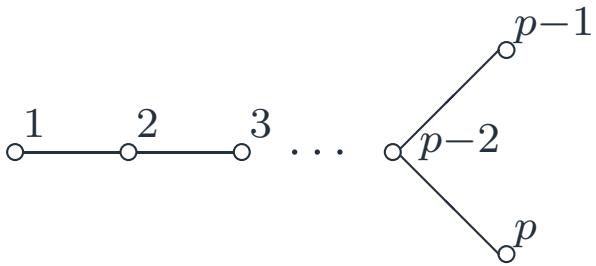

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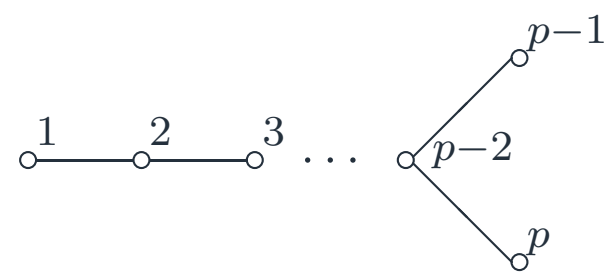

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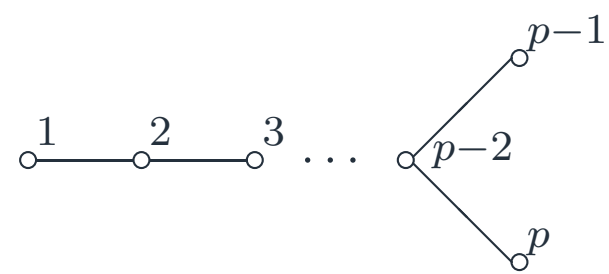

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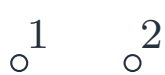

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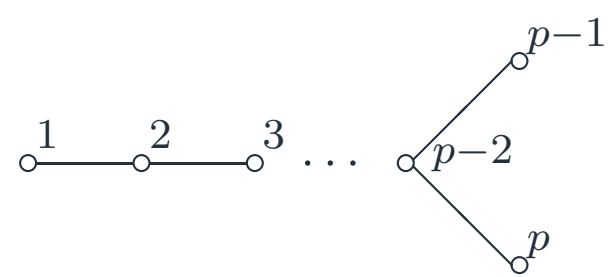

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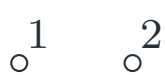

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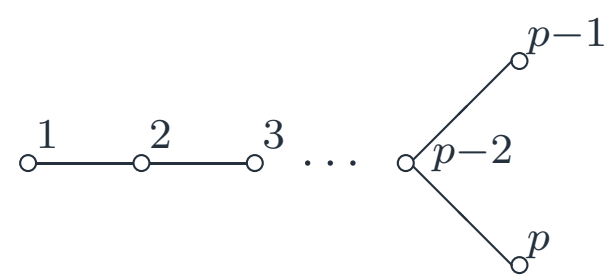

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Example: vacuum representation of the $c_{5,1} = -\frac{91}{5}$ LCFT

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- Bosonic-fermionic q-series identities

Fermionic Expressions for $c_{p,1}$ Characters ($p \geq 2$)

$$\chi_{\lambda,p} = \sum_{\substack{\vec{m}\in(\mathbb{Z}_{\geq 0})^p \\ m_{p-1}+m_p\equiv 0 \pmod 2}} \frac{q^{\vec{m}^t C_{D_p}^{-1}\vec{m}+\vec{b}^t_{\lambda,p}\vec{m}+c^\star_{\lambda,p}}}{(q)_{\vec{m}}} \tag{12}$$

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    $\Rightarrow$ interesting direction for further research

Appendix

Bosonic Character Expressions for $c_{p,1}$

Generalized Character Functions:

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Bosonic Expression

- *prototype:* free boson
  $\chi = q^{-\frac{1}{24}} \frac{1}{(q)_\infty} = \frac{1}{\eta(\tau)}$
- obtained by BRST projection of a (typically bosonic) free-field construction
- subsequent subtraction of submodules corr. to singular vectors, e.g.
  $\frac{1}{(q)_\infty} \sum_{n=-\infty}^{+\infty} (q^{h_n} - q^{h_{n+1}})$
- easily expressible in terms of Θ-functions

$$\frac{1}{(q)_\infty} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \dots \tag{15}$$

is the generating function for additive partitions of an integer n into an arbitrary number of integers

Fermionic Expression

- *prototype:* free fermion Fock space with PBC

$$q^{\frac{1}{48}}\chi = \prod_{n=0}^{\infty}(1+q^n) = \sum_{n=0}^{\infty}\frac{q^{\frac{n^2-n}{2}}}{(q)_n}$$

- $\frac{q^{\frac{n^2-n}{2}}}{(q)_n}$ is generating function for partitions of natural numbers into $n \in \mathbb{N}$ distinct parts
- admits interpretation of the theory in terms of quasi-particles

Theta Functions

- Jacobi-Riemann Θ-functions:

$$\Theta_{\lambda,k}(\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{(2kn+\lambda)^2}{4k}} \tag{16}$$

- Affine Θ-functions:

$$(\partial\Theta)_{\lambda,k}(\tau) = \sum_{n\in\mathbb{Z}} (2kn+\lambda) q^{\frac{(2kn+\lambda)^2}{4k}} \tag{17}$$

with $\lambda \in \mathbb{Z}/2$ and $k \in \mathbb{Z}_{\geq 1}/2$

- $$(\nabla\Theta)_{\lambda,k}(\tau) = \frac{\ln q}{2\pi \mathrm{i}} \sum_{n\in\mathbb{Z}} (2kn+\lambda) q^{\frac{(2kn+\lambda)^2}{4k}} \tag{18}$$