Explicit Formulas for the Scalar Modes in Seiberg–Witten Theory

with an Application to the Argyres–Douglas Point

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Overview

1. Lauricella $F_D^{(n)}$
2. Explicit Formulas for Scalar Modes
3. Application: The $SU(3)$ Argyres–Douglas Point

Lauricella $F_D^{(n)}$

Lauricella $F_D^{(n)}$ is a generalization of the Gaussian hypergeometric function ${}_2F_1$:

$$F_D^{(n)}(a, b_1, \dots, b_n; c; x_1, \dots, x_n)$$
$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1} \cdots (b_n)_{m_n}}{(c)_{m_1+\cdots+m_n} m_1! \cdots m_n!} \, x_1^{m_1} \cdots x_n^{m_n},$$

whenever $|x_1|, \dots, |x_n| < 1$. (Elsewhere by analytic continuation.)

Lauricella $F_D^{(n)}$

An important integral representation for $F_D^{(n)}$:

$$\int_0^1 t^{a-1}(1-t)^{c-a-1}\prod_{i=1}^{n}(1-tx_i)^{-b_i}\,\mathrm{d}t$$
$$=\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}\,F_D^{(n)}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_n).$$

This is proved using the binomial theorem.

Lauricella $F_D^{(n)}$

In the literature there are given a few continuation formulas for $F_D^{(n)}$.

However, there is lacking a complete overview of continuation formulas such as it exists for Gauss ${}_2F_1$.

Part of our work consisted of extending the set of known analytic continuations for $F_D^{(n)}$.

Scalar Modes

In $N = 2$ SYM theories there are the so-called scalar modes a_i, a_D^i.

They allow determination of the prepotential, which means 'solving' the theory.

Moreover, they give the mass spectrum of BPS states via

$$m_{(\mathbf{q},\mathbf{g})} \propto |q^i a_i + g_j a_D^j|.$$

Here $\mathbf{q}$, $\mathbf{g}$ denote the electric and magnetic charge vector of the state.

How to calculate a_i, a_D^i?

In the case of a simply laced gauge group, there is a hyperelliptic curve, e. g. for $SU(N)$,

$$y^2 = \left(x^N - \sum_{k=2}^{N} u_k x^{N-k} \right)^2 - \Lambda^{2N} = \prod_{i=1}^{2N} (x - e_i),$$

and a meromorphic 1-form on it, the SW differential,

$$\lambda_{\mathrm{SW}} = \frac{1}{2\pi\mathrm{i}} \frac{\prod_{\ell=0}^{N-1} (x - z_\ell)}{\prod_{i=1}^{2N} \sqrt{x - e_i}} \, \mathrm{d}x.$$

How to calculate a_i, a_D^i?

With these data, one has the formulas

$$a_i = \int_{\alpha_i} \lambda_{\mathrm{SW}}, \qquad a_D^j = \int_{\beta^j} \lambda_{\mathrm{SW}},$$

where $\{\alpha_i, \beta^i\}_{1 \leq i \leq N-1}$ denotes a canonical homology basis for the given curve, i. e. one for which $\alpha_i \cap \beta^j = \delta_i^j$.

Thus the calculation of the scalar modes reduces to the evaluation of period integrals.

Standard Approach

The commonest approach to the evaluation of these integrals is solving Picard–Fuchs equations.

The 'problem' with this is that it fails to yield general results and also is rather complicated.

As an example, for the case of gauge group $SU(3)$ one has to set up and solve the following system of partial differential equations for $a_i(u,v)$ $(i,j=1,2)$:

$$\big((27\Lambda^6-4u^3-27v^2)\partial_u^2-12u^2v\partial_u\partial_v-12u^2\partial_u-21uv\partial_v-4u\big)\partial_v a_i=0,$$

$$\big((27\Lambda^6-4u^3-27v^2)\partial_v^2-36uv\partial_u\partial_v-36u\partial_u-63v\partial_v-12\big)\partial_v a_i=0.$$

The same for $a_D^i(u,v)$.

Evaluation with $F_D^{(n)}$

However, using $F_D^{(n)}$ one can evaluate the period integrals directly. For the cycle around e_i, e_j one finds

$$a \quad \text{or} \quad a_D = (e_i - e_j)^{\frac{1}{2}} \prod_{k=0}^{N-1} (e_i - z_k) \prod_{\substack{\ell=1 \\ \ell \neq i,j}}^{2N} (e_i - e_\ell)^{-\frac{1}{2}} \times$$

$$\times F_D^{(3N-2)} \left(\frac{1}{2}, -1, \ldots, -1, \frac{1}{2}, \ldots, \frac{1}{2}; 1; \left\{ \frac{e_i - e_j}{e_i - z_k} \right\}_k, \left\{ \frac{e_i - e_j}{e_i - e_\ell} \right\}_{\ell \neq i,j} \right).$$

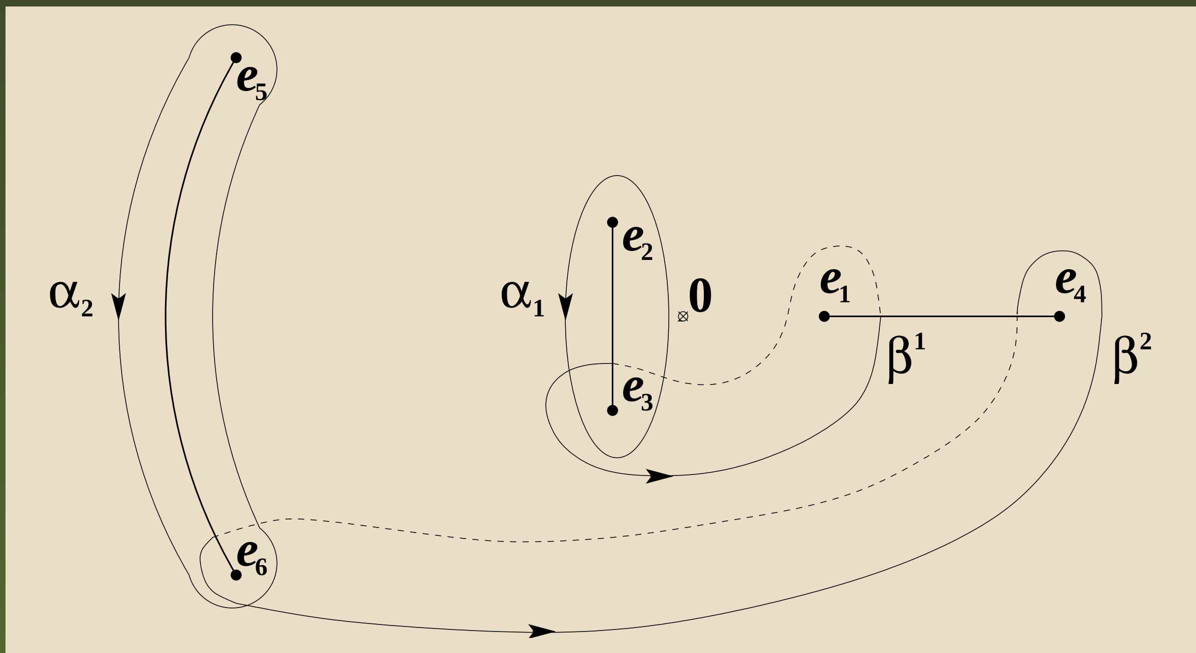
i, j in the above formula are related to the exact form of the chosen homology basis.

The Argyres–Douglas point

The AD point is a 'singular point' in moduli space. This means that three branch points coalesce at the AD point.

This entails two dual cycles shrinking to zero (which has various consequences).

A picture of the AD point:



a_i, a_D^i near Argyres–Douglas point

Applying our formulas we obtain for the scalar modes near the AD point:

$$a_1(\delta u, \delta v) = 2\int_{e_2}^{e_3} \lambda_{\mathrm{SW}} = \frac{e_2(e_2^2 - \delta u/3)}{(e_2-e_1)^{\frac{1}{2}}(e_2-e_4)^{\frac{1}{2}}(e_2-e_5)^{\frac{1}{2}}(e_2-e_6)^{\frac{1}{2}}} \times$$
$$\times F_D^{(7)}\Big(\ldots; \frac{e_2-e_3}{e_2-0}, \frac{e_2-e_3}{e_2-\sqrt{\delta u/3}}, \frac{e_2-e_3}{e_2+\sqrt{\delta u/3}}, \frac{e_2-e_3}{e_2-e_1}, \frac{e_2-e_3}{e_2-e_4}, \frac{e_2-e_3}{e_2-e_5}, \frac{e_2-e_3}{e_2-e_6}\Big),$$

and

$$a_D^1(\delta u, \delta v) = 2\int_{e_3}^{e_1} \lambda_{\mathrm{SW}} = \frac{e_3(e_3^2 - \delta u/3)}{(e_3-e_2)^{\frac{1}{2}}(e_3-e_4)^{\frac{1}{2}}(e_3-e_5)^{\frac{1}{2}}(e_3-e_6)^{\frac{1}{2}}} \times$$
$$\times F_D^{(7)}\Big(\ldots; \frac{e_3-e_1}{e_3-0}, \frac{e_3-e_1}{e_3-\sqrt{\delta u/3}}, \frac{e_3-e_1}{e_3+\sqrt{\delta u/3}}, \frac{e_3-e_1}{e_3-e_2}, \frac{e_3-e_1}{e_3-e_4}, \frac{e_3-e_1}{e_3-e_5}, \frac{e_3-e_1}{e_3-e_6}\Big).$$

(The branch points e_k are functions of $\delta u, \delta v$.)

Limit of scalar modes

Under the restricition of $\delta u < 0,\ \delta v = -\delta u$ one obtains the limits

$$\lim_{\delta u, \delta v \to 0} a_1(\delta u, \delta v) = 0,$$

and

$$\lim_{\delta u, \delta v \to 0} a_D^1(\delta u, \delta v) = 0.$$

To do this, one needs an analytic continuation formula for Lauricella $F_D^{(n)}$.

This reproduces the result stated by Argyres and Douglas (hep-th/9505062).

Closing words

The explicit formulas are a useful tool in studying such settings as the AD point in fine detail; especially in determining how general certain results really are.

In order to make the most of the derived formulas, the theory of Lauricella $F_D^{(n)}$ needs to be developed further.

Generalization to other simply laced gauge groups easily possible.

Generalization to $SU(N)$ with massive hypermultiplets extremely easy and already published along with the results reported in this talk.