Vertex Algebras, Operator Product Expansion, and C_2-cofiniteness*

Nils Carqueville

Uni Bonn

N. Carqueville, M. Flohr, J. Phys. A: Math. Gen. **39** (2006), 951–966, [math-ph/0508015]

Synopsis

- conformal field theory
- vertex operator algebras...
- ...and related structures
- meromorphic operator product expansion
- nonmeromorphic operator product expansion
- results on triplet $\mathcal{W}$-algebras
- C_2-cofiniteness and rationality

Conformal Field Theory

- A two-dimensional **conformal field theory*** is a projective functor $\mathscr{T} : \mathscr{C} \to \mathscr{V}$ (motivated by string theory).
- Constructing a general CFT is very difficult.
- For $g = 0$, the problem has been completely solved, building on the notion of **vertex operator algebra**.
- Historically, vertex operator algebras arose in the study of the **Monster** finite group and **infinite-dimensional Lie algebras**.

* Segal 1987

Vertex Operator Algebras

Definition. A vertex operator algebra* is a $\mathbb{Z}$-graded $\mathbb{C}$-vector space

$$V = \coprod_{m\in\mathbb{Z}} V_{(m)} \quad \text{with} \quad \dim V_{(m)} < \infty \quad \text{for all } m \in \mathbb{Z}$$

together with a linear vertex operator map

$$V \longrightarrow (\mathrm{End} V)[\![x, x^{-1}]\!] \, , \quad v \longmapsto Y(v,x) = \sum_{m\in\mathbb{Z}} v_m x^{-m-1} \, .$$

There are two special elements in V: the vacuum $\Omega \in V_{(0)}$ and the conformal vector $\omega \in V_{(2)}$. The following axioms hold for all $u, v \in V$:

(V1) the *truncation condition* $u_m v = 0$ for all $m \gg 0$;

(V2) the *vacuum property* $Y(\Omega, x) = \mathbb{1}_V$;

(V3) the *creation property* $Y(v,x)\Omega \in V[\![x]\!]$ and $Y(v,x)\Omega\big|_{x=0} = v$;

Vertex Operator Algebras

(V4) the *Jacobi identity*

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)Y(v,x_2)-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v,x_2)Y(u,x_1)$$
$$=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v,x_2)\ ;$$

(V5) the modes L_m of the *energy momentum operator* $Y(\omega,x)=\sum_{m\in\mathbb{Z}}L_m x^{-m-2}$ span a representation of the **Virasoro algebra**

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{c}{12}(m^3-m)\delta_{m+n,0}\ ,$$

and the homogeneous subspaces $V_{(m)}$ are exactly the eigenspaces of the operator L_0 with eigenvalues m;

(V6) the L_{-1}*-derivative property* $\frac{\mathrm{d}}{\mathrm{d}x}Y(v,x)=Y(L_{-1}v,x)$.

Modules for Vertex Operator Algebras

Definition. A (generalized) V-**module** is an $\mathbb{R}$-graded $\mathbb{C}$-vector space

$$W = \coprod_{h \in \mathbb{R}} W_{[h]} \quad \text{with} \quad \dim W_{[h]} < \infty \quad \text{for all } h \in \mathbb{R}$$

together with a linear **vertex operator** map

$$V \longrightarrow (\operatorname{End} W)[\![x, x^{-1}]\!] \, , \quad v \longmapsto Y_W(v, x) = \sum_{m \in \mathbb{Z}} v_m^W x^{-m-1} \, .$$

The following axioms hold for all $u, v \in V$ and $w \in W$:

(M1) the *truncation condition* $u_m^W w = 0$ for all $m \gg 0$;

(M2) the *vacuum property* $Y_W(\Omega, x) = \mathbb{1}_W$;

Modules for Vertex Operator Algebras

(M3) the *Jacobi identity*

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_W(u,x_1)Y_W(v,x_2) - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_W(v,x_2)Y_W(u,x_1)$$
$$= x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_W(Y(u,x_0)v,x_2) \ ;$$

(M4) the modes L_m^W of the *energy momentum operator*

$$Y_W(\omega,x) = \sum_{m\in\mathbb{Z}} L_m^W x^{-m-2}$$

span a representation of the **Virasoro algebra**, and the homogeneous subspaces $W_{[h]}$ are exactly the (generalized) eigenspaces of the operator L_0^W with (generalized) eigenvalues h;

(M5) the L_{-1}*-derivative property* $\frac{\mathrm{d}}{\mathrm{d}x}Y_W(v,x) = Y_W(L_{-1}v,x)$.

Modules for Vertex Operator Algebras

Important example: For a V-module W, the structure (W', Y') defined by $W' = \coprod_{h\in\mathbb{R}} W^*_{[h]}$ with the vertex operator

$$V \longrightarrow (\mathrm{End} W')[\![x, x^{-1}]\!] ,$$
$$v \longmapsto Y'(v,x) = \sum_{m\in\mathbb{Z}} v'_m x^{-m-1} ,$$

given by the relation

$$\left\langle Y'(v,x)w', w \right\rangle = \left\langle w', Y\left(\mathrm{e}^{xL_1}\left(-x^{-2}\right)^{L_0} v, x^{-1}\right) w \right\rangle$$

is the **contragredient module**, where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the natural pairing between W and W'.

$\Longrightarrow \langle \psi'_m w', w\rangle = \langle w', \psi_{-m} w\rangle$ for primary fields $\sum_{m\in\mathbb{Z}} \psi_m x^{-m-\mathrm{wt}\psi}$

(Logarithmic) Intertwining Operators

Definition. Let (W_i, Y_i), (W_j, Y_j) and (W_k, Y_k) be (generalized) V-modules. A **(logarithmic) intertwining operator** of type $\binom{W_k}{W_i\ W_j}$ is a linear map

$$W_i \longrightarrow (\mathrm{Hom}(W_j, W_k))[\log x]\{x\} \,,$$
$$w_{(i)} \longmapsto \mathcal{Y}_{ij}^k(w_{(i)}, x) = \sum_{m \in \mathbb{C}} \sum_{a \in \mathbb{N}} (w_{(i)})_{m,a}^{\mathcal{Y}} x^{-m-1} (\log x)^a \,.$$

The following axioms hold for all $v \in V$, $w_{(i)} \in W_i$ and $w_{(j)} \in W_j$:

(I01) the *truncation condition* $(w_{(i)})_{m,a}^{\mathcal{Y}} w_{(j)} = 0$ for all m with $\mathrm{Re}\, m \gg 0$, independently of a;

(Logarithmic) Intertwining Operators

(IO1) the *truncation condition* $(w_{(i)})^{\mathcal{Y}}_{m,a} w_{(j)} = 0$ for all m with $\mathrm{Re}\, m \gg 0$, independently of a;

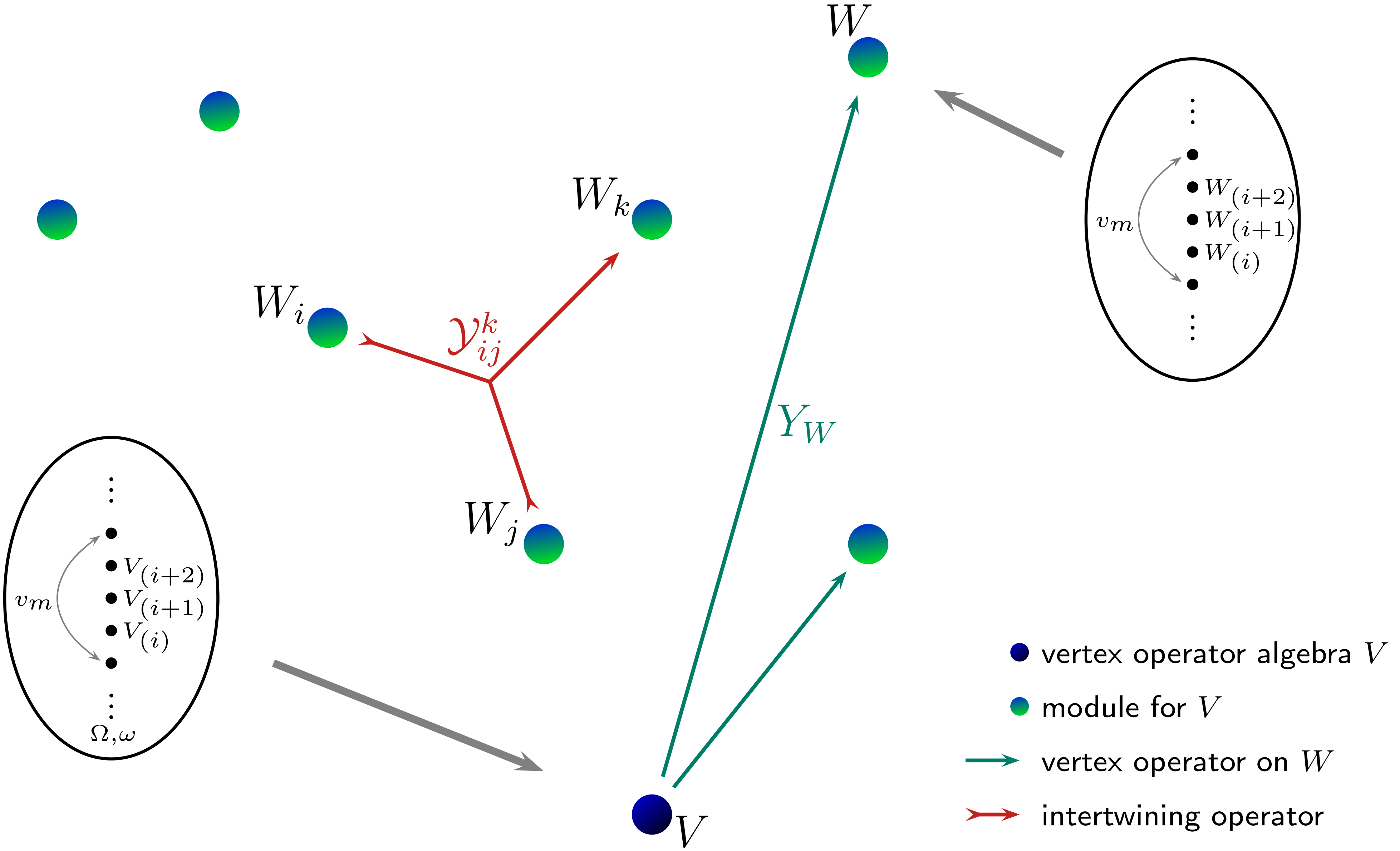
(IO2) the *Jacobi identity*

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_k(v, x_1)\mathcal{Y}^k_{ij}(w_{(i)}, x_2)w_{(j)}$$
$$- x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) \mathcal{Y}^k_{ij}(w_{(i)}, x_2)Y_j(v, x_1)w_{(j)}$$
$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) \mathcal{Y}^k_{ij}(Y_i(u, x_0)w_{(i)}, x_2)w_{(j)}\ ;$$

(IO3) the L_{-1}*-derivative property* $\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{Y}^k_{ij}(w_{(i)}, x) = \mathcal{Y}^k_{ij}(L^{W_i}_{-1} w_{(i)}, x)$.

The dimensions of the spaces of all intertwining operators $\mathcal{Y}^k_{ij}$ are called the **fusion rules** N^k_{ij}.

Visualization



Meromorphic Operator Product Expansion

An essential notion in field theories are **correlation functions** like

$$\langle w', Y(v_1, x_1)Y(v_2, x_2) \dots Y(v_n, x_n)w\rangle \ .$$

They can be used to compute *physical observables*.

For large n such computations are quite involved. **Operator product expansion** expresses the product of *two* fields as the sum of *single* fields. More precisely, the axioms yield

$$\iota_{12}^{-1} \langle w', Y(v_1, x_1)Y(v_2, x_2)w\rangle = \left(\iota_{20}^{-1} \langle w', Y(Y(v_1, x_0)v_2, x_2)w\rangle\right)\Big|_{x_0 = x_1 - x_2}$$

or expanded and shortened

$$Y(v_1, x_1)Y(v_2, x_2) \sim \sum_{i \geq 0} (x_1 - x_2)^{-i-1} Y\left((v_1)_i v_2, x_2\right) \ .$$

Nonmeromorphic Operator Product Expansion

Correlation functions for **intertwining operators** $\mathcal{Y}_{ij}^k$ are particularly important. Do they also satisfy an operator product expansion of the from

$$\iota_{12}^{-1}\langle w', Y(u,x_1)Y(v,x_2)w\rangle = \left(\iota_{20}^{-1}\langle w', Y(Y(u,x_0)v,x_2)w\rangle\right)\Big|_{x_0=x_1-x_2} ?$$

Problem: the existence of the *product* $\mathcal{Y}_1(w_1,x_1)\mathcal{Y}_2(w_2,x_2)$ does not guarantee the existence of the *iterate* $\mathcal{Y}_1(\mathcal{Y}_2(w_1,x_1-x_2)w_2,x_2)$.

This problem is solved by **$P(z)$-tensor product theory**,* replacing the ordinary vector space tensor product $\otimes$ by the more complicated operation $\boxtimes_{P(z)}$.

Motivation: Let W_1, W_2 be V-modules, then $W_1 \otimes W_2$ is a module for $V \otimes V$, but not for V. $W_1 \boxtimes_{P(z)} W_2$ *is* a module for V.

Huang, Lepowsky 1995; Huang 1995; Huang 2002; Huang, Lepowsky, Zhang 2003

Nonmeromorphic Operator Product Expansion

Theorem. Given two logarithmic intertwining maps $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of type $\binom{W_4}{W_1\ M}$ and $\binom{M}{W_2\ W_3}$, there exists a logarithmic intertwining map $\mathcal{Y}$ of type $\binom{W_4}{W_1 \boxtimes_{P(z_1-z_2)} W_2\ W_3}$ such that

$$\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle = \left\langle w_4', \mathcal{Y}\left(w_1 \boxtimes_{P(z_1-z_2)} w_2, z_2\right) w_3\right\rangle \ ,$$

if the following conditions are satisfied for a full subcategory $\mathcal{C}$ of generalized V-modules that is closed under the contragredient functor.

(1) All generalized V-modules W in $\operatorname{ob}\mathcal{C}$ are **C_1-cofinite**, i.e. $\dim(W/C_1(W)) < \infty$ with $C_1(W) = \operatorname{span}\{u_{-1}w \,|\, u \in \coprod_{m>0} V_{(m)},\ w \in W\}$.

(2) All generalized V-modules W in $\operatorname{ob}\mathcal{C}$ are quasi-finite-dimensional, i.e. $\dim \coprod_{m<R} W_{[m]} < \infty$ for all $R \in \mathbb{R}$.

(3) Every object which is a finitely generated lower-truncated generalized V-module, except that it may have infinite-dimensional homogeneous subspaces, is an object in $\mathcal{C}$.

Triplet Algebras

Many well-understood, *rational* vertex operator algebras satisfy the conditions of the theorem, e.g. the *minimal Virasoro models* and those associated to *Kac-Moody algebras*.

Also an infinite family of less-understood **logarithmic conformal field theories** can be shown to satisfy the conditions: the **triplet algebras** $\{\mathcal{W}(2,(2p-1)^{\times 3})\}_{p\geq 2}{}^{\star}$ with central charge $c_{p,1} = 1 - 6(p-1)^2/p$.

Definition. A **$\mathcal{W}$-algebra** of type $\mathcal{W}(2, h_1, \ldots, h_m)$ is a vertex operator algebra which has a minimal generating set consisting of the vacuum Ω, the conformal vector ω of weight 2 and m additional primary vectors W^i of weight h_i, $i \in \{1, \ldots, m\}$, with all singular vectors divided out.

Using further results* on vertex operator algebras, instead of studying C_1-cofiniteness of *all* modules, one "only" needs to check **C_2-cofiniteness** of the triplet algebras $V \in \{\mathcal{W}(2,(2p-1)^{\times 3})\}_{p\geq 2}$ themselves:

Triplet Algebras

$$\dim\left(V/C_2(V)\right) < \infty \quad \text{with} \quad C_2(V) = \operatorname{span}\left\{u_{-2}v \,\middle|\, u,v \in V\right\}$$

$\Longrightarrow u_{-m}v \in C_2(V)$ for all $m \geq 2$ and $v_{-m}C_2(V) \subset C_2(V)$ for all $m \geq 0$

C_2-cofiniteness is easily proven for the first triplet algebra with $p = 2$, as the commutators $[L_m, L_n]$, $[L_m, W^a_n]$, $[W^a_m, W^b_n]$ and the **singular vectors**

$$N^{ab} = W^a_{-3}W^b_{-3}\Omega - \delta_{ab}\left(\frac{8}{9}L^3_{-2} + \frac{19}{36}L^2_{-3} + \frac{14}{9}L_{-4}L_{-2} - \frac{16}{9}L_{-6}\right)\Omega$$
$$+ \mathrm{i}\varepsilon_{abc}\left(-2W^c_{-4}L_{-2} + \frac{5}{4}W^c_{-6}\right)\Omega$$

at level $4p - 2 = 6$ are explicitly known,* i.e.

$$L_m N^{ab} = 0 = W^c_m N^{ab} \quad \text{for all } m \in \mathbb{Z}_+ \,.$$

Triplet Algebras

Proposition. The vertex operator algebra $\mathcal{W}(2,3^{\times 3})$ is C_2-cofinite and the nonmeromorphic operator product expansion exists.

For V_{2p-1} with $p \geq 3$ neither commutators nor singular vectors are explicitly known, and computing them directly is very laborious — for each p separately!

Problem: Prove C_2-cofiniteness with very little information — for all p at once.

1^{st} *step*: Analyze the **characters***

$$\chi_{V_{2p-1}}(q) = \mathrm{tr}_{V_{2p-1}} q^{L_0 - c_{p,1}/24} = \frac{q^{-1/24}}{\varphi(q)} \sum_{n\in\mathbb{Z}} (2n+1) q^{(2np+p-1)^2/(4p)}$$

in detail to obtain information on singular vectors.

Triplet Algebras

$$\chi_{V_{2p-1}}(q) = \frac{q^{-1/24}}{\varphi(q)} \sum_{n\in\mathbb{Z}} (2n+1) q^{(2np+p-1)^2/(4p)}$$

This should be compared with the characters which pertain to the *vacuum Verma module* divided by the ideal generated by three pure Virasoro singular vectors:

$$\tilde{\chi}_{2p-1}(q) = q^{-c_{p,1}/24} \left(\frac{1}{\varphi_2(q)} + \frac{3q^{2p-1}(1-q^3)}{\varphi(q)(\varphi_{2p-1}(q))^2} \right)$$

where $\varphi_k(q) = \varphi(q) \prod_{l=1}^{k-1} (1-q^l)^{-1}$, $\varphi(q) = q^{-1/24}\eta(\tau)$ and $q = \mathrm{e}^{2\pi \mathrm{i} \tau}$.

$$\chi_{V_{2p-1}}(q) = \frac{q^{-c_{p,1}/24}}{\varphi(q)} \left(1 - q + 3q^{2p-1} - 3q^{2p+2} + \mathcal{O}(q^{6p-2})\right)$$

$$\tilde{\chi}_{2p-1}(q) = \frac{q^{-c_{p,1}/24}}{\varphi(q)} \left(1 - q + 3q^{2p-1} - 3q^{2p+2} + 6q^{4p-2} + \mathcal{O}(q^{4p-1})\right)$$

Triplet Algebras

For example:

$$q^{c_{2,1}/24}\,\chi_{V_3}(q) = 1 + q^2 + 4q^3 + 5q^4 + 8q^5 + 10q^6 + 16q^7 + 22q^8 + \dots$$

$$q^{c_{2,1}/24}\,\tilde{\chi}_{V_3}(q) = 1 + q^2 + 4q^3 + 5q^4 + 8q^5 + 16q^6 + 28q^7 + \dots$$

$$q^{c_{3,1}/24}\,\chi_{V_5}(q) = 1 + q^2 + q^3 + 2q^4 + 5q^5 + 7q^6 + 10q^7 + 13q^8 + 20q^9 + 27q^{10} + 38q^{11} + 51q^{12} + 69q^{13} + \dots$$

$$q^{c_{3,1}/24}\,\tilde{\chi}_{V_5}(q) = 1 + q^2 + q^3 + 2q^4 + 5q^5 + 7q^6 + 10q^7 + 13q^8 + 20q^9 + 33q^{10} + 50q^{11} + 75q^{12} + 105q^{13} + \dots$$

Proposition. For all $p \in \mathbb{Z}_+$, the triplet algebra $\mathcal{W}(2,(2p-1)^{\times 3})$ has six singular vectors at level $4p-2$ of the form

$$N^{ab} = W^a_{-2p+1} W^b_{-2p+1}\Omega + \delta_{ab}\big(\text{Virasoro polynomial}\big)\Omega + \varepsilon_{abc}\big(\text{Virasoro-}W^c_m \text{ polynomial}\big)\Omega\,.$$

Triplet Algebras

Proposition. For all $p \in \mathbb{Z}_+$, the triplet algebra $\mathcal{W}(2,(2p-1)^{\times 3})$ has six singular vectors at level $4p-2$ of the form

$$N^{ab} = W^a_{-2p+1} W^b_{-2p+1}\Omega + \delta_{ab}\big(\alpha L_{-2}^{2p-1} + \text{Virasoro polynomial}\big)\Omega$$
$$+ \varepsilon_{abc}\big(\text{Virasoro-}W^c_m \text{ polynomial}\big)\Omega\,.$$

2^{nd} *step.* Try to find out a little more about N^{ab} in order to generalize the proof for $p=2$.

Lemma. $\alpha \neq 0$.

Theorem. **For all $p \in \mathbb{Z}_{\geq 2}$, the nonmeromorphic operator product expansion exists and is associative for the vertex operator algebra $\mathcal{W}(2,(2p-1)^{\times 3})$. Furthermore, all these vertex operator algebras are C_2-cofinite.**

C_2-cofiniteness

$\dim \left(V/C_2(V) \right) < \infty$ with $C_2(V) = \text{span}\left\{ u_{-2}v \,\middle|\, u, v \in V \right\}$

Why is this property so interesting?

- crucial for convergence and *modular covariance of characters**
- crucial for Huang's proof of the *Verlinde conjecture**
- finite fusion rules*
- finitely many inequivalent irreducible modules*
- every weak module is a direct sum of generalized eigenspaces of L_0*

C_2-cofiniteness

$$\dim\left(V/C_2(V)\right) < \infty \quad \text{with} \quad C_2(V) = \operatorname{span}\left\{u_{-2}v \;\middle|\; u, v \in V\right\}$$

- vertex operator algebras are *very big* in many respects:
  - ▷ infinite-dimensional vector spaces $V = \coprod_{m\in\mathbb{Z}} V_{(m)}$
  - ▷ infinite-dimensional associated Lie algebras
  - ▷ infinitely many products $(u, v) \mapsto u_m v$, $m \in \mathbb{Z}$
  - ▷ ...
  - ▷ often infinitely many (big) modules

It is desirable that all modules can be concisely organized.

C_2-cofiniteness and Rationality

A vertex operator algebra V is called **rational** if

(i) there are only finitely many **irreducible** V-modules;

(ii) any V-module is completely reducible;

(iii) all *fusion rules* for V-modules are finite, i.e. $N_{ij}^k < \infty$.

Conjecture: C_2-cofiniteness and rationality are equivalent.

This conjecture is wrong, the triplet algebras present counter-examples.

New conjecture: C_2-cofiniteness is equivalent to "rationality" in the sense that a finite set $\mathcal{S}$ of (generalized) modules *closes under fusion*, i.e. for $W_i, W_j \in \mathcal{S}$, $N_{ij}^k = 0$ if $W_k \notin \mathcal{S}$.

Conclusion

Theorem. For all $p \in \mathbb{Z}_{\geq 2}$, the nonmeromorphic operator product expansion exists and is associative for the vertex operator algebra $\mathcal{W}(2, (2p-1)^{\times 3})$. Furthermore, all these vertex operator algebras are C_2-cofinite.

- Vertex operator algebras are fundamental in conformal field theory.
- Operator product expansion and correlation functions
- C_2-cofiniteness as an important finiteness property
- upper bounds on the dimensions of the Zhu algebras
- Many logarithmic conformal field theories have very nice properties!