Four-Point Functions in LCFT

Crossing Symmetry and SL(2, \(\mathbb{C}\)) covariance

Michael Flohr

Physics Institute
University of Bonn

Marco Krohn

Institute for Theoretical Physics
University of Hannover
Global Conformal Invariance
Ward Identities

Correlation functions have to satisfy the global conformal Ward identities, i.e. for $m = -1, 0, 1$ we must have

\[
0 = L_m \langle \Psi_1(z_1) \ldots \Psi_n(z_n) \rangle \\
= \sum_{i=1}^{n} \bar{z}_i^m \left[ z_i \partial_i + (m+1)(h_i + \hat{\delta}_h) \right] \langle \Psi_1(z_1) \ldots \Psi_n(z_n) \rangle.
\]
Ward Identities

Correlation functions have to satisfy the global conformal Ward identities, i.e. for \( m = -1, 0, 1 \) we must have

\[
0 = L_m \langle \Psi_1(z_1) \ldots \Psi_n(z_n) \rangle \\
= \sum_{i=1}^{n} z_i^m \left[ z_i \partial_i + (m + 1)(h_i + \hat{\delta}_{hi}) \right] \langle \Psi_1(z_1) \ldots \Psi_n(z_n) \rangle .
\]

In case of rank \( r > 1 \) Jordan cells of indecomposable representations with respect to \( Vir \), we have

\[
\hat{\delta}_{hi} \Psi_{(h_j; k_j)} = \begin{cases} 
\delta_{i,j} \Psi_{(h_j; k_j-1)} & \text{if } 1 \leq k_j \leq r - 1 , \\
0 & \text{if } k_j = 0 .
\end{cases}
\]
Ward Identities

- Correlation functions have to satisfy the *global conformal Ward identities*, i.e. for \( m = -1, 0, 1 \) we must have

\[
0 = L_m \langle \Psi_1(z_1) \ldots \Psi_n(z_n) \rangle
= \sum_{i=1}^{n} z_i^m \left[ z_i \partial_i + (m + 1)(h_i + \hat{\delta}_{h_i}) \right] \langle \Psi_1(z_1) \ldots \Psi_n(z_n) \rangle.
\]

- In case of rank \( r > 1 \) Jordan cells of indecomposable representations with respect to \( \text{Vir} \), we have

\[
\hat{\delta}_{h_i} \Psi(h_j; k_j) = \begin{cases} 
\delta_{i,j} \Psi(h_j; k_j - 1) & \text{if } 1 \leq k_j \leq r - 1, \\
0 & \text{if } k_j = 0.
\end{cases}
\]

- Equivalently, \( L_0 |h; k\rangle = h |h; k\rangle + (1 - \delta_{k,0}) |h; k - 1\rangle \).
Ward identities become *inhomogeneous* in LCFT. The inhomogeneities are given by correlation functions with *total Jordan-level* \( K = \sum_{i=1}^{n} k_i \) decreased by one,

\[
\langle \Psi(h_1;k_1)(z_1) \ldots \Psi(h_n;k_n)(z_n) \rangle \equiv \langle k_1 k_2 \ldots k_n \rangle ,
\]

\[
\frac{1}{(m + 1)} L'_m \langle k_1 k_2 \ldots k_n \rangle = - z_1^m \langle k_1 - 1, k_2 \ldots k_n \rangle - z_2^m \langle k_1, k_2 - 1, k_3 \ldots k_n \rangle - \ldots - z_n^m \langle k_1 \ldots k_{n-1}, k_n - 1 \rangle .
\]
Ward identities become \textit{inhomogeneous} in LCFT. The inhomogeneities are given by correlation functions with \textit{total Jordan-level} $K = \sum_{i=1}^{n} k_i$ decreased by one,

\[
\langle \Psi_{(h_1;k_1)}(z_1) \ldots \Psi_{(h_n;k_n)}(z_n) \rangle \equiv \langle k_1 k_2 \ldots k_n \rangle ,
\]

\[
\frac{1}{(m+1)} L'_m \langle k_1 k_2 \ldots k_n \rangle = - z_1^m \langle k_1 - 1, k_2 \ldots k_n \rangle
\]

\[
- z_2^m \langle k_1, k_2 - 1, k_3 \ldots k_n \rangle
\]

\[
- \ldots
\]

\[
- z_n^m \langle k_1 \ldots k_{n-1}, k_n - 1 \rangle .
\]

We obtain a hierarchical scheme of solutions, starting with correlators of total Jordan-level $K = r - 1$. 
Correlators

Generic form of 1, 2 and 3pt functions for fields forming Jordan cells, pre-logarithmic fields and fermionic fields in arbitrary rank $r$ LCFT uniquely fixed:
Correlators

- Generic form of 1, 2 and 3pt functions for fields forming Jordan cells, pre-logarithmic fields and fermionic fields in arbitrary rank $r$ LCFT uniquely fixed:

\[
\langle \Psi(h;k) \rangle = \delta_{h,0} \delta_{k,r-1},
\]

\[
\langle \Psi(h;k)(z)\Psi(h';k')(0) \rangle = \delta_{hh'} \sum_{j=r-1}^{k+k'} D_{(h;j)} \sum_{0 \leq i \leq k, 0 \leq i' \leq k', \atop i + i' = k + k' - j} \frac{(\partial h)^i}{i!} \frac{(\partial h')^{i'}}{i'!} z^{-h-h'},
\]

\[
\langle \Psi(h_1;k_1)(z_1)\Psi(h_2;k_2)(z_2)\Psi(h_3;k_3)(z_3) \rangle = \sum_{j=r-1}^{k_1+k_2+k_3} C_{(h_1 h_2 h_3;j)}
\]

\[
\times \sum_{0 \leq i_l \leq k_l, l=1,2,3, \atop i_1 + i_2 + i_3 = k_1 + k_2 + k_3 - j} \frac{(\partial h_1)^{i_1}}{i_1!} \frac{(\partial h_2)^{i_2}}{i_2!} \frac{(\partial h_3)^{i_3}}{i_3!} \prod_{\sigma \in S_3, \sigma(1) < \sigma(2)} (z_{\sigma(1)} \sigma(2))^{h_{\sigma(3)} - h_{\sigma(1)} - h_{\sigma(2)}},
\]
Four-Point Functions
Beyond 3pt Functions

- To find a useful algorithm to fix the generic form of 4pt functions, visualize a logarithmic field $\Psi_{(h;k)}$ by a vertex with $k$ outgoing lines.
To find a useful algorithm to fix the generic form of 4pt functions, visualize a logarithmic field $\Psi_{(h;k)}$ by a vertex with $k$ outgoing lines.

Contractions of logarithmic fields give rise to logarithms in the correlators. The possible powers with which $\log(z_{ij})$ may occur, can be determined by graph combinatorics.
To find a useful algorithm to fix the generic form of 4pt functions, visualize a logarithmic field $\Psi_{(h;k)}$ by a vertex with $k$ outgoing lines.

Contractions of logarithmic fields give rise to logarithms in the correlators. The possible powers with which $\log(z_{ij})$ may occur, can be determined by graph combinatorics.
Graphs

- Terms of generic form of $n$-pt function given by sum over all admissible graphs subject to the rules:
Graphs

- Terms of generic form of $\mathcal{N}$-pt function given by sum over all admissible graphs subject to the rules:

  - Each $k_{\text{out}}$-vertex may receive $k_{\text{in}}' \leq (r - 1)$ lines.
Graphs

- Terms of generic form of $n$-pt function given by sum over all admissible graphs subject to the rules:
  - Each $k_{\text{out}}$-vertex may receive $k_{\text{in}}' \leq (r - 1)$ lines.
  - Vertices with $k_{\text{out}} = 0$ (proper primary fields) do not receive any legs.
Graphs

Terms of generic form of \(n\)-pt function given by sum over all admissible graphs subject to the rules:

- Each \(k_{\text{out}}\)-vertex may receive \(k_{\text{in}}' \leq (r - 1)\) lines.
- Vertices with \(k_{\text{out}} = 0\) (proper primary fields) do not receive any legs.
- Vertex \(i\) can receive legs from vertex \(j\) only for \(j \neq i\).
Graphs

Terms of generic form of $n$-pt function given by sum over all admissible graphs subject to the rules:

- Each $k_{\text{out}}$-vertex may receive $k_{\text{in}}' \leq (r - 1)$ lines.
- Vertices with $k_{\text{out}} = 0$ (proper primary fields) do not receive any legs.
- Vertex $i$ can receive legs from vertex $j$ only for $j \neq i$.
- Precisely $r - 1$ lines in correlator remain open.
Terms of generic form of $n$-pt function given by sum over all admissible graphs subject to the rules:

- Each $k_{\text{out}}$-vertex may receive $k_{\text{in}}' \leq (r - 1)$ lines.
- Vertices with $k_{\text{out}} = 0$ (proper primary fields) do not receive any legs.
- Vertex $i$ can receive legs from vertex $j$ only for $j \neq i$.
- Precisely $r - 1$ lines in correlator remain open.

**Example:** 4pt function for $r = 2$ and all fields logarithmic yields, up to permutations, the graphs
The Algorithm

- Linking numbers $A_{ij}(g)$ of given graph $g$ yield upper bounds for power with which logarithms occur.
The Algorithm

- Linking numbers $A_{ij}(g)$ of given graph $g$ yield upper bounds for power with which logarithms occur.

- **Recursive procedure:** start with all ways $f_i$ to choose $r - 1$ free legs, find at each level $K'$ and for each configuration $f_i$ all graphs, which connect the remaining $K - K' - (r - 1)$ legs to vertices.

```latex
\text{Recursive procedure:} \quad \text{start with all ways } f_i \text{ to choose } r - 1 \\
\text{free legs, find at each level } K' \text{ and for each configuration } f_i \\
\text{all graphs, which connect the remaining } K - K' - (r - 1) \text{ legs to vertices.}
```
The Algorithm

- Linking numbers $A_{ij}(g)$ of given graph $g$ yield upper bounds for power with which logarithms occur.

- **Recursive procedure:** start with all ways $f_i$ to choose $r - 1$ free legs, find at each level $K'$ and for each configuration $f_i$ all graphs, which connect the remaining $K - K' - (r - 1)$ legs to vertices.

- Write down corresponding monomial in $\log(z_{ij})$, multiplied with an as yet undetermined constant $C(g)$ for each graph $g$. 
The Algorithm

- Linking numbers $A_{ij}(g)$ of given graph $g$ yield *upper bounds* for power with which logarithms occur.

- **Recursive procedure**: start with all ways $f_i$ to choose $r - 1$ free legs, find at each level $K'$ and for each configuration $f_i$ all graphs, which connect the remaining $K - K' - (r - 1)$ legs to vertices.

- Write down corresponding monomial in $\log(z_{ij})$, multiplied with an as yet undetermined constant $C(g)$ for each graph $g$.

- Determine some constants by imposing global conformal invariance.
The Algorithm

- Linking numbers $A_{ij}(g)$ of given graph $g$ yield upper bounds for power with which logarithms occur.

- **Recursive procedure:** start with all ways $f_i$ to choose $r - 1$ free legs, find at each level $K'$ and for each configuration $f_i$ all graphs, which connect the remaining $K - K' - (r - 1)$ legs to vertices.

- Write down corresponding monomial in $\log(z_{ij})$, multiplied with an as yet undetermined constant $C(g)$ for each graph $g$.

- Determine some constants by imposing global conformal invariance.

- Fix further constants by imposing admissible permutation symmetries among the Jordan-levels $k_i$. 
Generic Form

- Generic form of the LCFT 4pt functions is

\[
\langle k_1k_2k_3k_4 \rangle \equiv \langle \Psi(h_1;k_1)(z_1) \ldots \Psi(h_4;k_4)(z_4) \rangle = 
\]

\[
\prod_{i<j} \left( z_{ij} \right)^{\mu_{ij}} \sum_{(k'_1,k'_2,k'_3,k'_4)} \left[ \sum_{g \in G_{K-K'}} C(g) \left( \prod_{i<j} \log A_{ij}(g)(z_{ij}) \right) \right] F_{k'_1k'_2k'_3k'_4}(x),
\]

where
Generic form of the LCFT 4pt functions is

\[
\langle k_1 k_2 k_3 k_4 \rangle \equiv \langle \Psi(h_1;k_1)(z_1) \cdots \Psi(h_4;k_4)(z_4) \rangle = \\
\prod_{i<j} (z_{ij})^{\mu_{ij}} \sum_{(k'_1,k'_2,k'_3,k'_4)} \left[ \sum_{g \in G_{K-K'}} C(g) \left( \prod_{i<j} \log^A_{ij}(g)(z_{ij}) \right) \right] F_{k'_1 k'_2 k'_3 k'_4}(x),
\]

where

- \( G_{K-K'} \) is set of graphs for \((k_1 - k'_1, \ldots, k_4 - k'_4)\),

- \( \mu_{ij} \) is linking number of vertices \(i,j\) of graph \(g\),

- \( x = z_{12} z_{34} z_{14} z_{23} \),

- \( F_{k'_1 k'_2 k'_3 k'_4}(x) \) is the crossing ratio.
Generic form of the LCFT 4pt functions is

\[
\langle k_1 k_2 k_3 k_4 \rangle \equiv \langle \Psi(h_1;k_1)(z_1) \ldots \Psi(h_4;k_4)(z_4) \rangle =
\prod_{i<j} (z_{ij})^{\mu_{ij}} \sum_{(k'_1,k'_2,k'_3,k'_4)} \left[ \sum_{g \in G_{K-K'}} C(g) \left( \prod_{i<j} \log A_{ij}(g)(z_{ij}) \right) \right] F_{k'_1 k'_2 k'_3 k'_4}(x),
\]

where

- \(G_{K-K'}\) is set of graphs for \((k'_1 - k_1, \ldots, k'_4 - k_4)\),
- \(A_{ij}(g)\) is linking number of vertices \(i, j\) of graph \(g\),
Generic Form

Generic form of the LCFT 4pt functions is

\[
\langle k_1 k_2 k_3 k_4 \rangle \equiv \langle \Psi(h_1; k_1)(z_1) \ldots \Psi(h_4; k_4)(z_4) \rangle = 
\]

\[
\prod_{i<j} \langle z_{ij} \rangle^{\mu_{ij}} \sum_{(k_1', k_2', k_3', k_4')} \left[ \sum_{g \in G_K - K'} C(g) \left( \prod_{i<j} \log A_{ij}(g)(z_{ij}) \right) \right] F_{k_1' k_2' k_3' k_4'}(x),
\]

where

- \( G_K - K' \) is set of graphs for \((k_1 - k_1', \ldots, k_4 - k_4')\),
- \( A_{ij}(g) \) is linking number of vertices \(i, j\) of graph \(g\),
- \( x \) is the crossing ratio \(x = \frac{z_{12}z_{34}}{z_{14}z_{23}}\).
Generic form of the LCFT 4pt functions is

$$\langle k_1 k_2 k_3 k_4 \rangle \equiv \langle \Psi(h_1;k_1)(z_1) \ldots \Psi(h_4;k_4)(z_4) \rangle =$$

$$\prod_{i<j} (z_{ij})^{\mu_{ij}} \sum_{(k'_1,k'_2,k'_3,k'_4)} \left[ \sum_{g \in G_{K-K'}} C(g) \left( \prod_{i<j} \log A_{ij}(g)(z_{ij}) \right) \right] F_{k'_1 k'_2 k'_3 k'_4}(x),$$

where

- $G_{K-K'}$ is set of graphs for $(k_1 - k'_1, \ldots, k_4 - k'_4)$,
- $A_{ij}(g)$ is linking number of vertices $i, j$ of graph $g$,
- $x$ is the crossing ratio $x = \frac{z_{12} z_{34}}{z_{14} z_{23}},$
- $\mu_{ij}$ is typically $\mu_{ij} = \frac{1}{3} (\sum_k h_k) - h_i - h_j.$
Simplification and Recursion

- **Generic structure** \( \Rightarrow \) set \( h_i = 0 \).
- \( L_{-1} \) is translation invariance \( \Rightarrow \) \( z_{ij} \).
- **Initial conditions:** \( \langle k_1 k_2 k_3 k_4 \rangle = F_0(x) \) for \( \sum_i k_i = r - 1 \)
- and \( \langle k_1 k_2 k_3 k_4 \rangle = 0 \) for \( \sum_i k_i < r - 1 \).

\[
O_0 \langle \ldots \rangle \equiv \sum_i z_i \partial_i \langle \ldots \rangle = -\sum_i \hat{\delta} h_i \langle \ldots \rangle \\
O_1 \langle \ldots \rangle \equiv \sum_i z_i^2 \partial_i \langle \ldots \rangle = -2 \sum_i z_i \hat{\delta} h_i \langle \ldots \rangle
\]
**Global Conformal Invariance**

**Four-Point Functions**

- Beyond 3pt Functions
- Graphs
- The Algorithm
- Generic Form
- Simplification and Recursion
- Rank $r = 2$
- Redundancy

**Symmetries & Graphs**

### Simplification and Recursion

- **Generic structure** $\implies$ set $h_i = 0$.
- $L_{-1}$ is translation invariance $\implies z_{ij}$.
- **Initial conditions:**
  - $\langle k_1 k_2 k_3 k_4 \rangle = F_0(x)$ for $\sum_i k_i = r - 1$
  - and $\langle k_1 k_2 k_3 k_4 \rangle = 0$ for $\sum_i k_i < r - 1$.

\[
O_0 \langle \ldots \rangle \equiv \sum_i z_i \partial_i \langle \ldots \rangle = -\sum_i \delta h_i \langle \ldots \rangle
\]

\[
O_1 \langle \ldots \rangle \equiv \sum_i z_i^2 \partial_i \langle \ldots \rangle = -2 \sum_i z_i \delta h_i \langle \ldots \rangle
\]

\[
\langle 0110 \rangle = \langle 1100 \rangle = \langle 1010 \rangle = \langle 2000 \rangle = F_0 \quad (l=0)
\]

\[
\langle 1110 \rangle \quad \langle 2100 \rangle \sim \langle 2010 \rangle \quad (l=1)
\]

\[
\langle 2110 \rangle \quad (l=2)
\]
Rank $r = 2$

\[
\langle 1000 \rangle = F_0
\]

\[
\langle 1100 \rangle = \mathcal{P}_{S_2} \left\{ \frac{1}{2} F_{1100} - \bullet \quad \bullet \quad F_0 \right\}
\]

\[
\langle 1110 \rangle = \mathcal{P}_{S_3} \left\{ \frac{1}{6} F_{1110} + \left( \frac{1}{2} P_{(13)} - 1 \right) \bullet \quad \bullet \quad F_{0110} + \right.
\]
\[
\quad \left[ \bullet \quad \bullet \quad - \frac{1}{2} \quad \bullet \quad \bullet \right] F_0 \right\}
\]

\[
\langle 1111 \rangle = \mathcal{P}_{S_4} \left\{ \frac{1}{24} F_{1111} + \left( \frac{1}{6} P_{(13)} - \frac{1}{3} \right) \bullet \quad \bullet \quad F_{0111} + \right.
\]
\[
\quad \left[ \frac{1}{2} \left( P_{(24)} - 1 \right) \bullet \quad \bullet \quad + \left( 1 - \frac{1}{2} P_{(14)} \right) \bullet \quad \bullet \quad - \frac{1}{4} \quad \bullet \quad \bullet \right] F_{0011} + \right.
\]
\[
\quad \left[ \frac{1}{2} \quad \bullet \quad \bullet \quad + \frac{1}{3} \quad \bullet \quad \bullet \quad - \bullet \quad \bullet \right] F_0 \right\}.
\]
Interestingly, there remain \textit{free constants}, when all four fields are logarithmic!

\[
\langle 1111 \rangle = F_{1111} + \mathcal{P}_{(1234)} \left\{ \left[ (\ell_{12} - \ell_{34} + \ell_{23} + \ell_{14})C_1 + (\ell_{13} + \ell_{24} - \ell_{12} - \ell_{34})C_2 \\
- \ell_{14} + \ell_{34} - \ell_{13} \right] F_{0111} \right\} \\
+ \mathcal{P}_{(12)(34)} \left\{ \left[ (\ell_{13} + \ell_{24} - \ell_{14} - \ell_{23} + 2(-\ell_{13}\ell_{24} - \ell_{12}\ell_{24} + \ell_{34}\ell_{14} + \ell_{13}\ell_{24} \\
- \ell_{13}\ell_{34} + \ell_{23}\ell_{34} + \ell_{12}\ell_{23} - \ell_{12}\ell_{13} - \ell_{23}\ell_{14} + \ell_{12}\ell_{14})C_3 \\
+ (-\ell_{23} + \ell_{14})^2 + \ell_{23}\ell_{34} + \ell_{12}\ell_{14} - \ell_{13}\ell_{34} + \ell_{34}\ell_{14} + \ell_{13}\ell_{14} \\
- \ell_{34}\ell_{24} - \ell_{12}\ell_{13} - \ell_{12}\ell_{24} + \ell_{23}\ell_{24} + \ell_{23}\ell_{13} + \ell_{12}\ell_{23} + \ell_{24}\ell_{14})C_4 \\
- \ell_{23}^2 - \ell_{12}^2 - \ell_{14}^2 + 2\ell_{23}\ell_{34} + 2\ell_{34}\ell_{14} - 2\ell_{12}\ell_{34} - \ell_{23}\ell_{14} + \ell_{23}\ell_{24} \\
- \ell_{12}\ell_{13} + \ell_{12}\ell_{14} + \ell_{12}\ell_{23} - \ell_{12}\ell_{24} + \ell_{13}\ell_{14} + \ell_{13}\ell_{24} \right] F_{1100} \right\} \\
+ \left[ 2(\ell_{12}\ell_{24}\ell_{14} - \ell_{23}\ell_{13}\ell_{14} + \ell_{23}\ell_{34}\ell_{24} - \ell_{24}\ell_{13}\ell_{34} - \ell_{23}\ell_{34}\ell_{14} \\
- \ell_{12}\ell_{23}\ell_{34} - \ell_{12}\ell_{34}\ell_{24} - \ell_{23}\ell_{13}\ell_{24} + \ell_{12}\ell_{23}\ell_{13} + \ell_{13}\ell_{34}\ell_{14} \\
- \ell_{13}\ell_{14}\ell_{24} - \ell_{23}\ell_{24}\ell_{14} - \ell_{12}\ell_{13}\ell_{24} - \ell_{12}\ell_{23}\ell_{14} - \ell_{12}\ell_{13}\ell_{34} \\
- \ell_{12}\ell_{34}\ell_{14}) \\
+ 2(\ell_{13}\ell_{24} + \ell_{12}\ell_{34} + \ell_{14}\ell_{23} + \ell_{23}\ell_{14} + \ell_{34}\ell_{12} + \ell_{24}\ell_{13}) \right] F_0
\]
Symmetries & Graphs
Additional Constants

\[ O_0 \left< k_1 k_2 k_3 k_4 \right> = - \sum_i \hat{\delta}_{h_i} \left< k_1 k_2 k_3 k_4 \right>, \]
\[ O_1 \left< k_1 k_2 k_3 k_4 \right> = -2 \sum_i z_i \hat{\delta}_{h_i} \left< k_1 k_2 k_3 k_4 \right>. \]

Are there any \( f \in F_{\log} \) with \( O_0 f = 0 \) and \( O_1 f = 0 \)?
Additional Constants

\[ O_0 \langle k_1 k_2 k_3 k_4 \rangle = -\sum_i \delta_h \langle k_1 k_2 k_3 k_4 \rangle, \]
\[ O_1 \langle k_1 k_2 k_3 k_4 \rangle = -2 \sum_i z_i \delta_h \langle k_1 k_2 k_3 k_4 \rangle. \]

Are there any \( f \in F_{\log} \) with \( O_0 f = 0 \) and \( O_1 f = 0 \)?

\[ \ker_{F_{\log}, g} O = \left\{ \sum_{i=0}^{g} a_i K_1^i K_2^{g-i} : a_k \in \mathbb{R} \right\} \]

\[ K_1 \equiv l_{12} + l_{34} - l_{13} - l_{24} = \log |x| - \log |1 - x| \]
\[ K_2 \equiv l_{12} + l_{34} - l_{14} - l_{23} = \log |x|. \]

This means that all four fields have to be of logarithmic type. Symmetry considerations and combinatorial restrictions in fact constrain the number of additional constants further.
Additional Constants

\[ O_0 \langle k_1 k_2 k_3 k_4 \rangle = - \sum_i \hat{\delta}_{h_i} \langle k_1 k_2 k_3 k_4 \rangle, \]
\[ O_1 \langle k_1 k_2 k_3 k_4 \rangle = -2 \sum_i z_i \hat{\delta}_{h_i} \langle k_1 k_2 k_3 k_4 \rangle. \]

Are there any \( f \in \mathcal{F}_{\log} \) with \( O_0 f = 0 \) and \( O_1 f = 0 \)?

\[ \ker_{\mathcal{F}_{\log}, g} O = \left\{ \sum_{i=0}^{g} a_i K_1^i K_2^{g-i} : a_k \in \mathbb{R} \right\} \]
\[ K_1 \equiv l_{12} + l_{34} - l_{13} - l_{24} = \log |x| - \log |1 - x| \]
\[ K_2 \equiv l_{12} + l_{34} - l_{14} - l_{23} = \log |x|. \]

This means that all four fields have to be of logarithmic type. Symmetry considerations and combinatorical restrictions in fact constrain the number of additional constants further.

\[ \text{Ker}_{1111} = \mathcal{P}_{S_4} \left\{ (K_1^2 - K_1 K_2 + K_2^2)F_{0011} \right\}. \]
Now, the final results simply read

\[
\langle 1110 \rangle = F_{1110} - P_{S_3}(\ell_{12})F_{0011} + P_{S_3}(2\ell_{12}\ell_{23} - \ell_{12}^2)F_0 \\
= F_{1110} - P_{S_3}(\bullet \bullet \bullet)F_{0011} + P_{S_3}(2\bullet \bullet \bullet - \bigcirc \bullet \bullet \bullet)F_0 ,
\]

\[
\langle 1111 \rangle = F_{1111} - \frac{1}{6} P_{S_4}(\bullet \bullet \bullet \bullet)F_{0111} + \frac{1}{4} P_{S_4}(\bullet \bullet \bullet + K_{S_4}^{(2)})F_{0011} \\
+ P_{S_4}(\frac{1}{2} \bigcirc \bullet + \frac{1}{3} \bigcirc \bullet \bullet \bullet - \bullet \bullet \bullet \bullet )F_0 .
\]
Graphical Solution

Now, the final results simply read

\[
\langle 1110 \rangle = F_{1110} - \mathcal{P}_3(\ell_{12})F_{0011} + \mathcal{P}_3(2\ell_{12}\ell_{23} - \ell_{12}^2)F_0
\]

\[
= F_{1110} - \mathcal{P}_3(\cdots \cdots)F_{0011} + \mathcal{P}_3(2\cdots \cdots - \bullet \cdots \cdots)F_0 ,
\]

\[
\langle 1111 \rangle = F_{1111} - \frac{1}{6}\mathcal{P}_4(\cdots \cdots)F_{0111} + \frac{1}{4}\mathcal{P}_4(\cdots \cdots + K^{(2)}_{S_4})F_{0011}
\]

\[
+ \mathcal{P}_4(\frac{1}{2} \bullet \cdots + \frac{1}{3} \cdot \cdots \cdots)F_0 .
\]

Note the appearance of a term \( K \in \ker L^{\text{offdiag}}_m \) in the kernel of the nilpotent part of the Virasoro generators:

\[
\ker(L_m - L'_m) = \langle K_1 \equiv \log(x), \ K_2 \equiv -\log(1 - 1/x) \rangle
\]

\[
= \langle \ell_{12} + \ell_{34} - \ell_{14} - \ell_{23}, \ell_{12} + \ell_{34} - \ell_{13} - \ell_{24} \rangle ;
\]

\[
K^{(2)}_{S_4} = K_1^2 - K_1K_2 + K_2^2 .
\]
Again, if all $k_i > 0$, free constants remain:

$$\langle 1111 \rangle = F_{1111} + \mathcal{P}_{(1234)} \{ [(l_{13} - l_{12} + l_{24} - l_{34}) C_1 \\
+ (l_{23} + l_{14} - l_{34} - l_{12}) C_2 - l_{14} + l_{24} - l_{12}] F_{0111} \} + \left[ (l_{12}^2 + l_{24}^2 + l_{34}^2 + l_{13}^2 + 2(l_{12} l_{13} + l_{13} l_{24} - l_{13} l_{34} \\
- l_{34} l_{24} - l_{12} l_{24} + l_{12} l_{34})) C_3 \\
+ (-2 l_{13} l_{14} + l_{24}^2 + 2 l_{23} l_{14} - 2 l_{23} l_{24} + l_{23}^2 + 2 l_{13} l_{24} \\
+ l_{13}^2 - 2 l_{23} l_{13} + l_{14}^2 - 2 l_{24} l_{14}) C_4 \\
+ ((l_{24} + l_{13})^2 \\
+ l_{12} l_{14} - l_{23} l_{24} - l_{12} l_{24} - l_{24} l_{14} - l_{23} l_{13} + l_{34} l_{14} \\
- l_{13} l_{34} - l_{13} l_{14} + l_{23} l_{34} - l_{34} l_{24} + l_{12} l_{23} - l_{12} l_{13}) C_5 \\
+ 2(l_{13} l_{24} + l_{23} l_{14} + l_{12} l_{34}) \} F_0.$$
Again, if all $k_i > 0$, free constants remain:

\[
\langle 1111 \rangle = F_{1111} + \mathcal{P}_{(1234)} \{ [(l_{13} - l_{12} + l_{24} - l_{34})C_1 \\
+ (l_{23} + l_{14} - l_{34} - l_{12})C_2 - l_{14} + l_{24} - l_{12}]F_{0111} \} \\
+ \left[ (l_{12}^2 + l_{24}^2 + l_{34}^2 + l_{13}^2 + 2(l_{12}l_{13} + l_{13}l_{24} - l_{13}l_{34} \\
- l_{34}l_{24} - l_{12}l_{24} + l_{12}l_{34}))C_3 \\
+ \left( -2l_{13}l_{14} + l_{24}^2 + 2l_{23}l_{14} - 2l_{23}l_{24} + l_{23}^2 + 2l_{13}l_{24} \\
+ l_{13}^2 - 2l_{23}l_{13} + l_{14}^2 - 2l_{24}l_{14} \right)C_4 \\
+ \left( (l_{24} + l_{13})^2 \\
+ l_{12}l_{14} - l_{23}l_{24} - l_{12}l_{24} - l_{24}l_{14} - l_{23}l_{13} + l_{34}l_{14} \\
- l_{13}l_{34} - l_{13}l_{14} + l_{23}l_{34} - l_{34}l_{24} + l_{12}l_{23} - l_{12}l_{13} \right)C_5 \\
+ 2(l_{13}l_{24} + l_{23}l_{14} + l_{12}l_{34}) \} F_0 \} \\
\langle 1111 \rangle = F_{1111} - \frac{1}{6} \mathcal{P}_{S_4} (l_{12}) F_{0111} + \\
\left\{ \mathcal{P}_{S_4} \left[ -\frac{1}{4}l_{34}^2 + \frac{1}{2}l_{34}l_{24} \right] + K_{S_4}^{(2)} \right\} F_0 \}. 
\]
\begin{align*}
\langle 2222 \rangle &= \\
\mathcal{P}_{S_4} \left\{ \frac{1}{24} F_{2222} + \left( \frac{1}{6} P_{(13)} - \frac{1}{3} \right) \bullet \bullet F_{1222} + \\
\left[ \left( \frac{1}{3} P_{(12)} + \frac{1}{6} P_{(14)} \right) \bullet \bullet - \frac{1}{3} \bullet \bullet - \frac{1}{12} P_{(13)} \circ \bullet \bullet \right] F_{0222} + \\
\left[ \frac{1}{2} P_{(24)} \bullet \bullet + \left( \frac{1}{2} P_{(23)} - P_{(24)} \right) \bullet \bullet + \frac{1}{4} P_{(13)(24)} \bullet \bullet \bullet \right] F_{1122} + \\
\left[ \left( 3 P_{(34)} + 3 + P_{(14)} \right) \circ \bullet \bullet - 5 \bullet \bullet - \left( \frac{13}{6} + \frac{5}{2} P_{(34)} \right) \bullet \bullet \bullet + \right. \\
\left. \left( 5 + 3 P_{(24)} \right) \bullet \bullet - \circ \bullet \bullet \bullet - \left( 3 + \frac{3}{2} P_{(14)} \right) \bullet \bullet \bullet \right] F_{1112} + \\
\left[ \frac{1}{2} \left( P_{(124)} - 11 - 9 P_{(12)} - 7 P_{(123)} - P_{(132)} - 3 P_{(142)} - 7 P_{(14)} - 7 P_{(13)(24)} + \\
P_{(13)} - 8 P_{(14)(23)} \right) \circ \bullet \bullet + \left( 5 + \frac{3}{2} P_{(24)} + 3 P_{(14)} + \frac{9}{2} P_{(13)(24)} \right) \bullet \bullet \bullet + \right. \\
\left. \left( 6 + 7 P_{(23)} + 5 P_{(12)} \right) \bullet \bullet + \left( \frac{3}{2} + P_{(13)} + \frac{5}{4} P_{(13)(24)} \right) \bullet \bullet \bullet + \ldots \right. \\
\end{align*}
\[ \ldots + \]

\[ (2P_{(34)} + P_{(234)} - P_{(243)} + P_{(13)} - P_{(14)}) - (1 + \frac{3}{4}P_{(13)(24)}) + \]

\[ (3 + \frac{1}{4}P_{(234)} - P_{(243)} + \frac{1}{2}P_{(13)}) - (P_{(24)} + P_{(1324)}) + \]

\[ \frac{1}{2}P_{(1324)} - \frac{3}{8} - P_{(13)} + \] \[ F_{0012} \]

\[ \left\{ \begin{array}{l}
+ \quad + \quad - \quad + \quad - \quad - \quad + \\
+ \quad \frac{1}{4} \quad - \quad - \quad \frac{1}{4} \quad - \quad - \quad + \\
+ \quad \frac{1}{2} \quad - \quad - \quad \frac{1}{2} \quad - \quad - \quad + \\
+ \quad 5 \quad - \quad - \quad - \quad - \quad + \\
+ \quad \frac{1}{2} \quad + \quad + \quad + \quad + \quad + \\
\end{array} \right\} F_0 \]
Towards $c = 0$

- That is all very elegant, very short, and very simple ; – )
Towards $c = 0$

- That is all very elegant, very short, and very simple ;–)

- Remember initial conditions:
  - $\langle k_1 k_2 k_3 k_4 \rangle = 0 \quad \forall k_1 + k_2 + k_3 + k_4 < r - 1$,  
  - $\langle k_1 k_2 k_3 k_4 \rangle = \langle k'_1 k'_2 k'_3 k'_4 \rangle \quad \forall k_1 + k_2 + k_3 + k_4 = k'_1 + k'_2 + k'_3 + k'_4 = r - 1$.
  - All $\Psi_{(h;0)}$ are proper primaries.
Towards $c = 0$

- That is all very elegant, very short, and very simple ;-

- Remember initial conditions:
  - $\langle k_1 k_2 k_3 k_4 \rangle = 0 \quad \forall k_1 + k_2 + k_3 + k_4 < r - 1$
  - $\langle k'_1 k'_2 k'_3 k'_4 \rangle = \langle k'_1 k'_2 k'_3 k'_4 \rangle \quad \forall k'_1 + k'_2 + k'_3 + k'_4 = k_1 + k_2 + k_3 + k_4 = r - 1$
  - All $\Psi_{(h;0)}$ are proper primaries.

- Generalize to pre-logarithmic fields.
Towards $c = 0$

- That is all very elegant, very short, and very simple ;-

- Remember initial conditions:
  - $\langle k_1 k_2 k_3 k_4 \rangle = 0 \ \forall \ k_1 + k_2 + k_3 + k_4 < r - 1,$
  - $\langle k_1 k_2 k_3 k_4 \rangle = \langle k'_1 k'_2 k'_3 k'_4 \rangle \ \forall \ k_1 + k_2 + k_3 + k_4 = k'_1 + k'_2 + k'_3 + k'_4 = r - 1.$
  - All $\Psi_{(h;0)}$ are proper primaries.

- Generalize to pre-logarithmic fields.

- Generalize to case where not all Jordan cells have the same rank, since $c = 0$ seems to be that complicated
  [Kogan,Nichols; Gurarie,Ludwig].

That is all very elegant, very short, and very simple ;(-
Remember initial conditions:
- $\langle k_1 k_2 k_3 k_4 \rangle = 0 \ \forall \ k_1 + k_2 + k_3 + k_4 < r - 1,$
- $\langle k_1 k_2 k_3 k_4 \rangle = \langle k'_1 k'_2 k'_3 k'_4 \rangle \ \forall \ k_1 + k_2 + k_3 + k_4 = k'_1 + k'_2 + k'_3 + k'_4 = r - 1.$
- All $\Psi_{(h;0)}$ are proper primaries.

Generalize to pre-logarithmic fields.

Generalize to case where not all Jordan cells have the same rank, since $c = 0$ seems to be that complicated
[Kogan,Nichols; Gurarie,Ludwig].
Towards $c = 0$

- That is all very elegant, very short, and very simple ;-

- Remember initial conditions:
  - $\langle k_1 k_2 k_3 k_4 \rangle = 0 \quad \forall k_1 + k_2 + k_3 + k_4 < r - 1,$
  - $\langle k_1 k_2 k_3 k_4 \rangle = \langle k'_1 k'_2 k'_3 k'_4 \rangle \quad \forall k_1 + k_2 + k_3 + k_4 = k'_1 + k'_2 + k'_3 + k'_4 = r - 1.$
  - All $\Psi_{(h;0)}$ are proper primaries.

- Generalize to pre-logarithmic fields.

- Generalize to case where not all Jordan cells have the same rank, since $c = 0$ seems to be that complicated [Kogan, Nichols; Gurarie, Ludwig].

- Finally, allow for mixing of Jordan cells of different sizes [Nagi; Rasmussen].
Towards $c = 0$

- That is all very elegant, very short, and very simple ;–)
- Remember initial conditions:
  - $\langle k_1 k_2 k_3 k_4 \rangle = 0 \quad \forall \ k_1 + k_2 + k_3 + k_4 < r - 1$,
  - $\langle k_1 k_2 k_3 k_4 \rangle = \langle k'_1 k'_2 k'_3 k'_4 \rangle \quad \forall \ k_1 + k_2 + k_3 + k_4 = k'_1 + k'_2 + k'_3 + k'_4 = r - 1$.
  - All $\Psi_{(h;0)}$ are proper primaries.
- Generalize to pre-logarithmic fields.
- Generalize to case where not all Jordan cells have the same rank, since $c = 0$ seems to be that complicated [Kogan,Nichols; Gurarie,Ludwig].
- Finally, allow for mixing of Jordan cells of different sizes [Nagi; Rasmussen].
- What is the structure of the $h = 0$ sector in $c = 0$ theories?