Boundary States in Logarithmic Conformal Field Theory

— A novel Approach —

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Abstract

In this thesis, a constructive method is presented to obtain boundary states in conformal field theory. It is compatible to the usual approach via Ishibashi states in ordinary conformal field theories but extendible to cases that have a more complicated structure, such as rank-2 indecomposable Jordan cells as in logarithmic conformal field theories. In particular, it allows to study boundary states keeping the structure of the underlying bulk theory visible. Using this method the logarithmic conformal field theory with central charge $c = -2$ is studied in detail, deriving the maximal set of boundary states in this case. The analysis shows the existence of states corresponding to indecomposable representations as well as their irreducible subrepresentations. Furthermore, a new kind of boundary states emerges. So-called mixed boundary states glue together the two different irreducible representations of the $c = -2$ theory at the boundary. A relation between the boundary states is deduced that implies a deeper connection to the unique local logarithmic conformal field theory studied by M. R. Gaberdiel and H. G. Kausch. Both, the three-dimensional and the five-dimensional representation of the modular group are found when calculating the cylinder amplitudes, the latter one by introducing additional states that serve as duals to certain boundary states. Finally, the symplectic fermion model is studied to link the results to the coherent state approach of S. Kawai and J. F. Wheater.

The scientific results underlying this thesis are content of two publications that can be found in [56, 57].

Zusammenfassung

In dieser Arbeit wird eine konstruktive Methode zur Gewinnung von Randzuständen in konformen Feldtheorien aufgezeigt. Diese Methode ist kompatibel zu der Herangehensweise mittels Ishibashi-Zuständen in gewöhnlicher konformer Feldtheorie. Sie kann jedoch auch auf komplizierter strukturierte Fälle, beispielsweise unzerlegbare Rang-2 Jordanzellen, wie sie in logarithmisch konformer Feldtheorie auftreten, angewendet werden. Insbesondere ist es möglich, bei der Untersuchung von Randzuständen die innere Struktur der zugrundeliegenden Theorie zu berücksichtigen. Diese Methode wird im weiteren Verlauf auf die logarithmisch konforme Feldtheorie mit zentraler Ladung $c = -2$ zur Gewinnung des maximalen Satzes an Randzuständen angewendet. Damit kann die Existenz von Randzuständen für die unzerlegbaren Darstellungen und solchen, die den darin enthaltenen irreduziblen Unterdarstellungen zuzuordnen sind, gezeigt werden. Desweiteren tritt eine neuartige Klasse von Zuständen in Erscheinung. Sogenannte gemischte Randzustände „verkleben“ die beiden unzerlegbaren Darstellungen der $c = -2$-Theorie auf dem Rand. Der relative Bezug der gefundenen Zustände zueinander läßt einen tieferen Zusammenhang zwischen der Randtheorie und der einzigartigen lokalen logarithmisch konformen Feldtheorie nach M. R. Gaberdiel und H. G. Kausch vermuten. Sowohl die dreidimensionale als auch die fünf-dimensionale Darstellung der modularen Gruppe können über die Zylinderamplituden identifiziert werden, die letztere jedoch nur unter Zuhilfenahme zusätzlicher Zustände, die als Duale zu einigen der Randzustände eingeführt werden. Schließlich wird das Modell der symplektischen Fermionen untersucht, um eine Verbindung zwischen den präsentierten Ergebnissen und dem Lösungsansatz von S. Kawai und J. F. Wheater über kohärente Zustände herzustellen.

Contents

Introduction

Since its discovery, conformal field theory in two dimensions [1] has become one of the most important tools in modern theoretical physics. It has a huge number of applications both in string theory and in condensed matter physics, describing, e. g. two-dimensional critical phenomena [2]. The current interests in conformal field theory can be divided into two different directions:

In reality, physical systems are finite. Therefore, it is interesting to study theories on surfaces with a boundary [3, 4, 5]. Also, in string theory, these so-called boundary conformal field theories are assumed to provide the spectra of D-branes [6].

A second, independent field of interest are conformal field theories that contain logarithmic operators leading to divergent correlation functions [7, 8, 9]. Reasons for the emergence of such operators can be found, for example, in various condensed matter problems. In 1992, H. Saleur analyzed dense polymers and observed that systems involving disorder can contain density fields of scaling dimension zero [10]. These fields allow the existence of logarithmic operators. On the other hand, there are also contributions from string theory. V. Knishnik revealed already in 1987 that twist fields in ghost systems exhibit logarithmically diverging correlators [11]. Finally, also puncture operators in Liouville theory show such a behaviour as pointed out by I. I. Kogan and A. Lewis in the discussion of Coulomb gas models [12]. Applications were found also in conformal field theory approaches to the quantum Hall effect [13, 14, 15]. In particular, the famous Haldane-Rezayi quantum Hall state is described by a logarithmic conformal field theory at central charge $c = -2$ [16]. Two-dimensional turbulence yields such a behaviour [17] and recently, logarithmic correlators appeared in two-dimensional abelian sandpile models studied by S. Mathieu and P. Ruelle [18, 19]. Finally, there are contributions to Seiberg-Witten models [20]. A very extensive list of references is given by A. Nichols [21]. Lecture notes can be found in [8, 9, 22, 23].

The aim of this thesis is to combine these two different directions to a single boundary logarithmic conformal field theory. In this context, rationality plays an important role. For ordinary (rational) conformal field theories, N. Ishibashi [5] prescribed a generic way towards the computation of boundary states: A cylinder, for example, has two boundaries. These boundaries can be understood as the initial and final state of a propagating closed string. This was extended by J. L. Cardy [3] in order to obtain the set of physical relevant boundary conditions relating the coefficients of the boundary partition functions to the fusion rules of the bulk theory. Many concepts of rationality can be generalised to logarithmic conformal field theories, such as characters, fusion rules, partition function, etc. Detailed discussions are conducted in [24, 25, 26, 27, 28, 29, 30, 31, 32, 33].

Boundary states in a $c = -2$ logarithmic conformal field theory were first examined by I. I. Kogan and J. F. Wheater [34]. More recently, S. Kawai and again J. F. Wheater [35] studied boundary states of the same model using the method of symplectic fermions [30], see also [36]. A third approach towards this problem was investigated by Y. Ishimoto [37]. Logarithmic conformal field theories in the vicinity of a boundary were also discussed by A. Lewis [38] and by S. Moghimi-Araghi and S. Rouhani [39].

The results of the former three works on boundaries in the $c = -2$ case are all different and partially contradictory. This demonstrates that the much more complicated representation theory of *logarithmic* conformal field theories poses a major obstacle to a rigorous and consistent description of boundary states. For example, they naturally contain zero-norm states which cannot be neglected. This and their non-trivial inner structure make it impossible to obtain a normalised orthogonal basis of states which usually is assumed in Ishibashi-like constructions of boundary states.

This work is positioned exactly at this point. It is intended to provide a general procedure for deriving the set of boundary states in the rational $c = -2$ logarithmic conformal field theory in a mathematically consistent way. However, it is applicable to other conformal field theories as well.

Chapter 1 presents an overview of the main ideas and concepts of both logarithmic and boundary conformal field theory.

Then, chapter 2 focusses on the construction of boundary states. Therein, the above-mentioned Ishibashi and Cardy approach for ordinary rational conformal field theories is shortly revised. In the second part, this chapter is devoted to introduce an algorithm to directly calculate a complete set of boundary states from first principles, i. e. very basic features of conformal field theories. It does not make explicit use of any special features of logarithmic conformal field theory. Also, assumptions are made neither on normalisability nor orthogonality of a basis of states. Thus, it should be appropriate for applications towards generic logarithmic conformal field theories. The way the algorithm is designed, a boundary state is computed iteratively. Finally, it is shown that it produces results consistent to the Ishibashi approach in the case of ordinary conformal field theories. It is important to mention that the states derived by both Ishibashi and with the help of the presented method do not correspond to the physical boundary conditions themselves but rather form a suitable basis.

Chapter 3 begins with a short introduction to the $c = -2$ logarithmic conformal field theory under the influence of the maximally extended chiral symmetry algebra $\mathcal{W}(2, 3, 3, 3)$. This theory is closest to the notions of a rational theory. In two different approaches, the boundary states for this theory are derived: The first one is intended to give a very intuitive picture on how to deal with boundaries in this theory and leads to a direct generalisation of the Ishibashi basis of boundary states. By applying the machinery invented in chapter 2, it turns out that this theory contains more boundary states than assumed up to now. So-called *mixed* boundary states emerge which interconnect two different representations at the boundary. Starting in section 5, a deeper discussion of the structure of the newly derived boundary states is conducted. Their existence is manifested and a one-to-one correspondence between the structure of the boundary states on the one side and the bulk states on the other side is derived. This yields a possible relation between the boundary theory and the structure of the unique local logarithmic $c = -2$ conformal field theory constructed by M. R. Gaberdiel and H. G. Kausch [29]. It is remarkable that the set of boundary states shows a very similar structure, although for the construction, no relations to the local theory were assumed. The problem of a degenerate metric of the natural pairings in the space of boundary states is attempted to be fixed by introducing additional so-called *weak* boundary states. These serve as duals to some of the proper boundary states. Two propositions are made for the application of Cardy's formalism in this situation. First, a subset of four boundary states is considered that corresponds to the three-dimensional space spanned by the characters of the rational $c = -2$ theory. This set is well defined and the induced metric is non-degenerate. For this setup, it is shown that Cardy's formalism can be applied precisely as for ordinary rational conformal field theories. Secondly, taking the full space of boundary states and the above-mentioned additional states into account, Cardy's formalism still works to some extent. The partition functions are now related not to the physical characters, but to functions forming a five-dimensional representation of the modular group, the elements of which can presumably be interpreted as the torus amplitudes [40]. Interestingly, Cardy's formalism fails at precisely the same point where a Verlinde formula like computation of fusion coefficients within the five-dimensional representation of the modular group breaks down. A way out seems to be a limiting procedure, which eliminates the weak boundary states. Unfortunately, the result of this limit is a bit ambiguous and its physical interpretation is not yet completely clear.

Chapter 4 is devoted to boundary states in the symplectic fermion model [30, 35]. After a short introduction to this theory, the results of Kawai and Wheater [35] are presented. Then, the derivation of the boundary states obeying the fermion symmetry is performed using the general procedure introduced in chapter 2. It is shown that the space of boundary states is exactly the same as the one derived for the $\mathcal{W}(2, 3, 3, 3)$ algebra. Thus, the two theories are equivalent. Furthermore, the result is connected to Kawai/Wheater in order to show that the two different approaches are equal.

A brief discussion concludes this work, where especially the results are compared to earlier works. Open questions and directions for future research are also included.

Chapter 1

Modern conformal field theory

During the last 20 years two-dimensional conformal field theory has become an important and powerful tool in modern theoretical physics. This chapter provides a short introduction to logarithmic conformal field theory and boundary conformal field theory. In this work it is assumed that the reader has good knowledge of the basics of conformal field theory in two dimensions. Good introductions can be found in various books, lecture notes, articles, and reviews, see [41, 42, 43, 44, 45].

1.1 Logarithmic conformal field theory

The discovery of logarithmically divergent four-point correlation functions in 1993 by V. Gurarie [7] was the starting point for the development of a new concept in modern mathematical physics, called logarithmic conformal field theory. Gurarie studied the properties of a primary operator $\mu(z)$ with conformal weight $h = -\frac{1}{8}$ in a two-dimensional conformal field theory at central charge $c = -2$. He computed the four-point correlation function[1]

$$\langle \mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4)\rangle = (z_1 - z_3)^{\frac{1}{4}} (z_2 - z_4)^{\frac{1}{4}} F(x). \tag{1.1}$$

This is the well known result for any such correlator due to the constraints of global conformal invariance. $F(x)$ is a holomorphic function of the anharmonic ratio

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \tag{1.2}$$

that has to be deduced from dynamical constraints. Denoting the vacuum state by $\big|0\big\rangle$ the highest weight state $\big|\mu\big\rangle = \mu(0)\big|0\big\rangle$ satisfies the condition that $\left(L_{-2} - 2L_{-1}^2\right)\big|\mu\big\rangle$ is a null state: Its scalar products with all states in the theory vanish. This induces a second order differential equation for $F(x)$:

$$x(1-x)F''(x) + (1-2x)F'(x) - \frac{1}{4}F(x) = 0. \tag{1.3}$$

It has two solutions: One is the hypergeometric function $F(x) = F(\frac{1}{2}, \frac{1}{2}; 1; x)$ that is regular in the vicinity of $x = 0$ but has a logarithmic singularity at $x = 1$. The second fundamental solution cannot be found via a power series ansatz: $F(x) = F(\frac{1}{2}, \frac{1}{2}; 1; x)\log(x) +$

[1] In this thesis, formulas are always given in terms of a chiral Euclidean conformal field theory on the complex plane unless not annotated otherwise.

$H(x)$ where $H(x)$ is some analytic function both at 0 and 1. This solution is regular at $x = 1$ but logarithmically divergent at $x = 0$. Thus, it is not possible to get rid of the logarithmic divergences by discarding one of the two solutions.

Consequently, there exist two different operator product expansions of the field μ with itself depending on how the contour integrations are performed. The regular solution implies

$$\mu(z)\mu(0) = z^{\frac{1}{4}}\Omega(0) + \dots\,, \tag{1.4}$$

where $\Omega(z)$ is the identity operator. The second solution leads to a different operator product expansion:

$$\mu(z)\mu(0) = z^{\frac{1}{4}}\left(\Omega(0)\log(z) + \omega(0)\right) + \dots\,. \tag{1.5}$$

Here, $\Omega(z)$ is the same as in (1.4) but $\omega(z)$ is another operator with new features.

In a different approach one can study the properties of the representation module $\mathcal{V}_{-1/8}$ built on the highest weight state $\big|\mu\big\rangle = \mu(0)\big|0\big\rangle$. $\mathcal{V}_{-1/8}$ is an admissable highest weight representation. The fusion product of this representation with itself is an indecomposable rank-2 representation. The fusion product is not simply a tensor product but rather a complicated procedure [46]: If one inserts two representations ψ and χ at the points z_1 and z_2, respectively, the contour integral with the energy-momentum tensor around both insertion points defines a representation, the fusion product $(\psi \otimes_{\mathrm{f}} \chi)$. Even though it differs from the tensor product it is common to use the same symbol $\otimes$. In order to suppress confusions, it is here used either with an index f or when concerning the Verma modules, a cross ($\times$) is used. $\otimes$ refers to the usual tensor product. The action of the Virasoro modes on this representation is given by a so-called comultiplication formula:

$$\begin{aligned}&\oint_0 dw\, w^{m+1}\big\langle\phi\big|T(w)\psi(z_1)\chi(z_2)\big|\Omega\big\rangle\\ &\qquad = \big\langle\phi\big|(\Delta^{(1)}_{z_1,z_2}(L_m)\psi)(z_1)(\Delta^{(2)}_{z_1,z_2}(L_m)\chi)(z_2)\big|\Omega\big\rangle\end{aligned} \tag{1.6}$$

Here, ϕ is an arbitrary state inserted at infinity. The symbol Δ is given in [46]:

$$\begin{aligned}&\Delta_{z_1,z_2}(L_n) =\\ &\underbrace{\sum_{m=-1}^{n}\begin{pmatrix} n+1\\ m+1\end{pmatrix} z_1^{n-m}\left(L_m\otimes\mathbb{1}\right)}_{\left(\Delta^{(1)}\otimes\mathbb{1}\right)} + \underbrace{\sum_{l=-1}^{n}\begin{pmatrix} n+1\\ l+1\end{pmatrix} z_2^{n-l}\left(\mathbb{1}\otimes L_l\right)}_{\left(\mathbb{1}\otimes\Delta^{(2)}\right)},\quad (n\geq -1)\end{aligned} \tag{1.7}$$

For $n \leq -2$ there are two different possible choices:

$$\begin{aligned}\Delta_{z_1,z_2}(L_n) &= \sum_{m=-1}^{\infty}\begin{pmatrix} m-n-1\\ m+1\end{pmatrix}(-1)^{m+1}z_1^{n-m}\left(L_m\otimes\mathbb{1}\right)\\ &\quad+\sum_{l=-n}^{\infty}\begin{pmatrix} l-2\\ -n-2\end{pmatrix}(-z_2)^{l+n}\left(\mathbb{1}\otimes L_{-l}\right)\end{aligned} \tag{1.8}$$

and $\widetilde{\Delta}_{z_1,z_2}(L_n)$ where the tensoring is the other way round:

$$\begin{aligned}\widetilde{\Delta}_{z_1,z_2}(L_n) = &\sum_{m=-1}^{\infty} \begin{pmatrix} m-n-1 \\ m+1 \end{pmatrix} (-1)^{m+1} z_1^{n-m} \left(\mathbb{1} \otimes L_m\right) \\ &+ \sum_{l=-n}^{\infty} \begin{pmatrix} l-2 \\ -n-2 \end{pmatrix} (-z_2)^{l+n} \left(L_{-l} \otimes \mathbb{1}\right).\end{aligned} \tag{1.9}$$

Due to the fact that the fusion product should be uniquely defined states of the form $\left[\Delta(L_m) - \widetilde{\Delta}(L_m)\right] (\psi_1 \otimes_{\mathrm{f}} \psi_2)$ are discarded from the product space of the two representations. An explicit formulation in the case of the c $= -2$ theory is given in [8]. The main result is the existence two states

$$\omega = (\mu \otimes_{\mathrm{f}} \mu) \quad \text{and} \quad \varOmega = -\frac{1}{4} (\mu \otimes_{\mathrm{f}} \mu) + (L_{-1}\mu \otimes_{\mathrm{f}} \mu)\,. \tag{1.10}$$

They have the property

$$L_0 \omega = \varOmega \quad \text{and} \quad L_0 \varOmega = 0. \tag{1.11}$$

They form a rank-2 indecomposable Jordan block in the L_0 mode and can be identified with the states corresponding to the operators $\varOmega(z)$ and $\omega(z)$ in the operator product expansions (1.4) and (1.5). In (1.11), L_0 belongs to the realisation of the Virasoro modes on the fusion representation and is given by the comultiplication formula (1.7). The symbol Δ will be omitted from now on unless confusions arise.

1.2 Boundaries, boundary conditions and boundary conformal field theory

In all experimental setups the probes that are examined occupy a finite area, i. e. they have a finite length. One simply cannot study a two-dimensional system of infinite size in a laboratory. This naturally implies the existence of physical boundaries and boundary conditions. Conformal field theory found many applications in modern theoretical physics, just to give an example in the studies of the critical behaviour of two-dimensional lattice systems at the phase transition point. Therefore, it had been an obvious quest to study the consequences of conformal invariance in the vicinity of a boundary and to connect the results to the knowledge one had had for the bulk theories, i. e. theories without boundaries. The subject of boundary conformal field theory emerged first at the end of the 1980's simultaneously in the field of open string theory [5] and in connection with critically behaving systems in condensed matter physics. During the last years, boundary conformal field theory has found a renewed interest in both high energy and condensed matter subjects. In this section a very brief overview leading to the basic conceptual ideas is presented. An extensive description can be found in [4].

It turned out that a conformal field theory on a surface with boundaries is deeply connected to the prescription of the same theory on the corresponding surface without boundaries, just as in string theory there is a deep relation between closed and open strings. In this work, these two surfaces will always be considered to be the cylinder and the torus, respectively. P. di Francesco and J.-B. Zuber [47] as well as R. E. Behrend, P. A. Pearce, V. B. Petkova, and J.-B. Zuber [48] formulated this correspondence in terms of graphs. On the cylinder, the graphs' nodes denote the physically consistent boundary conditions and the partition functions are described by their adjacency matrices, while for the bulk theory, i. e. on the torus, the nodes yield the modular invariants.

In a conformal field theory with boundaries, the conditions on the boundary have to obey conformal invariance. In general cases, the symmetry algebra is bigger than just the conformal symmetry algebra. It may contain the energy-momentum tensor and some additional field(s) W of some weight $h \in \mathbb{Z}_+/2$. Considering for the moment a semi-infinite cylinder which can be equally regarded as the upper half complex plane, i. e. a setup with only one boundary, then one has to impose the absence of energy-momentum flow across the boundary and corresponding continuity equations for the fields $W(z)$ and $\overline{W}(\overline{z})$:

$$T(z) = \overline{T}(\overline{z}) \quad \text{and} \quad W(z) = \overline{W}(\overline{z}) \quad \text{for } z = \overline{z}. \tag{1.12}$$

$\overline{T}(\overline{z})$ and $\overline{W}(\overline{z})$ are exactly the anti-holomorphic fields corresponding to $T(z)$ and $W(z)$. This situation can be mapped to an annulus in the complex plane, which is topological a (finite) cylinder, by the transformation

$$\zeta = \mathrm{e}^{-2\mathrm{i}\pi\frac{L}{T}\cdot\log(z)}. \tag{1.13}$$

In these coordinates, (1.12) reads

$$\zeta^2 T(\zeta) = \overline{\zeta}^2\overline{T}(\overline{\zeta}) \quad \text{and} \quad \zeta^s W(\zeta) = (-\overline{\zeta})^s\overline{W}(\overline{\zeta}) \quad \text{for} \quad |\zeta| \in \big\{1, \mathrm{e}^{2\pi\frac{L}{T}}\big\}. \tag{1.14}$$

Here, s denotes the spin of the field $W(z)$ and its anti-holomorphic partner. In the given picture a boundary condition applied to one side of the cylinder can be understood as a closed string state $\big|B\big\rangle$ that propagates to another state $\big|C\big\rangle$ corresponding to the boundary condition on the other side of the cylinder. After radial quantization the above-mentioned conditions (1.14) transform to constraints on these *boundary states* $\big|B\big\rangle$ and $\big|C\big\rangle$ that are given in terms of the Virasoro modes and the mode expansion of the field $W(z)$, $W(z) = \sum_n z^{-n-h} W_n$:

$$\big(L_n - \overline{L}_{-n}\big)\,\big|B\big\rangle = 0 \quad \text{and} \tag{1.15}$$

$$\big(W_n - (-1)^s\overline{W}_{-n}\big)\,\big|B\big\rangle = 0, \quad (n \in \mathbb{Z}). \tag{1.16}$$

(1.15) can be understood as an equation of motion for the open string background and thus itself does not completely determine the boundary states. Therefore, it is natural to consider the influence of a larger symmetry algebra.

The general solution to these equations in the case of ordinary rational conformal field theories containing only irreducible representations in their spectra was analyzed first by N. Ishibashi [5]. Later, J. L. Cardy [3] connected the results of Ishibashi to the properties of the corresponding bulk theory. This standard approach to boundary conformal field theory is shortly reviewed in section 2.1 and exemplarily applied to a very simple and well-known system, the two-dimensional lattice Ising model on an annulus.

Chapter 2

Construction of boundary states: A general procedure

In this chapter, the constraints on the boundary states are solved, leading to the complete set of allowed boundary states. Firstly, the “standard approach” is presented. Its main results in mind, a procedure is invented that is based on only a small set of fundamental assumptions concerning the bulk state properties. Thus, this method should be applicable to a wide range of conformal field theories.

2.1 Standard approach in ordinary conformal field theories

It was already mentioned in the last chapter that in ordinary rational conformal field theories with boundaries there exists a generic way for obtaining the set of physical boundary conditions in terms of boundary states. The span of boundary states is derived by the so-called Ishibashi construction [5]: It was discussed in section 1.2 that conformal invariance gives strong constraints on a boundary state $\big|B\big\rangle$:

$$\big(L_n - \overline{L}_{-n}\big)\,\big|B\big\rangle = 0. \tag{2.1}$$

This equation is not sufficient enough to determine $\big|B\big\rangle$ completely. Therefore, one analyzes boundary operators with respect to the maximally extended chiral symmetry algebra $\mathcal{W}$ with additional N fields $W^r,\ r = 1, \ldots, N$. The boundary state for this algebra has to obey in addition:

$$\big(W^r_n - (-1)^{s_r}\overline{W}^r_{-n}\big)\,\big|B\big\rangle = 0, \tag{2.2}$$

where s_r labels the spin of the field W^r. A basis over a given bulk representation module $\mathcal{M}_h$ may be denoted by

$$\big\{\big|l, n\big\rangle \,\big|\, l = h,\, h+1, \ldots;\, n = 1, \ldots, n_l\big\}, \tag{2.3}$$

where l counts the levels beginning from the highest weight h of the module and n labels a suitable basis on each level. This might look as in the following diagram:

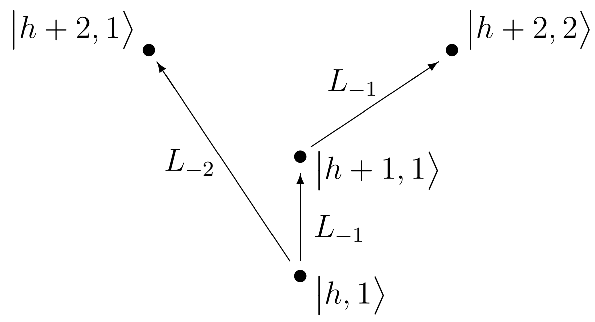

figure 2.1: Denoting the basis of a Virasoro module $\mathcal{M}_h$ with highest weight state $|h,1\rangle \equiv |h\rangle$ of weight h

The first thing is to compute a suitable basis of boundary states. For a representation module $\mathcal{M}_h$ of the chiral bulk theory and its anti-chiral partner module $\overline{\mathcal{M}}_h$ a sufficient boundary state is given by the sum over tensor products $|l,n;l,n\rangle \equiv |l,n\rangle \otimes \overline{|l,n\rangle}$ of a complete orthonormal basis, the *Ishibashi states*:

$$|M_h\rangle = \sum_{l,n} \left(\mathbb{1} \otimes \overline{\mathcal{U}}\right) |l,n;l,n\rangle. \tag{2.4}$$

Here, $\mathcal{U}$ is the anti-unitary operator acting on the modes as

$$\mathcal{U}L_n = L_n\mathcal{U} \quad \text{and} \quad \mathcal{U}W_n^r = (-1)^{s_r} W_n^r \mathcal{U}. \tag{2.5}$$

For a two-boundary system the physical relevant boundary conditions were analyzed by J. L. Cardy [3]. The partition function in such a theory with boundary conditions α and β, respectively, is not a sum over bilinears of the characters as in pure bulk theories but rather a linear combination

$$Z_{\alpha\beta}(q) = \sum_i n^i_{\alpha\beta}\chi_i\,(q) = \sum_i n^i_{\alpha\beta}\mathcal{S}_i^j\chi_j\,(\tilde{q}) \tag{2.6}$$

with coefficients $n^i_{\alpha\beta}$. Here, the index i runs over the representations and $q \equiv \mathrm{e}^{2\pi\mathrm{i}\tau}$, τ being the modular parameter. $\mathcal{S}$ is the transformation matrix of the characters under $q \longrightarrow \tilde{q} = \mathrm{e}^{-2\pi\mathrm{i}/\tau}$, $\tau \longrightarrow -1/\tau$. An open string that propagates periodically in time can be regarded as a propagating closed string. With the help of this $\mathcal{T}$-duality $Z_{\alpha\beta}(q)$ can be rewritten in terms of a closed string amplitude, i. e. a closed string propagating from the initial state $|\beta\rangle$ to a final state $|\alpha\rangle$:

$$Z_{\alpha\beta}(q) = \langle\alpha|\tilde{q}^{\mathcal{H}}|\beta\rangle. \tag{2.7}$$

Here, $\mathcal{H} \equiv \frac{1}{2}(L_0 + \overline{L}_0 - c/24)$ is the Hamiltonian. The two equations (2.6) and (2.7) combine to Cardy's consistency equation:

$$\sum_{i,j} n^i_{\alpha\beta}\mathcal{S}_i^j\chi_j(\tilde{q}) \equiv \langle\alpha|\tilde{q}^{\mathcal{H}}|\beta\rangle. \tag{2.8}$$

In ordinary theories the Ishibashi boundary states diagonalise the closed string amplitudes (2.7)

$$\langle i|\tilde{q}^{\mathcal{H}}|j\rangle = \delta_{ij}\chi_i(\tilde{q}). \tag{2.9}$$

Here, i counts the Ishibashi states. Therefore, the identity can be expressed in terms of the complete set of Ishibashi boundary states (2.4) as

$$\mathbb{1} = \sum_i |i\rangle\langle i|. \tag{2.10}$$

This leads to a relation between (2.6) and (2.7):

$$Z_{\alpha\beta}(q) = \sum_i \langle\alpha|i\rangle\langle i|\beta\rangle\chi_i(\tilde{q}). \tag{2.11}$$

If the characters $\chi_i(q)$ are linearly independent (as they are for ordinary conformal field theories) then a comparison to (2.6) shows that

$$\langle\alpha|i\rangle\langle i|\beta\rangle \equiv n^j_{\alpha\beta}\mathcal{S}^i_j. \tag{2.12}$$

Cardy used this equation to relate the coefficients $n^i_{\alpha\beta}$ to the fusion rules of the underlying bulk theory obtaining the set of physical relevant boundary conditions. They are derived by looking at conditions where the spectrum of the bulk Hamiltonian consists of the vacuum representation Ω and the representation k. This corresponds to boundary conditions $(\boldsymbol{\Omega}, \boldsymbol{k})$, i. e. $\boldsymbol{\Omega}$ on one side of the cylinder and $\boldsymbol{k}$ on the other side:

$$|\boldsymbol{k}\rangle = \sum_i \frac{\mathcal{S}^i_k}{\sqrt{\mathcal{S}^i_0}}|i\rangle. \tag{2.13}$$

Here, the row $\mathcal{S}^i_0$ is associated with the transformation of the vacuum character under $\tau \longrightarrow -1/\tau$. The dual states are not simply the adjoints. The boundary conditions are symmetric, i. e. for boundary conditions $(\boldsymbol{k}, \boldsymbol{\Omega})$ the Hamiltonian's spectrum contains the representations k and Ω. In order to stick to these considerations, the correct bra states are

$$\langle\boldsymbol{k}^\vee| = \sum_i \frac{\mathcal{S}^i_k}{\sqrt{\mathcal{S}^i_0}}\langle i|. \tag{2.14}$$

As an example, the two dimensional Ising model on a square lattice with two boundaries is briefly reviewed here. The Ising model at the critical point can be described by a rational conformal field theory with central charge $c = \frac{1}{2}$. It is equivalent to the free fermion system. There are three representations that close under fusion: $\mathcal{V}_0$ (the vacuum representation), $\mathcal{V}_{1/2}$ (belonging to the energy), and $\mathcal{V}_{1/16}$ (coming from a twist operator that changes the spin). The fusion rules $\mathcal{V}_i \times \mathcal{V}_j = \sum_k N^k_{ij}\mathcal{V}_k$ read:

$$\begin{aligned} \mathcal{V}_0 \times \Psi &= \Psi & \mathcal{V}_{1/2} \times \mathcal{V}_{1/16} &= \mathcal{V}_{1/16} \\ \mathcal{V}_{1/2} \times \mathcal{V}_{1/2} &= \mathcal{V}_0 & \mathcal{V}_{1/16} \times \mathcal{V}_{1/16} &= \mathcal{V}_0 + \mathcal{V}_{1/2} \end{aligned} \tag{2.15}$$

The $\mathcal{S}$ matrix for the characters $\{\chi_0(q), \chi_{1/2}(q), \chi_{1/16}(q)\}$ of the representation modules is given by

$$\mathcal{S} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}. \tag{2.16}$$

There exist three relevant boundary conditions for each boundary: All spins up, all down and a condition where the spins are free. The boundary states derived by the Ishibashi and Cardy construction are

$$\begin{aligned} \big|\mathbf{0}\big\rangle &= \frac{1}{\sqrt{2}}\big|V_0\big\rangle + \frac{1}{\sqrt{2}}\big|V_{1/2}\big\rangle + \frac{1}{\sqrt[4]{2}}\big|V_{1/16}\big\rangle, \\ \big|\tfrac{\mathbf{1}}{\mathbf{2}}\big\rangle &= \frac{1}{\sqrt{2}}\big|V_0\big\rangle + \frac{1}{\sqrt{2}}\big|V_{1/2}\big\rangle - \frac{1}{\sqrt[4]{2}}\big|V_{1/16}\big\rangle, \\ \big|\tfrac{\mathbf{1}}{\mathbf{16}}\big\rangle &= \big|V_0\big\rangle - \big|V_{1/16}\big\rangle. \end{aligned} \tag{2.17}$$

The first two states can be identified with the all-spin-up and all-spin-down state, respectively, according to their $\mathbb{Z}_2$ symmetry. Naturally, $\big|\tfrac{\mathbf{1}}{\mathbf{16}}\big\rangle$ has to be the state corresponding to free boundary conditions. This is not unexpected, since the operator related to the representation $\mathcal{V}_{1/16}$ is the twist operator that interchanges the spin content. One should stress here that boundary operators in general do not have the same weight as the corresponding bulk operators: The twist state, for example, has weight $h = 0$ while in the bulk the weight is equal to 1/16.

In principle there can be other choices for the proper boundary conditions as well. A necessary condition is that the coefficients $n^i_{\alpha\beta}$ interpreted as matrices $(n_\alpha)^i{}_\beta$ obey a matrix multiplication law equal to the fusion rules, the Verlinde algebra:

$$(n_k)^r{}_s \cdot (n_l)^s{}_t = \sum_i N^i_{kl} \cdot (n_i)^r{}_t. \tag{2.18}$$

Evidently, for n being equal to the fusion coefficients N themselves this condition is trivially satisfied. In ordinary rational conformal field theory the Cardy condition is equal to finding so-called non-negative integer matrix (NIM) representations of the Verlinde algebra [4, 54]. It is still not clear if and how more general solutions have to be taken into account, especially in logarithmic conformal field theories.

2.2 A general procedure

The standard method to gain boundary states works well in ordinary conformal field theories but it fails in cases that have a more complicated structure like logarithmic conformal field theories. There, one has to cope with the natural existence of zero-norm states. This prohibits the use of an orthonormal basis. In principle, the Ishibashi construction does not make use of orthonormality but in most of its applications to logarithmic conformal field theory this far it was in fact always assumed. Another feature is that the L_0 mode contains non-trivial Jordan blocks and is therefore not diagonalisable. A possible way out is presented here: The implementation of a procedure that derives the set of boundary states from bottom-up by using rather general properties of the bulk theory. Considering an arbitrary not necessarily orthonormal basis $\{|l,n\rangle\}$ of a representation $\mathcal{M}_h$ the metric elements g_{mn} are given by the always well-defined and symmetric Shapovalov forms [49]:

$$\delta_{ll'} g_{mn} \equiv \langle l,m|l',n\rangle \equiv \lim_{z\to\infty}\lim_{w\to 0} z^{2l}\langle \phi_{l,m}(z)\phi_{l',n}(w)\rangle. \tag{2.19}$$

Here, $\phi_{l,m}$ is the field corresponding to the state $|l,m\rangle$. A prescription for the case of logarithmic conformal field theories was given by F. Rohsiepe [27]. It is assumed here, that $\mathcal{M}_h$ contains an element of highest weight h. This does not have to be the cyclic state that generates the complete representation as in ordinary theories. One might, for example, think of more general setups in which there exists a state ψ that generates the module and has the property that $L_1^2\psi = 0$ while $L_1\psi \neq 0$. This is in fact the case in the $\mathrm{c} = -2$ triplet model. Keeping in mind the standard approach, a generalised ansatz for a boundary state can be written as a sum of product states $|l,m;\bar{l},n\rangle \equiv |l,m\rangle \otimes \overline{|\bar{l},n\rangle}$ of a holomorphic representation $\mathcal{M}_h$ and an anti-holomorphic module $\overline{\mathcal{M}}_{\bar{h}}$:

$$|B\rangle = \sum_{l,m,\bar{l},n} c^{l\bar{l}}_{mn}|l,m;\bar{l},n\rangle. \tag{2.20}$$

Here, $l \neq \bar{l}$ is in principle allowed. Of course, it will turn out that only states of the type $|l,m;l,n\rangle$ contribute to the solution just as in the Ishibashi approach. The modes L_n in equation (2.1) obey the Virasoro algebra

$$[L_m, L_n] = (m-n)\,L_{m+n} + \frac{\mathrm{c}}{12}\left(m^3 - m\right)\delta_{m+n,0}. \tag{2.21}$$

For $n \neq 2$ a simple calculation shows that

$$L_n = \frac{1}{n-2}\,[L_{n-1}, L_1] \quad\text{and}\quad L_{-n} = \frac{1}{2-n}\,[L_{1-n}, L_{-1}]\,. \tag{2.22}$$

Thus, it is enough to check condition (2.1) for $n = -2, \ldots, 2$ because the equation automatically holds for $|n| \geq 3$. For a boundary state that is built on two copies of the same representation module in its holomorphic and anti-holomorphic part it is equivalent to demand that (2.1) is valid for $n = 0,1,2$ and to choose the coefficients symmetrically in m and n, i. e. $c^{l\bar{l}}_{mn} = c^{l\bar{l}}_{nm}$.

From now on it is always assumed that L_0 and $\overline{L}_0$ are in Jordan form and the decomposition into their diagonal and off-diagonal parts is given by $\hat{h}$ ($\hat{\bar{h}}$) and $\hat{\delta}$ ($\hat{\bar{\delta}}$), respectively, such that

$$L_0 = \hat{h} + \hat{\delta} \quad\text{and}\quad \overline{L}_0 = \hat{\bar{h}} + \hat{\bar{\delta}}. \tag{2.23}$$

Equation (2.1) transforms into

$$\begin{aligned} 0 &= \left(L_0 - \overline{L}_0\right) \big|B\big\rangle \\ &= \left(\hat{h} - \hat{\bar{h}} + \hat{\delta} - \hat{\bar{\delta}}\right) \sum_{l,m,\bar{l},n} c^{l\bar{l}}_{mn} \big|l,m;\bar{l},n\big\rangle \\ &= \sum_{l,m,\bar{l},n} c^{l\bar{l}}_{mn} \left(l - \bar{l} + \hat{\delta} - \hat{\bar{\delta}}\right) \big|l,m;\bar{l},n\big\rangle. \end{aligned} \tag{2.24}$$

Because the states $\big|l,m;\bar{l},n\big\rangle$ form a basis of $\mathcal{M}_h \otimes \overline{\mathcal{M}_{\bar{h}}}$, the off-diagonal part $(\hat{\delta} - \hat{\bar{\delta}})\big|B\big\rangle$ has to vanish and the holomorphic and anti-holomorphic weight should coincide, $l = \bar{l}$:

$$\big|B\big\rangle = \sum_{l,m,n} c^{l}_{mn} \big|l,m;l,n\big\rangle. \tag{2.25}$$

This is already part of the result given by N. Ishibashi for ordinary conformal field theories, see section 2.1. In the same manner the treatment of the $n = 1$ case in (2.1) yields the following equations:

$$\begin{aligned} 0 &= \left(L_1 - \overline{L}_{-1}\right) \big|B\big\rangle \\ &= \left(L_1 - \overline{L}_{-1}\right) \sum_{l,m,n} c^{l}_{mn} \big|l,m;l,n\big\rangle \\ &= \sum_{l,m,n} c^{l}_{mn} \Big(\sum_a \alpha^{l\,m}{}_a \big|l-1,a;l,n\big\rangle - \sum_b \beta^{l\,n}{}_b \big|l,m;l+1,b\big\rangle \Big) \\ &= \sum_{l,m,n} \left(\alpha^{l\,a}{}_m c^{l}_{an} - c^{l-1}_{mb} \beta^{l-1\,b}{}_n \right) \big|l-1,m;l,n\big\rangle. \end{aligned} \tag{2.26}$$

In the last line of (2.26) and in the following, Einstein's summing convention is used for the indices a and b. Here, α and β denote the coefficients in the expansion of $L_1\big|l,m\big\rangle$ and $\overline{L_{-1}\big|l,n\big\rangle}$ in terms of the basis states $\big|l-1,a\big\rangle$ and $\overline{\big|l+1,b\big\rangle}$ at level $l-1$ and $l+1$, respectively:

$$L_1\big|l,m\big\rangle = \sum_a \alpha^{l\,m}{}_a \big|l-1,a\big\rangle \quad \text{and} \quad \overline{L_{-1}\big|l,n\big\rangle} = \sum_b \beta^{l\,n}{}_b \overline{\big|l+1,b\big\rangle}. \tag{2.27}$$

To fulfil the condition (2.26) the coefficients have to vanish identically because $\{\big|l,m\big\rangle\}$ is a basis:

$$\alpha^{l\,a}{}_m c^{l}_{an} - c^{l-1}_{mb} \beta^{l-1\,b}{}_n = 0. \tag{2.28}$$

For $n = 2$ it follows analogously that

$$\varrho^{l\,a}{}_m c^{l}_{an} - c^{l-2}_{mb} \sigma^{l-2\,b}{}_n = 0. \tag{2.29}$$

As in (2.28), ϱ and σ are the expansion coefficients for the states $L_2\big|l,m\big\rangle$ and $\overline{L_{-2}\big|l,n\big\rangle}$. The conditions (2.2) for the extended symmetry algebra $\mathcal{W}$ have to be treated in the same way. These equations reduce to a finite set as well. The additional N fields in the extended chiral symmetry algebra are primary with respect to the energy momentum tensor. If h is the conformal weight of W^r, then:

$$\left[L_n, W^r_m\right] = ((h-1)m - n)\, W^r_{m+n}. \tag{2.30}$$

As in (2.22) this implies

$$W^a_n = \begin{cases} \frac{1}{(h-1)n}\left[L_n, W^r_0\right] & (n \neq 0,\ h \neq 1) \\ \left[L_{n-1}, W^r_1\right] & (h = 1) \end{cases} . \tag{2.31}$$

Therefore, it is enough to check (2.2) for $n = 0$ (or $n \in \{0, \pm 1\}$ if $h = 1$), since the remaining conditions are treated implicitly with the help of (2.1):

$$\left(W^r_0 - (-1)^{s_r}\overline{W}^r_0\right)\left|B\right\rangle = 0,\ r = 1, \ldots, N. \tag{2.32}$$

This leads to additional $N + 2k$ equations, where N is the total number of additional fields W^a and k is the number of fields of conformal weight 1 among these. In particular, in the special case of the $\mathcal{W}(2,3,3,3)$ algebra in the $\mathrm{c} = -2$ theory the three additional fields W^a are spin-3 fields and (2.32) reduces to the three equations

$$\left(W^a_0 + \overline{W}^a_0\right)\left|B\right\rangle = 0, \tag{2.33}$$

where a is the spinor index of $su(2)$ and takes three different values.

By solving all the conditions for the coefficients c^l_{mn} and all possible combinations of holomorphic and anti-holomorphic representations contributing to $\left|B\right\rangle$ one gains a complete basis of boundary states. After having derived the coefficients for the first three levels without any inconsistencies, there are no contradictions occurring on higher levels. This is assured by the algebraic structure ((2.21) and (2.30)). Especially, the states are well-defined solutions of (2.1) and (2.2). Indeed, this is the case for the $\mathrm{c} = -2$ rational logarithmic conformal field theory considered here as will be shown explicitly later on. In particular, the coefficients c^l_{mn} for any arbitrary given finite level l can be calculated iteratively and in a finite number of steps.

Given a set of representation modules $\mathcal{M}^m_h,\ m = 1, \ldots, M$ that build a Jordan block of rank M in the zero mode of the Virasoro algebra, the number of boundary states built on $\mathcal{M}^1_h \otimes \overline{\mathcal{M}}^1_h$ derived by the presented method is M. There is one state that is associated to the whole representation $\mathcal{M}^1_h \otimes \overline{\mathcal{M}}^1_h$ and for each submodule there exists a state that has only contributions coming from the particular subrepresentation. To see this, one might look at a highest weight module $\mathcal{M}^m_h$ that is embedded as a subrepresentation in $\mathcal{M}^1_h$. Exemplarily, one can consider an irreducible subrepresentation. Then, there exists a highest weight state ζ with weight h such that $L_n\zeta = 0$ for $n > 0$. Therefore, the coefficient $c^h_{\zeta\zeta}$ in the expansion (2.25) of the boundary state for $\mathcal{M}^1_h \otimes \overline{\mathcal{M}}^1_h$ is independent due to (2.1) and (2.2). By acting with creators and annihilators on the states $\left|l, m; l, n\right\rangle \in \mathcal{M}^m_h \otimes \overline{\mathcal{M}}^m_h$ one cannot leave the representation. Therefore, this coefficient only affects states in this subrepresentation.

It is obvious that in ordinary conformal field theories this formalism reproduces the usual Ishibashi results: Let $\{|l,m\rangle\}$ be an orthonormal basis of an irreducible highest weight module $\mathcal{V}$. The corresponding Ishibashi boundary state reads:

$$|V\rangle = \sum_{l,m} \left(\mathbb{1} \otimes \overline{\mathcal{U}}\right) |l,m;l,m\rangle. \tag{2.34}$$

$\mathcal{U}$ is the anti-unitary operation defined in the previous section with the feature that it acts on the modes of the extended chiral algebra as $W_n^r\mathcal{U} = (-1)^{s_r}\mathcal{U}W_n^r$ and commutes with all the Virasoro modes. Since the state given in (2.34) satisfies the two equations (2.1) and (2.2), it has to fulfil the coefficient equations (2.24), (2.26), and (2.29) as well as the corresponding ones for the modes of the extended algebra (2.33). This means that $|V\rangle$ can be constructed by the formalism presented above. The difference is that it does not make explicit use of the unitary operation $\mathcal{U}$. If $\mathcal{U}$ is split off by hand afterwards, the well-known results of ordinary conformal field theories are exactly reproduced.

The big advantage compared to other ansatzes is that this method does not make any use of the properties of the states themselves other than the expansion with respect to an arbitrary but fixed basis. On the other hand, it is not possible with this method to derive the boundary states in a closed form because that would mean to deduce an infinite number of coefficients. Therefore, it is necessary to identify the states afterwards. This is the price one has to pay for the sake of the method's simplicity that results in the possibility to keep track of non-trivial inner structures of complicated representations and a better handling of non-normalisable states. Both are a specialty of logarithmic conformal field theories. This features its application in a wide range of boundary conformal field theories.

Chapter 3

Boundary states in c = -2 logarithmic conformal field theory

This chapter deals with the derivation of the span of boundary states for the c = -2 triplet model. After a short introduction of this rational conformal field theory the results of previous works on this topic are briefly reviewed. In section 3.3 it is shown to what extent it is possible to generalise the standard Ishibashi formalism. In section 3.4 the method presented in chapter 2 is applied to the triplet model. It turns out that the rational c = -2 theory containts more boundary states than has been assumed up to now. The solution is discussed in section 3.5. Exploring its structure a connection to the unique local theory analyzed by M. R. Gaberdiel and H. G. Kausch [29] arises. Finally, the cylinder amplitudes are calculated to see how the physical relevant boundary states *à la* J. L. Cardy can be obtained and how this standard approach has to be modified for logarithmic conformal field theories.

3.1 The triplet model

Before considering boundary states, this section tries to give a short overview of the c = -2 triplet model. M. R. Gaberdiel and H. G. Kausch spent a lot of work analyzing this rational logarithmic conformal (bulk) theory [25, 26, 29, 50]. F. Rohsiepe examined the physical characters of the representation modules [27]. They form the three-dimensional representation of the modular group. The torus amplitudes on the other hand seem to be related to a five-dimensional representation of the same group that was analyzed by M. Flohr [28]. The latter representation contains the smaller one as a subrepresentation. Even though it is not quite well understood, it is suspected to be an essential and interesting new feature of logarithmic conformal field theories that characters and torus amplitudes form different representations. Work in this direction is still in progress [40]. In ordinary theories the two sets — characters and torus amplitudes — are exactly equal. The theory is based on a $\mathcal{W}(2,3,3,3)$ triplet algebra which is generated by the Virasoro modes L_n and the modes W^a_n of a triplet of spin-3 fields $W^a(z)$. With the help of two quasi-primary normal ordered fields $\Lambda = :L^2: -3/10\,\partial^2 L$ and $V^a = :LW^a: -3/14\,\partial^2 W^a$

the commutation relations read

$$\begin{aligned}
\left[L_m, L_n\right] =&(m-n)L_{m+n} - \frac{1}{6}\left(m^3 - m\right)\delta_{m+n,0},\\
\left[L_m, W^a_n\right] =&(2m-n)W^a_{m+n},\\
\left[W^a_m, W^b_n\right] =&\hat{g}^{\,ab}\Big(2\,(m-n)\,\Lambda_{m+n} + \frac{1}{20}\,(m-n)\left(2m^2+2n^2-mn-8\right)L_{m+n}\\
&-\frac{1}{120}m\left(m^2-1\right)\left(m^2-4\right)\delta_{m+n,0}\Big)\\
&+\hat{f}^{ab}_{c}\left(\frac{5}{14}\left(2m^2+2n^2-3mn-4\right)W^c_{m+n} + \frac{12}{5}V^c_{m+n}\right).
\end{aligned} \tag{3.1}$$

Here, $\hat{g}^{\,ab}$ is the metric and $\hat{f}^{ab}_c$ are the structure constants of $su(2)$. For the further discussion it is suitable to choose a Cartan-Weyl basis for $su(2)$ which reads W^0 and $W^\pm$. The metric is given by $\hat{g}^{\,00} = 1$, $\hat{g}^{\,+-} = \hat{g}^{\,-+} = 2$ and the non-vanishing structure constants read $\hat{f}^{0\pm}_{\pm} = -\hat{f}^{\pm 0}_{\pm} = \pm 1$ and $\hat{f}^{+-}_0 = -\hat{f}^{-+}_0 = 2$. The anti-unitary operator $\mathcal{U}$ of chapter 2 acts in this setup as a spin flip: $\mathcal{U}W^0 = -W^0\mathcal{U}$ and $\mathcal{U}W^\pm = -W^\mp\mathcal{U}$. Commutators involving the operators $\hat{\delta}$ and $\hat{h}$ introduced as the decomposition of L_0 (2.23) read[1]

$$\left[\hat{h}, \mathcal{O}_n\right] = \left[L_0, \mathcal{O}_n\right] \quad \text{and} \quad \left[\hat{\delta}, \mathcal{O}_n\right] = 0, \tag{3.2}$$

where $\mathcal{O}_n$ is an arbitrary mode of the algebra. There are six representations that close under fusion. Four of them are irreducible: two singlet representations, the vacuum representation $\mathcal{V}_0$ and $\mathcal{V}_{-1/8}$ with highest weight states Ω at $h = 0$ and μ at $h = -1/8$, respectively, and two doublet representations $\mathcal{V}_1$ and $\mathcal{V}_{3/8}$ with highest weight states at $h = 1$ and $h = 3/8$. Furthermore there exist two reducible but indecomposable representations: $\mathcal{R}_0$ is generated by a cyclic state ω at level 0 that builds a Jordan block in L_0 together with the vacuum highest weight state Ω of $\mathcal{V}_0$ and $\mathcal{R}_1$ is generated by a doublet of level 1 cyclic states $\psi^\pm$ that form Jordan blocks together with the highest weight states $\phi^\pm$ of the representation $\mathcal{V}_1$. μ, Ω and ω are the states discussed in chapter 1. The states ω and $\psi^\pm$ are no highest weight states, i. e. $\xi^\pm \equiv -\frac{1}{2}L_1\psi^\pm$ is not zero and L_0 acts non-diagonal. The representations $\mathcal{V}_0$ and $\mathcal{V}_1$ are subrepresentations of the modules $\mathcal{R}_0$ and $\mathcal{R}_1$, respectively. It follows that the states in the two (sub-)representations $\mathcal{V}_0$ and $\mathcal{V}_1$ are zero-norm states [27]. Due to the fact that the highest occurring weight in both of these indecomposable representations is $h = 0$, these representations are also known as generalised highest weight representations[2]. Furthermore, $\mathcal{R}_0$ contains two subrepresentations of type $\mathcal{V}_1$ built on the two doublet states $\Psi^\pm_1$ and $\Psi^\pm_2$ (here, the conventions of [26] are used):

$$\begin{aligned}
\Psi^+_1 &= W^+_{-1}\omega, & \Psi^+_2 &= \left(W^0_{-1} + \tfrac{1}{2}L_{-1}\right)\omega,\\
\Psi^-_1 &= \left(-W^0_{-1} + \tfrac{1}{2}L_{-1}\right)\omega, & \Psi^-_2 &= W^-_{-1}\omega.
\end{aligned} \tag{3.3}$$

[1] $\mathcal{O}_{-n}\hat{h}|h\rangle \simeq h|h+n\rangle$, $\hat{h}\mathcal{O}_{-n}|h\rangle \simeq \hat{h}|h+n\rangle \simeq (h+n)|h+n\rangle$

[2] i. e. their spectra are bounded from below as in irreducible representations

The structure of the indecomposable modules $\mathcal{R}_0$ and $\mathcal{R}_1$ can be drawn schematically:

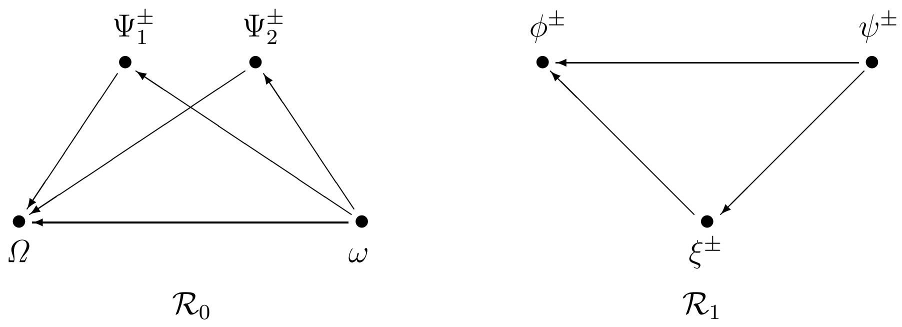

figure 3.1: Generalised highest weight modules $\mathcal{R}_0$ and $\mathcal{R}_1$

The points in figure 3.1 refer to the states on which the different (sub-)representations are built and the lines denote the action of the $\mathcal{W}$-algebra.

These six representations form a finite set that close under fusion. Therefore among others, one refers to this as a rational theory, even though rationality has to be seen in a weaker sense than in ordinary rational conformal field theories. The fusion rules $\mathcal{M}_i \times \mathcal{M}_j = \sum_k N_{ij}^k \mathcal{M}_k$ for the triplet model were analyzed by M. R. Gaberdiel and H. G. Kausch [25, 26]:

$$\begin{aligned}
\mathcal{V}_0 \times \Psi &= \Psi, & \mathcal{V}_{3/8} \times \mathcal{V}_1 &= \mathcal{V}_{-1/8}, \\
\mathcal{V}_{-1/8} \times \mathcal{V}_{-1/8} &= \mathcal{R}_0, & \mathcal{V}_{3/8} \times \mathcal{R}_m &= 2\,\mathcal{V}_{-1/8} + 2\,\mathcal{V}_{3/8}, \\
\mathcal{V}_{-1/8} \times \mathcal{V}_{3/8} &= \mathcal{R}_1, & \mathcal{V}_1 \times \mathcal{V}_1 &= \mathcal{V}_0, \\
\mathcal{V}_{-1/8} \times \mathcal{V}_1 &= \mathcal{V}_{3/8}, & \mathcal{V}_1 \times \mathcal{R}_0 &= \mathcal{R}_1, \\
\mathcal{V}_{-1/8} \times \mathcal{R}_m &= 2\,\mathcal{V}_{-1/8} + 2\,\mathcal{V}_{3/8}, & \mathcal{V}_1 \times \mathcal{R}_1 &= \mathcal{R}_0, \\
\mathcal{V}_{3/8} \times \mathcal{V}_{3/8} &= \mathcal{R}_0, & \mathcal{R}_m \times \mathcal{R}_n &= 2\,\mathcal{R}_0 + 2\,\mathcal{R}_1.
\end{aligned} \tag{3.4}$$

Here, m and n can take the values 0, 1. There exists a sub-group with respect to fusion: The set $\left\{\mathcal{R}_0, \mathcal{R}_1, \mathcal{V}_{-1/8}, \mathcal{V}_{3/8}\right\}$ closes under fusion as well. By looking at this reduced set the non-trivial inner structures of the representation $\mathcal{R}_0$ and $\mathcal{R}_1$ are surely lost. The characters $\chi_i(q)$ associated to the representations read

$$\begin{aligned}
\chi_{\mathcal{V}_0}(q) &= \frac{1}{2\eta(q)}\left(\Theta_{1,2}(q) + (\partial\Theta)_{1,2}(q)\right), \\
\chi_{\mathcal{V}_1}(q) &= \frac{1}{2\eta(q)}\left(\Theta_{1,2}(q) - (\partial\Theta)_{1,2}(q)\right), \\
\chi_{\mathcal{V}_{-1/8}}(q) &= \frac{1}{\eta(q)}\Theta_{0,2}(q), \\
\chi_{\mathcal{V}_{3/8}}(q) &= \frac{1}{\eta(q)}\Theta_{2,2}(q), \\
\chi_{\mathcal{R}}(q) \equiv \chi_{\mathcal{R}_0}(q) = \chi_{\mathcal{R}_1}(q) &= \frac{2}{\eta(q)}\Theta_{1,2}(q).
\end{aligned} \tag{3.5}$$

Here, $\eta(q) = q^{1/24} \prod_{n\in N} (1-q^n)$ is the Dedekind eta function and $\Theta_{r,2}(q)$ and $(\partial\Theta)_{1,2}(q)$ are the ordinary and affine Riemann-Jacobi theta functions:

$$\begin{aligned}
\Theta_{r,k}(q) &= \sum_{n\in\mathbb{Z}} q^{(2kn+r)^2/4k}, \\
(\partial\Theta)_{r,k}(q) &= \sum_{n\in\mathbb{Z}} (2kn+r)\, q^{(2kn+r)^2/4k}, \\
(\nabla\Theta)_{r,k}(q) &= \frac{1}{2\pi}\log(q)\,(\partial\Theta)_{r,k}(q) = \mathrm{i}\tau\,(\partial\Theta)_{r,k}(q).
\end{aligned} \tag{3.6}$$

Y. Ishimoto showed that the metric for the two representations $\mathcal{R}_0$ and $\mathcal{R}_1$ can be chosen in the following form [37]:

$$\begin{aligned}
\langle \Omega|\Omega\rangle &= 0, & \langle \Omega|\omega\rangle &= 1, & \langle \omega|\omega\rangle &= d, \\
\langle \phi^+|\phi^-\rangle &= 0, & \langle \phi^+|\psi^-\rangle &= -1, & \langle \psi^+|\psi^-\rangle &= -t,
\end{aligned} \tag{3.7}$$

where d and t are in principal arbitrary real numbers. This determines the metric completely (see appendix A).

3.2 Boundary states – previous works

The field of boundary logarithmic conformal field theory has not been much developed up to now. The first works on this topic appeared first in 2000 when I. I. Kogan and J. F. Wheater tried to solve it via the standard Ishibashi approach [34]. They restricted themselves to the case of only one rank-2 indecomposable representation $\mathcal{R}$ containing the (irreducible) sub-representation $\mathcal{V}$. To get rid of the zero norm states they introduced some regularization, such that the states in $\mathcal{V}$ were normalised to a polynomial in some small value ϵ. Here, the orthogonal basis states of $\mathcal{V}$ are denoted by $|l,n\rangle$. The additional states in order to complete this set of states to a basis of $\mathcal{R}$ are labeled by $|l,n\rangle\rangle$. Their proposal for the Ishibashi boundary states was

$$|R\rangle = \frac{1}{\epsilon}\sum_{l,n} |l,n;l,n\rangle\rangle + |l,n;l,n\rangle \quad\text{and}\quad |V\rangle = \epsilon\sum_{l,n}|l,n;l,n\rangle. \tag{3.8}$$

The cylinder amplitudes that arise from these states are

$$\begin{aligned}
\langle R|\hat{q}|R\rangle &= \chi_0(q)\log(q) + \chi_1(q), \\
\langle R|\hat{q}|V\rangle &= \chi_0(q), \\
\langle V|\hat{q}|V\rangle &= 0.
\end{aligned} \tag{3.9}$$

In the c $= -2$ case, these characters were identified in the following way in order to stick to the lines of Cardy to construct the physical relevant boundary conditions:

$$\chi_0(q) = \frac{(\partial\Theta)_{1,2}(q)}{\eta(q)} \quad\text{and}\quad \chi_1(q) = \frac{1}{\eta(q)}\Big(\Theta_{1,2}(q) - (\partial\Theta)_{1,2}(q)\Big). \tag{3.10}$$

Obviously, it is not very satisfying that these "characters" are not exactly the physical ones but rather different linear combinations of (3.5). Anyhow, together with the two ordinary Ishibashi states $\big|V_{-1/8}\big\rangle$ and $\big|V_{3/8}\big\rangle$ for the irreducible representation modules $\mathcal{V}_{-1/8}$ and $\mathcal{V}_{3/8}$ the physical relevant boundary states were identified:

$$\begin{aligned} \big|1\big\rangle &= a\big|\,V\big\rangle + \big|\,V_{-1/8}\big\rangle + \big|\,V_{3/8}\big\rangle, \\ \big|2\big\rangle &= b\big|\,V\big\rangle + \sqrt{\frac{3}{2}}\big|\,V_{-1/8}\big\rangle + \sqrt{\frac{1}{2}}\big|\,V_{3/8}\big\rangle, \\ \big|3\big\rangle &= c\big|\,V\big\rangle + \sqrt{2}\big|\,V_{-1/8}\big\rangle. \end{aligned} \tag{3.11}$$

Here, a, b and c are arbitrary (!) constants. It is interesting that $\big|R\,\big\rangle$ does not show up at all.

Another proposition for the Ishibashi construction in the triplet model was given by Y. Ishimoto [37]. He ignored the fact that the states in $\mathcal{V}_0$ and $\mathcal{V}_1$, respectively, have zero norm but rather assumed an orthonormal basis for the modules $\mathcal{R}_0$ and $\mathcal{R}_1$. It was derived that the only possible boundary states for the rank-2 indecomposable representations were those based on the subrepresentations. From this, Ishimoto conjectured that it is indeed true for rank-2 indecomposable representations in general: There can only exist one corresponding boundary state. This conjecture is discussed further below. For the states connected to the fusion rules coefficients[3] a wide range of possibilities was given.

A third approach towards a proper and consistent prescription of the boundary states in the triplet model was proposed by S. Kawai and J. F. Wheater [35]. Their construction exploits the fact that the triplet model can be described in terms of symplectic fermions by starting from the fermionic (η, ξ)-ghost system with central charge $\mathrm{c} = -2$. The main idea was that the boundary equations with respect to this algebra are satisfied by certain coherent states. Chapter 4 concentrates explicitly on the boundary states obeying the symplectic fermion symmetry. There, this approach is described in detail.

3.3 A generalisation of the standard formalism – The matrix approach

This section deals with a matrix formulation of the boundary states. Those for the $\mathrm{c} = -2$ triplet model are derived by a direct generalisation of the Ishibashi states. It will turn out in the next section that this approach is not sufficient enough to cover the complete set of boundary states. Anyhow, the idea of this section is to give a very intuitive prescription for dealing with the derivation of boundary states.

The boundary conditions implied by conformal invariance and the ansatz for the boundary states are:

$$\left(\mathcal{O}^r_n - (-1)^{s_r}\overline{\mathcal{O}}^r_{-n}\right)\big|B\big\rangle = 0, \tag{3.12}$$

$$\big|B\big\rangle = \sum_{l,m,n} c^l_{mn}\left(\mathbb{1}\otimes\overline{\mathcal{U}}\right)\big|l,m;l,n\big\rangle. \tag{3.13}$$

[3] i. e. obtained by the Cardy construction

In (3.13) it is already implemented that only those product states survive that come with the same weight $l = \bar{l}$ as discussed in section 2.2.
The dual basis is given by the states $\langle l_1, \alpha; l_2, \beta| \left(\mathbb{1} \otimes \overline{\mathcal{U}}^\dagger\right)$. By multiplying such a state from the left, equation (3.12) transforms into

$$\begin{aligned}
0 &= \langle l_1, \alpha; l_2, \beta| \left(\mathcal{O}^r_n - \overline{\mathcal{O}}^r_{-n}\right) \sum_{l,p,q} c^l_{pq} |l, p; l, q\rangle \\
&= \sum_{l,p,q} \Big(\langle l_1, \alpha|\mathcal{O}^r_n|l, p\rangle c^l_{pq} \langle l, q|l_2, \beta\rangle - \langle l_1, \alpha|l, p\rangle c^l_{pq} \langle l, q|\mathcal{O}^r_n|l_2, \beta\rangle \Big) \qquad (3.14) \\
&= \langle l_1, \alpha|\mathcal{O}^r_n|l, p\rangle c^l_{pq} \delta_{l,l_2} g^{(l)}_{q\beta} - \delta_{l_1,l} g^{(l)}_{\alpha p} c^l_{pq} \langle l, q|\mathcal{O}^r_n|l_2, \beta\rangle.
\end{aligned}$$

Herein, g^l_{pq} is the metric on level l. Introducing the matrix notation $\mathcal{O}^{r\,\alpha}_{n\;\;\beta} \equiv \langle \alpha|\mathcal{O}^r_n|\beta\rangle$ the last line can be written in a very short way:

$$0 = \mathcal{O}^{r\,\alpha}_{n\;\;p} c_{pq} g_{q\beta} - g_{\alpha p} c_{pq} \mathcal{O}^{r\,q}_{n\;\;\beta}. \qquad (3.15)$$

By multiplying from left and right with the inverse metric g^{-1} and dropping the indices the final result is the matrix identity

$$0 = g^{-1} \cdot \mathcal{O}^r_n \cdot c - c \cdot \mathcal{O}^r_n \cdot g^{-1}. \qquad (3.16)$$

Obviously, for the coefficient matrix being the inverse metric, $c \equiv g^{-1}$, this equation is trivially satisfied. If the considered basis is orthonormal this reproduces the usual Ishibashi result exactly. The best one can do in the c $= -2$ theory, however, is to choose the metric as in (3.7) with some structure constants d and t for the two representations $\mathcal{R}_0$ and $\mathcal{R}_1$.
For simplicity, only $\mathcal{R}_0$ is treated here. All considerations are valid in the $\mathcal{R}_1$ case as well. The matrix $\mathcal{O}^r_n$ is connected to the metric g via multiplication (either from left or from right) with a matrix that is constant with respect to d:

$$A = \mathcal{O}^r_n \cdot g^{-1} \quad \text{and} \quad B = g^{-1} \cdot \mathcal{O}^r_n. \qquad (3.17)$$

This is seen as follows. The action of $\mathcal{O}^r_n$ on a state $|l + n, \alpha\rangle$ can always be expanded in terms of the given basis: $\mathcal{O}^r_n|l + n, \alpha\rangle = \sum_\beta \varrho^\alpha_\beta |\beta, n\rangle$. Since ϱ is completely determined by the commutation relations (3.1), it does not depend on d.

$$\begin{aligned}
B &= g^{-1} \cdot \mathcal{O}^r_n = g^{-1}_{pq} \mathcal{O}^{r\,q}_{n\;\;s} \\
&= g^{-1}_{pq} \langle q|\mathcal{O}^r_n|s\rangle \qquad (3.18) \\
&= g^{-1}_{pq} \langle q| \sum_\alpha \varrho^s_\alpha |\alpha\rangle \\
&= g^{-1}_{pq} g_{q\alpha} \varrho^s_\alpha = \varrho^p_s \equiv \varrho,
\end{aligned}$$

and together with the analogous calculation for A the statement is proven. Inserting this into (3.16) gives $A{\cdot}c - c{\cdot}B = 0$. Since $c = g^{-1}$ is a solution, it follow that $A{\cdot}g^{-1} - g^{-1}{\cdot}B = 0$. The inverse metric depends linear[4] on the structure constant d. Because A and B are

[4] For an explicit derivation, see appendix A.

constant matrices, the derivation ∂_d only acts on g^{-1} in this equation. Obviously $\partial_d g^{-1}$ is not equal to zero and since $A \cdot (\partial_d g^{-1}) - (\partial_d g^{-1}) \cdot B = 0$, there exists another proper boundary state with the coefficient matrix being equal to $c = -\partial_d g^{-1}$. The minus sign is added to normalise the coefficient of $\big(\Omega \otimes \Omega\big)$ in the sum to be positive.

Altogether, two proper independent boundary states were identified for the indecomposable representation $\mathcal{R}_0$:

$$\big|R_0\big\rangle = \big|c = g^{-1}\big\rangle \quad \text{and} \quad \big|V_0\big\rangle = \big|c = -\partial_d g^{-1}\big\rangle. \tag{3.19}$$

Here, g^{-1} is the metric on the representation $\mathcal{R}_0$.

The same considerations for the representation $\mathcal{R}_1$ lead to analogous results. Again two boundary states are found:

$$\big|R_1\big\rangle = \big|c = g^{-1}\big\rangle \quad \text{and} \quad \big|V_1\big\rangle = \big|c = \partial_t g^{-1}\big\rangle, \tag{3.20}$$

with g now being the metric on $\mathcal{R}_1$. The representations $\mathcal{V}_{-1/8}$ and $\mathcal{V}_{3/8}$ are ordinary, irreducible ones. They produce the usual Ishibashi states for these cases. Thus, a total number of six boundary states could be identified in this section.

Evidently, the two states $\big|R_0\big\rangle$ and $\big|R_1\big\rangle$ are *generalised* Ishibashi boundary states. They are a generalisation of the usual Ishibashi states to an arbitrary basis and to indecomposable rank-2 and higher representations. The states $\big|V_0\big\rangle$ and $\big|V_1\big\rangle$ are only build from states lying in the subrepresentations $\mathcal{V}_0$ and $\mathcal{V}_1$, respectively. This statement is proven first in the next chapter, but it can also be read off from the explicit calculations in appendix A. These boundary states may be assigned the name *level-2* Ishibashi states.

3.4 Via the general procedure

Here, the boundary states in the case of the rational c $= -2$ logarithmic conformal field theory are solved following the construction of chapter 2. A total number of ten proper boundary states is identified:

- A state $\big|V_{-1/8}\big\rangle$ for the pairing $\mathcal{V}_{-1/8} \otimes \overline{\mathcal{V}_{-1/8}}$,
- Another state $\big|V_{3/8}\big\rangle$ for $\mathcal{V}_{3/8} \otimes \overline{\mathcal{V}_{3/8}}$.

These states are the usual Ishibashi states for the representation modules $\mathcal{V}_{-1/8}$ and $\overline{\mathcal{V}_{3/8}}$ and coincide with the results of 3.3. The situation is different for states built on $\mathcal{R}_0 \otimes \overline{\mathcal{R}_0}$. There are two independent coefficients $c_{\Omega\Omega} \equiv c^0_{\Omega\Omega}$ and $c_{\Omega\omega} \equiv c^0_{\Omega\omega}$ and hence two different solutions. One state is derived for the complete module $\mathcal{R}_0$ and one for the submodule $\mathcal{V}_0$:

$$\begin{aligned} \big|R_0\big\rangle &= \big|c_{\Omega\Omega} = -d,\ c_{\Omega\omega} = 1\big\rangle = -d\big|\Omega, \Omega\big\rangle + \big|\Omega, \omega\big\rangle + \big|\omega, \Omega\big\rangle + \dots\,, \\ \big|V_0\big\rangle &= \big|c_{\Omega\Omega} = 1,\ c_{\Omega\omega} = 0\big\rangle = \big|\Omega, \Omega\big\rangle + \dots\,. \end{aligned} \tag{3.21}$$

The notation $\big|\alpha, \beta\big\rangle \equiv \big(\alpha \otimes \overline{\beta}\big)$ is introduced as a short-hand. The boundary states are defined via the given choice of the two free parameters, fixing all other coefficients in (2.25).

To be more explicit, the first few terms of the infinite sum are also given, i. e. the first level contributions[5]. In $\big|R_0\big\rangle$ the parameter $c_{\Omega\Omega} = -d$ is chosen for a convenient further discussion and to be compatible with the results of the previous section. Remember that d is a structure constant fixing the metric (3.7). Analogously, there are two boundary states for $\mathcal{R}_1 \otimes \overline{\mathcal{R}_1}$:

$$\begin{aligned}\big|R_1\big\rangle &= \big|c_{\xi^+\xi^-} = -t,\ c_{\phi^+\psi^-} = 1\big\rangle \\ &= t\Big\{-\big|\xi^+,\xi^-\big\rangle + \big|\xi^-,\xi^+\big\rangle + \big|\phi^+,\phi^-\big\rangle - \big|\phi^-,\phi^+\big\rangle\Big\} \\ &\qquad\qquad + \big|\phi^+,\psi^-\big\rangle - \big|\phi^-,\psi^+\big\rangle + \dots\,, \\ \big|V_1\big\rangle &= \big|c_{\xi^+\xi^-} = 1,\ c_{\phi^+\psi^-} = 0\big\rangle \\ &= \big|\xi^+,\xi^-\big\rangle - \big|\xi^-,\xi^+\big\rangle - \big|\phi^+,\phi^-\big\rangle + \big|\phi^-,\phi^+\big\rangle + \dots\,,\end{aligned} \tag{3.22}$$

where in $\big|R_1\big\rangle$ again $c_{\xi^+\xi^-} = -t$ is chosen for convenience. Herein, the coefficients are antisymmetric with respect to interchanging the $su(2)$-spin indices.

These results coincide with the ones of the previous section: Indeed, one can expand the states $\big|R_0\big\rangle$ and $\big|R_1\big\rangle$ into the sums (2.25) and re-introduce the anti-unitary operation $\mathcal{U}$ which acts as $W_n^{\pm}\mathcal{U} = -\mathcal{U}W_n^{\mp}$ and $W_n^0\mathcal{U} = -\mathcal{U}W_n^0$. Finally, the coefficient matrix γ can implicitly be defined by $\gamma \cdot (1 \otimes \overline{\mathcal{U}}) \equiv c$. The two states take the following form:

$$\big|R_\lambda\big\rangle = \sum_{l,m,n} \gamma^{\lambda}{}^{l}_{mn}\big(\mathbb{1} \otimes \overline{\mathcal{U}}\big)\big|l,m;l,n\big\rangle, \quad \lambda = 0,1. \tag{3.23}$$

The coefficient matrices γ^λ are exactly the inverse metrics on the corresponding representations $\mathcal{R}_\lambda$. Thus, $\big|R_\lambda\big\rangle$ are the generalised Ishibashi states introduced in the previous section. They are well-defined:

$$\begin{aligned} 0 &= \big(\mathcal{O}_n \pm \overline{\mathcal{O}}_{-n}\big)\,\big|R_\lambda\big\rangle \\ &= \big\langle l_1,a\big| \otimes \overline{\mathcal{U}\big\langle l_1,b\big|}\,\big(\mathcal{O}_n \pm \overline{\mathcal{O}}_{-n}\big) \sum_{l,m,n} \gamma^{\lambda}{}^{l}_{mn}\big|l,m\big\rangle \otimes \overline{\mathcal{U}\big|l,n\big\rangle} \\ &= \sum_{l,m,n}\Big(\big\langle l_1,a\big|\mathcal{O}_n\gamma^{\lambda}{}^{l}_{mn}\big|l,m\big\rangle\big\langle l,m\big|l_2,b\big\rangle - \big\langle l_1,a\big|\gamma^{\lambda}{}^{l}_{mn}\big|l,m\big\rangle\big\langle l,m\big|\mathcal{O}_n\big|l_2,b\big\rangle\Big) \\ &= \big\langle l_1,a\big|\,\big[\mathcal{O}_n, \mathbb{1}^\lambda\big]\,\big|l_2,b\big\rangle \end{aligned} \tag{3.24}$$

for the modes $\mathcal{O}_n$ of the chiral algebra. The operator $\mathbb{1}^\lambda$ defined by

$$\mathbb{1}^\lambda \equiv \sum_{l,m,n} \gamma^{\lambda}{}^{l}_{mn}\big|l,m\big\rangle\big\langle l,n\big| \tag{3.25}$$

[5]Explicit calculations can be found in appendix A.1. There, the first few level contributions of the boundary states for the Virasoro sector of $\mathcal{R}_0$ are derived.

is the projector onto the representation module $\mathcal{R}_\lambda$:

$$\begin{aligned}\left(\mathbb{1}^\lambda\right)^2 &= \sum_{l,m,n}\sum_{k,a,b} \big|l,m\big\rangle \gamma^{\lambda\,l}_{\;\;mn} \underbrace{\big\langle l,n\big|k,a\big\rangle}_{\delta_{lk}g_{na}} \gamma^{\lambda\,k}_{\;\;ab} \big\langle k,b\big| \\ &= \sum_{l,m,n}\sum_{k,a,b} \big|l,m\big\rangle \delta_{lk}\delta_{ma} \gamma^{\lambda\,k}_{\;\;ab} \big\langle k,b\big| \\ &= \sum_{l,m,b} \gamma^{\lambda\,l}_{\;\;mb} \big|l,m\big\rangle\big\langle l,b\big| \;=\; \mathbb{1}^\lambda.\end{aligned} \tag{3.26}$$

Hence, it commutes with the action of the algebra.

In ordinary conformal field theories there are no boundary states based on product states of different representations in their holomorphic and anti-holomorphic part because the weights of two different representations are usually disjunct sets. Here, the representations $\mathcal{R}_0$ and $\mathcal{R}_1$ contain the same weights and even their characters coincide [26]. Indeed, there exist another two doublets of boundary states. For the combination $\mathcal{R}_0 \otimes \overline{\mathcal{R}_1}$ one obtains:

$$\begin{aligned}\big|R_{01}^{+}\big\rangle &= \big|c_{\Omega\xi^+} = 1\big\rangle = \big|\Omega,\xi^+\big\rangle + \big|\Psi_1^-,\phi^-\big\rangle - \big|\Psi_1^+,\phi^+\big\rangle + \dots\,, \\ \big|R_{01}^{-}\big\rangle &= \big|c_{\Omega\xi^-} = 1\big\rangle = \big|\Omega,\xi^-\big\rangle + \big|\Psi_2^+,\phi^-\big\rangle - \big|\Psi_2^-,\phi^+\big\rangle + \dots\,.\end{aligned} \tag{3.27}$$

Analogously, for $\mathcal{R}_1 \otimes \overline{\mathcal{R}_0}$ one finds:

$$\big|R_{10}^{\pm}\big\rangle = \big|c_{\xi^\pm\Omega} = 1\big\rangle. \tag{3.28}$$

These states glue together different bulk representations at the boundary. Since they are built on different representations they are called *mixed* boundary states. They seem to be a specialty of logarithmic conformal field theories.

After all, six solutions for the boundary states could be identified that have a one-to-one correspondence to the representations. Furthermore, there are two doublet solutions that relate the two generalised highest weight modules to each other. These ten states span the space of all possible boundary states in the rational c $= -2$ logarithmic conformal field theory.

3.5 Structural properties

This section deals with the analysis of the properties of the newly derived boundary states. It is shown that they are related to each other by the action of certain operators. The off-diagonal part of the Hamiltonian $\mathcal{H} \simeq L_0 + \overline{L}_0$ occurring in the "scalar products" of the boundary states as in (2.9)

$$\big\langle B\big|q^{\mathcal{H}}\big|C\big\rangle = \big\langle B\big|q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})}\big|C\big\rangle \tag{3.29}$$

defines an operator $\hat{\mathcal{N}}$

$$\hat{\mathcal{N}} \equiv \hat{\delta} + \hat{\bar{\delta}}. \tag{3.30}$$

Recalling $L_0 = \hat{h} + \hat{\delta}$ and that in the rational c $= -2$ theory L_0 has Jordan blocks of dimension 2 at maximum for the $\mathcal{R}_0$ and $\mathcal{R}_1$ representations, i. e. $\hat{\delta}^2 = 0$, it follows that the operator $\hat{\mathcal{N}}$ is nilpotent of degree three: $\hat{\mathcal{N}}^3 = 0$. Given this operator it is clear by use of (3.2) that given a boundary state $\big|B\big\rangle$ then $\hat{\mathcal{N}}\big|B\big\rangle$ either vanishes or is a boundary state itself. It turns out that

$$\big|\,V_\lambda\big\rangle = \frac{1}{2}\hat{\mathcal{N}}\big|R_\lambda\big\rangle = -\partial\big|R_\lambda\big\rangle, \quad (\lambda = 0,\, 1), \tag{3.31}$$

where $\partial \equiv \partial_d$ if acting on $\big|R_0\big\rangle$ and $\partial \equiv -\partial_t$ if acting on $\big|R_1\big\rangle$. This shows that the two states $\big|\,V_\lambda\big\rangle$ are identical to the level-2 Ishibashi states derived in section 3.2. The structure implied by (3.31) is very similar to the bulk theory:

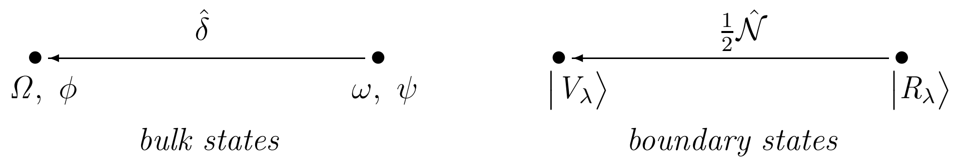

figure 3.2: Similarity of $\hat{\delta}$ and $\hat{\mathcal{N}}$

Every boundary state $\big|B\big\rangle$ satisfies

$$\hat{\mathcal{N}}^2\big|B\big\rangle = \partial^2\big|B\big\rangle = 0. \tag{3.32}$$

This is seen with the help of equation (2.24): $(\hat{\delta} - \hat{\hat{\delta}})\big|B\big\rangle = 0$ for a boundary state $\big|B\big\rangle$. The nilpotency of $\hat{\delta}$ and $\hat{\hat{\delta}}$ yields

$$0 = (\hat{\delta} - \hat{\hat{\delta}})^2\big|B\big\rangle = -2\hat{\delta}\hat{\hat{\delta}}\big|B\big\rangle = -\hat{\mathcal{N}}^2\big|B\big\rangle. \tag{3.33}$$

Equation (3.29) implies the definition of an operator $\hat{q}$:

$$\hat{q} \equiv q^{\frac{1}{2}\left(L_0+\overline{L}_0+\frac{1}{6}\right)} = q^{\frac{1}{2}\left(\hat{h}+\hat{\hat{h}}+\frac{1}{6}\right)}\left[1 + \log(q)\cdot\frac{1}{2}\hat{\mathcal{N}} + \log(q)^2\cdot\frac{1}{4}\hat{\mathcal{N}}^2\right]. \tag{3.34}$$

Here, c $= -2$ and $q \equiv \mathrm{e}^{2\pi\mathrm{i}\tau}$. The last equality is verified using the nilpotency property of $\hat{\mathcal{N}}$ and $L_0 + \overline{L}_0 \equiv \hat{h} + \hat{\hat{h}} + \hat{\mathcal{N}}$. By means of equation (3.32) this implies that pairings $\big\langle B\big|\hat{q}\big|C\big\rangle$ of boundary states can contain logarithmic terms proportional of order one at maximum, but never of higher order. This is not surprising, since usually, these pairings reproduce the torus amplitudes or equivalently, the characters. In ordinary conformal field theories the torus amplitudes and the characters span exactly the same representation of the modular group. Here, there are two different representations instead, a three-dimensional one for the physical characters and a five-dimensional one presumably for the torus amplitudes including the smaller one. The main result of [25, 26, 28, 29] is that the latter representation possesses elements of order $(\log(q))^1$ but neither of them contains $(\log(q))^2$ or higher order contributions. Unfortunately, pairings of the boundary states, i. e. the cylinder amplitudes, happen to contain no logarithmic terms at all. With the help of (3.5) they read

$$\begin{aligned} \big\langle V_{-1/8}\big|\hat{q}\big|V_{-1/8}\big\rangle &= \chi_{\mathcal{V}_{-1/8}}(q), & \big\langle V_{3/8}\big|\hat{q}\big|V_{3/8}\big\rangle &= \chi_{\mathcal{V}_{3/8}}(q), \\ \big\langle R_0\big|\hat{q}\big|R_0\big\rangle &= \chi_{\mathcal{R}}(q), & \big\langle R_1\big|\hat{q}\big|R_1\big\rangle &= \chi_{\mathcal{R}}(q). \end{aligned} \tag{3.35}$$

All other combinations vanish. In particular, the six states $\big|V_0\big\rangle$, $\big|V_1\big\rangle$, $\big|R_{01}^{\pm}\big\rangle$, and $\big|R_{10}^{\pm}\big\rangle$ are null-states with respect to the span of boundary states. Thus, the boundary states reproduce the three-dimensional representation of the modular group[6]. Coming back to Ishimoto's conjecture, from the cylinder amplitudes point of view there exists exactly one non-trivial boundary state for each of the rank-2 representations. In this sense, the conjecture holds, though presumably not in the way it was meant to. Due to the fact that most of the derived boundary states vanish the properties related to the inner structure of the indecomposable representations are not visible at this state. Therefore, it is necessary to study the boundary states in more detail.

In the following the main focus is on the structural relations of the states to each other. In figure 3.2 the similarity of $\hat{\delta}$ in the bulk theory and $\hat{\mathcal{N}}$ for boundary states was shown. Since $\hat{\mathcal{N}}$ has nilpotency degree three, the question arises if it is possible to construct states, such that the boundary states $\big|R_\lambda\big\rangle$ are the image of these states under the action of $\hat{\mathcal{N}}$. Of course, those additional states cannot be proper boundary states since they violate the boundary equations (2.1) and (2.2) and especially (3.32). Unfortunately, one can only find two states $\big|X_\lambda\big\rangle$ and $\big|Y_\lambda\big\rangle$ such that

$$\big|R_\lambda\big\rangle = \hat{\mathcal{N}}\big|X_\lambda\big\rangle + \big|Y_\lambda\big\rangle \quad \text{and} \quad \big|V_\lambda\big\rangle = \frac{1}{2}\hat{\mathcal{N}}\big|R_\lambda\big\rangle = \frac{1}{2}\hat{\mathcal{N}}^2\big|X_\lambda\big\rangle \quad (\lambda = 0,\, 1). \tag{3.36}$$

$$\begin{tikzcd} \big|X_\lambda\big\rangle \arrow[r, "\hat{\mathcal{N}}"'] & \oplus \arrow[r] & \big|R_\lambda\big\rangle \arrow[r, "\frac{1}{2}\hat{\mathcal{N}}"'] & \big|V_\lambda\big\rangle \\ & \big|Y_\lambda\big\rangle \arrow[u] & & \end{tikzcd}$$

figure 3.3: weak boundary states

The choice of $\big|X_\lambda\big\rangle$ and $\big|Y_\lambda\big\rangle$ is not unique. It is possible to add states belonging to the kernel of $\hat{\mathcal{N}}$ to $\big|X_\lambda\big\rangle$ without changing anything as well as one could subtract states from $\big|X_\lambda\big\rangle$ that belong to the kernel of $\hat{\mathcal{N}}^2$ and add their images under the $\hat{\mathcal{N}}$-operation to $\big|Y_\lambda\big\rangle$. $\big|X_\lambda\big\rangle$ and $\big|Y_\lambda\big\rangle$ generate the boundary states and can be called *weak* boundary states. This is justified by looking at their scalar products with the original boundary

[6] the set of physical characters

states. They can be chosen uniquely[7] to produce

$$\begin{aligned}
\langle X_\lambda|\hat{q}|V_\lambda\rangle &= \chi_{\mathcal{V}_\lambda}(q), & \langle X_\lambda|\hat{q}|R_\lambda\rangle &= \log(q)\cdot\chi_{\mathcal{V}_\lambda}(q),\\
\langle X_\lambda|\hat{q}|Y_\lambda\rangle &= 0, & \langle Y_\lambda|\hat{q}|R_\lambda\rangle &= \chi_{\mathcal{R}}(q) - 2\chi_{\mathcal{V}_\lambda}(q),\\
\langle R_\lambda|\hat{q}|R_\lambda\rangle &= \chi_{\mathcal{R}}(q).
\end{aligned} \tag{3.37}$$

The set of boundary states and the two states $|X_\lambda\rangle$ together reproduce the elements of the five-dimensional representation of the modular group. In the pairings, logarithmic terms occur only of order one at maximum. Unfortunately, they contain terms proportional to $\log(q)\Theta_{1,2}(q)$ as well which are not physical and do not belong to the representation. Luckily, they occur in such a way that they are suppressed in certain linear combinations of the boundary states. At this state the physical meaning of the additional states $|X_\lambda\rangle$ and $|Y_\lambda\rangle$ remains open. Before going further into this question the states $|R^\pm_{01}\rangle$ and $|R^\pm_{10}\rangle$ that relate the two generalised highest weight representations to each other are studied. Similarly to the definition of $|X_\lambda\rangle$ states $|Z^\pm_{01}\rangle$ and $|Z^\pm_{10}\rangle$ can be constructed in such a way that

$$\langle Z^\pm_{01}|\hat{q}|R^\pm_{01}\rangle = \langle Z^\pm_{10}|\hat{q}|R^\pm_{10}\rangle = \frac{1}{2}\chi_{\mathcal{R}}(q). \tag{3.38}$$

These states have the property $\hat{\mathcal{N}}|Z^\pm_{\lambda(1-\lambda)}\rangle = |R^\pm_{\lambda(1-\lambda)}\rangle$ and fulfil $\hat{\mathcal{N}}^2|Z^\pm_{\lambda(1-\lambda)}\rangle = 0$. They can be interpreted as weak boundary states again.

Boundary states are associated to propagators that connect the holomorphic part (e. g. the upper half complex plane, in a very simple setting) to the formal anti-holomorphic one (the lower half plane):

$$\begin{aligned}
|R^\pm_{01}\rangle &= \sum_{l,m,n} c^l_{mn}|l,m\rangle\otimes\overline{|l,n\rangle} \quad\Leftrightarrow\quad \hat{\mathcal{P}}_\pm\mathcal{U}^\dagger \equiv \sum_{l,m,n} c^l_{mn}|l,m\rangle\langle l,n| \quad\text{and}\\
|R^\pm_{10}\rangle &= \sum_{l,m,n} c^l_{mn}|l,n\rangle\otimes\overline{|l,m\rangle} \quad\Leftrightarrow\quad \hat{\mathcal{P}}^\dagger_\pm\mathcal{U}^\dagger \equiv \sum_{l,m,n} c^l_{mn}|l,n\rangle\langle l,m|.
\end{aligned} \tag{3.39}$$

Here, $\mathcal{U}$ is the usual anti-unitary operator. Because the corresponding boundary states satisfy the Ishibashi equations (2.1) and (2.2) the operators $\hat{\mathcal{P}}_\pm$ and $\hat{\mathcal{P}}^\dagger_\pm$ commute with the action of the chiral algebra:

$$\begin{aligned}
0 &= \langle l_1,a|\otimes\overline{\langle l_2,b|}\left(L_n - \overline{L}_{-n}\right)|R_{01}\rangle\\
&= \sum_{l,r,s}\langle l_1,a|\otimes\overline{\langle l_2,b|}\left(L_n - \overline{L}_{-n}\right)c^l_{rs}|l,r\rangle\otimes\overline{|l,s\rangle}\\
&= \sum_{l,r,s} c^l_{rs}\Big\{\langle l_1,a|L_n|l,r\rangle\langle l,s|l_2,b\rangle - \langle l_1,a|l,r\rangle\langle l,s|L_n|l_2,b\rangle\Big\}\\
&= \langle l_1,a|\big[L_n,\hat{\mathcal{P}}\mathcal{U}^\dagger\big]|l_2,b\rangle\\
&= \langle l_1,a|\big[L_n,\hat{\mathcal{P}}\big]\mathcal{U}^\dagger|l_2,b\rangle.
\end{aligned} \tag{3.40}$$

[7] The explicit construction is found in appendix A.4.

Thus, $\hat{\mathcal{P}}$ commutes with the Virasoro modes. Analogously, $\big[\hat{\mathcal{P}}, W_n^a\big]$ is calculated and the statement is proven. This implies that for every boundary state $\big|B\big\rangle$ the states $\hat{\mathcal{P}}\big|B\big\rangle$ and $\hat{\mathcal{P}}^\dagger\big|B\big\rangle$ are again boundary states or equal to zero. The action of the operators $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}^\dagger$ on the bulk states in the representation modules $\mathcal{R}_0$ and $\mathcal{R}_1$ is given by:

$$\begin{array}{lll} \hat{\mathcal{P}}^\dagger_\pm\big|\omega\big\rangle = \big|\xi^\pm\big\rangle, & \hat{\mathcal{P}}^\dagger_\pm\big|\Omega\big\rangle = 0, & \hat{\mathcal{P}}_+\big|\psi^\pm\big\rangle = -\big|\Psi_2^\pm\big\rangle, \\ \hat{\mathcal{P}}_\pm\big|\xi^\mp\big\rangle = \pm\big|\Omega\big\rangle, & \hat{\mathcal{P}}_\pm\big|\phi^\pm\big\rangle = 0, & \hat{\mathcal{P}}_-\big|\psi^\pm\big\rangle = \big|\Psi_1^\pm\big\rangle, \\ \hat{\mathcal{P}}^\dagger_-\big|\Psi_2^\pm\big\rangle = -\big|\phi^\pm\big\rangle, & \hat{\mathcal{P}}^\dagger_+\big|\Psi_1^\pm\big\rangle = \big|\phi^\pm\big\rangle. & \end{array} \tag{3.41}$$

Especially, these operators decompose the off-diagonal part $\hat{\delta}$ of L_0:[8]

$$\hat{\delta} = \begin{cases} \hat{\mathcal{P}}\hat{\mathcal{P}}^\dagger & \text{on} \quad \mathcal{R}_0 \\ \hat{\mathcal{P}}^\dagger\hat{\mathcal{P}} & \text{on} \quad \mathcal{R}_1 \end{cases} . \tag{3.42}$$

Using this equality it is easy to agree on the existence of the mixed states:

$$\begin{array}{ll} \big|R_{01}^\pm\big\rangle = \hat{\mathcal{P}}_\pm\big|R_1\big\rangle = \hat{\overline{\mathcal{P}}}{}^\dagger_\pm\big|R_0\big\rangle, & \big|V_0\big\rangle = \hat{\overline{\mathcal{P}}}_\mp\big|R_{01}^\pm\big\rangle = \hat{\mathcal{P}}_\mp\big|R_{10}^\pm\big\rangle, \\ \big|R_{10}^\pm\big\rangle = \hat{\overline{\mathcal{P}}}_\pm\big|R_1\big\rangle = \hat{\mathcal{P}}^\dagger_\pm\big|R_0\big\rangle, & \big|V_1\big\rangle = \hat{\mathcal{P}}^\dagger_\mp\big|R_{01}^\pm\big\rangle = \hat{\overline{\mathcal{P}}}{}^\dagger_\mp\big|R_{10}^\pm\big\rangle. \end{array} \tag{3.43}$$

The action of the operators $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}^\dagger$ and their anti-holomorphic partners on the states $\big|X_\lambda\big\rangle$ and $\big|Y_\lambda\big\rangle$ relate them to each other by

$$\big|Y_0\big\rangle = \hat{\mathcal{P}}\hat{\overline{\mathcal{P}}}\big|X_1\big\rangle \quad \text{and} \quad \big|Y_1\big\rangle = \hat{\mathcal{P}}^\dagger\hat{\overline{\mathcal{P}}}{}^\dagger\big|X_0\big\rangle. \tag{3.44}$$

Therefore, the states $\big|X_\lambda\big\rangle$ are the generating states for the boundary states involving the indecomposable representations. On the other hand, the denomination *weak* boundary states is now justified in the sense that they fulfil a slightly weaker boundary condition that is derived from (2.1) and (2.2) by the action of certain operators $\hat{\mathcal{A}} \in \{\hat{\mathcal{N}}, \hat{\mathcal{CD}}\}$, where $\mathcal{C}, \mathcal{D} \in \{\hat{\mathcal{P}}, \hat{\overline{\mathcal{P}}}, \hat{\mathcal{P}}^\dagger, \hat{\overline{\mathcal{P}}}{}^\dagger\}$:

$$\hat{\mathcal{A}}\left(\mathcal{O}_n \pm \overline{\mathcal{O}}_{-n}\right)\big|X_\lambda\big\rangle = 0. \tag{3.45}$$

The relations between the boundary states under the action of $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}^\dagger$ and their anti-holomorphic partners look schematically like the following. On the right hand side the embedding scheme of the representation $\mathcal{R}$ of the local logarithmic conformal field theory

[8] From now on, the spin index is omitted wherever it is possible and unless confusions arise. The reader is encouraged to add the indices in the appropriate way.

with central charge c = −2 [29] is cited which looks exactly the same:

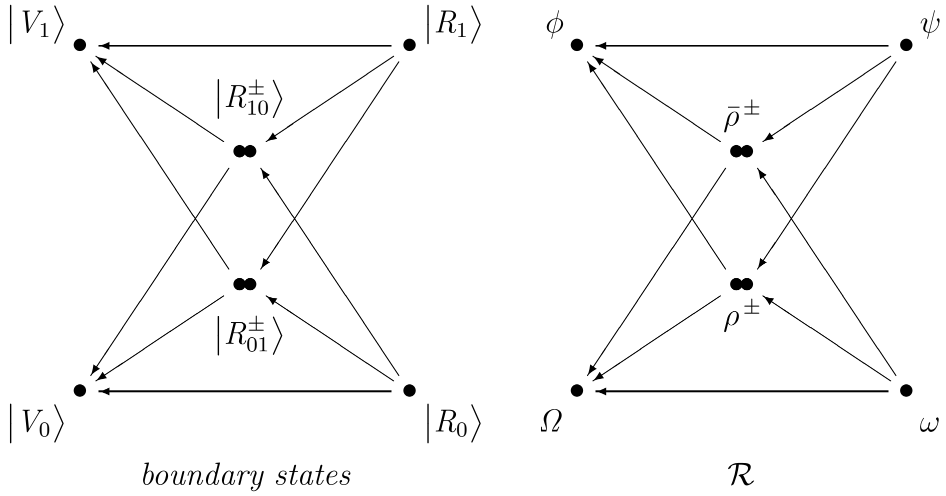

figure 3.4: boundary states vs. local theory

The lines in the left picture refer to the action of $\hat{\mathcal{P}}$, $\hat{\mathcal{P}}^\dagger$, $\hat{\overline{\mathcal{P}}}$, and $\hat{\overline{\mathcal{P}}}^\dagger$ while in the right picture they denote the action of the symmetry algebra. The perfect one-to-one correspondence between these two diagrams suggests that there is a deeper relation between the local theory and the chiral one with boundaries. This is quite remarkable, especially because the boundary states were derived completely independent of the local theory[9]. On the other hand, it is already clear from the very beginning that there has to be at least a link between the two theories. The local theory fuses together a chiral and an anti-chiral copy of the rational c = −2 theory. To keep all correlators local, certain states are discarded: the image of $L_0 - \overline{L}_0$. Equation (2.1) treated for $n = 0$ yields that exactly the states with non-vanishing norm in the range of $L_0 - \overline{L}_0$ are not allowed to contribute to the boundary states.

3.6 Modular properties (1)

The pairings of the non-vanishing boundary states (3.35) reproduce the elements of the three-dimensional representation of the modular group. This section investigates to what extent Cardy's formula can be applied to find linear combinations of the boundary states that satisfy Cardy's consistency equation (2.8) and are related to the physical boundary conditions.

For the set $\{\chi_{\mathcal{R}_0}, \chi_{\mathcal{R}_1}, \chi_{\mathcal{V}_{-1/8}}, \chi_{\mathcal{V}_{3/8}}\}$, the $\mathcal{S}$ and $\mathcal{T}$ matrices[10] are derived in [27]. There are six proper choices due to the fact that there are four independent orthogonal representation modules whose characters build only a three-dimensional representation of the

[9] See appendix C for a short overview, a detailed prescription can be found in [8, 29].

[10] giving the transformations of the characters under the modular transformations $\tau \to -1/\tau$ and $\tau \to \tau + 1$

modular group. One of the possibilities is:

$$\mathcal{S} = \begin{pmatrix} \frac{\mathrm{i}}{2} & -\frac{\mathrm{i}}{2} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{\mathrm{i}}{2} & \frac{\mathrm{i}}{2} & \frac{1}{4} & -\frac{1}{4} \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\pi/6} & 0 & 0 \\ \mathrm{e}^{\mathrm{i}\pi/6} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-\mathrm{i}\pi/12} & 0 \\ 0 & 0 & 0 & -\mathrm{e}^{-\mathrm{i}\pi/12} \end{pmatrix}. \tag{3.46}$$

The associated charge conjugation matrix $\mathcal{C}$ is a permutation matrix and has the following form:

$$\mathcal{C} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.47}$$

Altogether, these matrices satisfy $\mathcal{S}^4 = 1$ and $(\mathcal{ST})^3 = \mathcal{S}^2 = \mathcal{C}$. The set of boundary basis states $\{|R_0\rangle, |R_1\rangle, |V_{-1/8}\rangle, |V_{3/8}\rangle\}$[11] is the starting point for Cardy's method. Firstly, the vacuum boundary state $|\boldsymbol{\omega}\rangle$ has to be constructed. It can be written in terms of the basis states as

$$|\boldsymbol{\omega}\rangle = A|R_0\rangle + B|R_1\rangle + C|V_{-1/8}\rangle + D|V_{3/8}\rangle. \tag{3.48}$$

The remaining problem is that the Cardy formulation requires that $\mathcal{S}_0^j$, the line corresponding to the transformation of the vacuum character, has positive valued entries because the vacuum state should be self-conjugate. This is obviously not the case. Moreover, the coefficients are complex. A possible way out is to abandon self-conjugation, introduce a conjugate vacuum representation boundary state $|\boldsymbol{\omega}^\vee\rangle$,

$$|\boldsymbol{\omega}^\vee\rangle = A^*|R_0\rangle + B^*|R_1\rangle + C^*|V_{-1/8}\rangle + D^*|V_{3/8}\rangle, \tag{3.49}$$

and allow only boundary conditions of the form $(\alpha^\vee, \beta)$. This means to apply the condition $\alpha^\vee$ on one side of the cylinder instead of α and on the other side the condition β. Now, one can calculate the boundary states *à la* Cardy:

$$|\boldsymbol{i}\rangle = \sum_j \frac{S_i^j}{\sqrt{S_0^j}} |j\rangle. \tag{3.50}$$

Here, $|j\rangle$ denotes the boundary basis state belonging to the representation j and $|\boldsymbol{i}\rangle$ is the physical relevant boundary state corresponding to a bulk Hamiltonian that contains only the representation i and the vacuum in its spectrum. The physical boundary states are finally:

$$\begin{aligned} |\boldsymbol{\omega}\rangle &= \tfrac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\pi/4}|R_0\rangle - \tfrac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\pi/4}|R_1\rangle + \tfrac{1}{2}|V_{-1/8}\rangle + \tfrac{\mathrm{i}}{2}|V_{3/8}\rangle, \\ |\boldsymbol{\psi}\rangle &= -\tfrac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\pi/4}|R_0\rangle + \tfrac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\pi/4}|R_1\rangle + \tfrac{1}{2}|V_{-1/8}\rangle + \tfrac{\mathrm{i}}{2}|V_{3/8}\rangle, \\ |\boldsymbol{\mu}\rangle &= \sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}|R_0\rangle - \sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}|R_1\rangle + |V_{-1/8}\rangle - \mathrm{i}|V_{3/8}\rangle, \\ |\boldsymbol{\nu}\rangle &= -\sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}|R_0\rangle + \sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}|R_1\rangle + |V_{-1/8}\rangle - \mathrm{i}|V_{3/8}\rangle. \end{aligned} \tag{3.51}$$

[11] i. e. neglecting the null states

Here, the states are denoted in correspondence to the cyclic states of the underlying bulk representation. The conjugate states are given by complex conjugation of the coefficients. The boundary states (3.51) are not uniquely defined but rather chosen up to a $\mathbb{Z}_4$ symmetry in the coefficient phases. Furthermore, the pairs $\big|\boldsymbol{\omega}\big\rangle$, $\big|\boldsymbol{\psi}\big\rangle$ and $\big|\boldsymbol{\mu}\big\rangle$, $\big|\boldsymbol{\nu}\big\rangle$ are related by a $\mathbb{Z}_2$ symmetry. This is already implemented in the $\mathcal{S}$ matrix (3.46). Of course, it is also possible to start from any other of the five proper definitions of the $\mathcal{S}$ and $\mathcal{T}$ matrices. This leads to the same solutions up to the discussed symmetries. It is clear by construction that the partition function coefficients corresponding to these states are equal to the fusion rules that are related to the elements of the $\mathcal{S}$ matrix by the Verlinde formula [51]:

$$n^{k}_{i^\vee j} = N^{k}_{ij} = \sum_r \frac{\mathcal{S}^i_r \mathcal{S}^j_r \mathcal{S}^r_k}{\mathcal{S}^r_0}. \tag{3.52}$$

In ordinary conformal field theories, the $\mathcal{S}$ matrix diagonalises the fusion rules. As indicated in [27] this is not the case here. Instead, the fusion matrices are transformed into block-diagonal form.

This section showed that the standard Cardy procedure works perfectly well in the c = −2 theory on the character representation of the modular group. Only a minor sacrifice has to be accepted, the abandonment of a self-conjugate boundary state that corresponds to the vacuum representation. This is connected to the fact that the vacuum representation is taken to be $\mathcal{R}_0$ which embeds the vacuum into a rank-2 indecomposable representation and cannot exactly be identified with the identity operator as the true vacuum would. It seems natural that this can be generalised to other such theories.

3.7 Modular properties (2)

Here, the considerations of the previous section are repeated for the five-dimensional representation of the modular group, deriving again linear combinations of the boundary states following the lines of Cardy. Therefore, the complete set of boundary states plus the recently introduced dual states $\big|X_\lambda\big\rangle$ and $\big|Z_{\lambda(1-\lambda)}\big\rangle$ is studied. The representation was investigated in [25, 26, 28, 29].

In [28], an approach based on ideas of S. D. Mathur *et al.* [52, 53] was used, which is followed here: The linearly independent set of characters is given by

$$\Big\{\chi_{\mathcal{V}_0},\, \chi_{\mathcal{V}_{-1/8}},\, \chi_{\mathcal{V}_1}, \chi_{\mathcal{V}_{3/8}},\, 2\chi_{\widetilde{\mathcal{R}}} \equiv \frac{2}{\eta}\left[\Theta_{1,2} + \mathrm{i}\alpha\, (\nabla\Theta)_{1,2}\right]\Big\}, \tag{3.53}$$

where $\alpha \in \mathbb{R}$ is arbitrary. The corresponding $\mathcal{S}$ matrix that transforms the characters under $\tau \to -1/\tau$ is

$$\mathcal{S} = \begin{pmatrix} \frac{1}{2\alpha} & \frac{1}{4} & \frac{1}{2\alpha} & -\frac{1}{4} & -\frac{1}{4\alpha} \\ 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ -\frac{1}{2\alpha} & \frac{1}{4} & -\frac{1}{2\alpha} & -\frac{1}{4} & \frac{1}{4\alpha} \\ -1 & \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ -2\alpha & 1 & 2\alpha & -1 & 0 \end{pmatrix}. \tag{3.54}$$

In order to obtain the elements of the five-dimensional representation from the cylinder amplitudes partners to the boundary null-states were introduced in section 3.5 that serve as duals for the null-states. However, taking into account these states they should at least vanish in a physical limit because they are no proper boundary states. This can be implemented by renormalising the two states $\big|V_\lambda\big\rangle$ and $\big|X_\lambda\big\rangle$ such that

$$\big|V_\lambda\big\rangle \longrightarrow \frac{2\pi}{\sqrt{2}\alpha}\big|V_\lambda\big\rangle \quad\text{and}\quad \big\langle X_\lambda\big| \longrightarrow \frac{\sqrt{2}\alpha}{2\pi}\big\langle X_\lambda\big|. \tag{3.55}$$

The pairings $\big\langle X_\lambda\big|\hat{q}\big|V_\lambda\big\rangle$ do not change for any choice of α. In particular for $\alpha = 2\pi/\sqrt{2}$ the original states are obtained. On the other hand, the pairings $\big\langle X_\lambda\big|\hat{q}\big|R_\lambda\big\rangle$ get an additional pre-factor $(\sqrt{2}\alpha)/(2\pi)$:

$$\big\langle X_\lambda\big|\hat{q}\big|R_\lambda\big\rangle = \frac{\sqrt{2}\alpha}{2\pi}\log(q)\chi_{\mathcal{V}_\lambda}(q) \quad\text{and}\quad \big\langle X_\lambda\big|\hat{q}\big|V_\lambda\big\rangle = \chi_{\mathcal{V}_\lambda}(q). \tag{3.56}$$

From now on, $\big|V_\lambda\big\rangle$ and $\big|X_\lambda\big\rangle$ always refer to these renormalised states.
Following the Cardy formalism the physical vacuum boundary state is, in ordinary conformal field theory and up to a choice of phases, given by (3.50) for $i = 0$. As in the three-dimensional case, this definition has to be treated more carefully. First of all, since the concerned elements of the $\mathcal{S}$ matrix are not positive it is again necessary to introduce a conjugate vacuum representation boundary state $\big|\boldsymbol{\Omega}^\vee\big\rangle$ in order to be able to follow Cardy's arguments. The naively computed boundary states do not exactly reproduce the characters (3.5). Instead, the cylinder amplitudes incorporate certain linear combinations of them and terms proportional to $\log(q)\Theta_{1,2}(q)$ which are not physical. Luckily, $\big|Z_{\lambda(1-\lambda)}\big\rangle$ and $\big|R_{\lambda(1-\lambda)}\big\rangle$ are not used up to now, because they do not correspond to a unique representation. Only for those, of course, the $\mathcal{S}$ matrix yields the transformations. Adding these states by hand in the correct way they can serve as counter terms to adjust the boundary states such that they give the correct results:

$$\begin{aligned}
\big|\boldsymbol{\Omega}\big\rangle &= \frac{1}{2\sqrt{\alpha}}\Big\{\sqrt{2}\left(\big|V_0\big\rangle + \big|V_1\big\rangle\right) + \mathrm{i}\left(\big|R_0\big\rangle - \big|R_1\big\rangle\right) \\
&\qquad + \left(\big|R_{01}\big\rangle + \big|R_{10}\big\rangle\right)\Big\} + \frac{1}{2}\left(\big|V_{-1/8}\big\rangle + \mathrm{i}\big|V_{3/8}\big\rangle\right), \\
\big|\boldsymbol{\mu}\big\rangle &= \sqrt{2\alpha}\left\{\big|V_0\big\rangle + \big|V_1\big\rangle\right\} + \left(\big|V_{-1/8}\big\rangle - \mathrm{i}\big|V_{3/8}\big\rangle\right), \\
\big|\boldsymbol{\phi}\big\rangle &= \frac{1}{2\sqrt{\alpha}}\Big\{-\sqrt{2}\left(\big|V_0\big\rangle + \big|V_1\big\rangle\right) + \mathrm{i}\left(-\big|R_0\big\rangle + \big|R_1\big\rangle\right) \\
&\qquad + \left(-\big|R_{01}\big\rangle + \big|R_{10}\big\rangle\right)\Big\} + \frac{1}{2}\left(\big|V_{-1/8}\big\rangle + \mathrm{i}\big|V_{3/8}\big\rangle\right), \\
\big|\boldsymbol{\nu}\big\rangle &= -\sqrt{2\alpha}\left\{\big|V_0\big\rangle + \big|V_1\big\rangle\right\} + \left(\big|V_{-1/8}\big\rangle - \mathrm{i}\big|V_{3/8}\big\rangle\right), \\
\big|\boldsymbol{\omega}\big\rangle &= 2\sqrt{2\alpha^3}\Big\{-\big|V_0\big\rangle + \big|V_1\big\rangle\Big\} + 2\left(\big|V_{-1/8}\big\rangle + \mathrm{i}\big|V_{3/8}\big\rangle\right),
\end{aligned} \tag{3.57}$$

The coefficients for the six boundary basis states $\big|V_\lambda\big\rangle$ ($\lambda = -1/8, 0, 3/8, 1$) and $\big|R_\lambda\big\rangle$ ($\lambda = 0, 1$) fulfil the Cardy law[12]: Up to phases they are given by $\mathcal{S}_k^j/\sqrt{\mathcal{S}_0^j}$. The two

[12] all except the mixed boundary states

states corresponding to the indecomposable representations are treated by the same matrix elements except for a minus sign. Secondly, the mixed states guarantee that the partition function coefficients satisfy the Verlinde formula in the same way the $\mathcal{S}$ matrix elements reproduce the fusion rules. Indeed, calculating the partition functions for boundary conditions $(i^\vee, j)$ yields

$$Z_{i^\vee j}(q) = \sum_k n^k_{i^\vee j} \chi_k(q) \tag{3.58}$$

with $n^k_{i^\vee j}$ being *nearly* equal to the fusion coefficients derived from the given $\mathcal{S}$ matrix via the Verlinde formula (3.52). These fusion coefficients are equal to the physical ones up to some identifications[13] and in the physical limit $\alpha \longrightarrow 0$ under which the contributions of the non-boundary states and the mixed states vanish. For particular choices of boundary conditions the partition function contains terms proportional to $\log(q)\Theta_{1,2}(q)$. Fortunately, they have a pre-factor α and thus vanish in the limit $\alpha \longrightarrow 0$. One even finds that under this limit $n^k_{i^\vee j} = N^k_{ij}$. Again, the fusion matrices are not diagonalised by the $\mathcal{S}$ matrix but instead transformed in block-diagonal form. However, it is not possible to apply this limit to the boundary states themselves since they get singular at this point. The same problem was seen for the $\mathcal{S}$ matrix in [28]: Even though the fusion coefficients are reproduced by the Verlinde formula in the limit $\alpha \longrightarrow 0$, it is not possible to apply this limit to the matrix itself due to the fact that the set of characters gets linearly dependent and $\mathcal{S}$ (3.54) yields a singular behaviour.

[13] $2\mathcal{V}_0 + 2\mathcal{V}_1 \equiv \mathcal{R}_0 \equiv \mathcal{R}_1$ concerning the number of states on each level

Chapter 4

Symplectic fermion model

The concept of symplectic fermions was first introduced by H. G. Kausch [30] in order to describe the rational c = −2 logarithmic conformal field theory. After a short introduction the boundary states corresponding to the symplectic fermion symmetry algebra are derived and compared to the results of chapter 3 and the results of S. Kawai and J. F. Wheater [35]. It turns out that the boundary states based on this model are exactly the same as those derived under the restriction of the $\mathcal{W}$-algebra. They are identical to those of [35] which means that the two approaches really are compatible. This corresponds to the presumption of Kawai [55] that the coherent state approach is indeed as good as taking the usual Ishibashi states.

4.1 Symplectic fermions

Before deriving boundary states under the fermion symmetry, this section is intended to give a brief overview of the method of symplectic fermions in the c = −2 theory.

The theory has an explicit Lagrangian formulation based on two fermionic fields η and ξ of scaling dimension 1 and 0, respectively:

$$S = \frac{1}{\pi} \int \mathrm{d}^2 z \left(\eta \bar{\partial} \xi + \bar{\eta} \partial \bar{\xi} \right) . \tag{4.1}$$

This is the fermionic ghost system at c = −2 with the operator product expansions

$$\eta(z)\xi(w) = \xi(z)\eta(w) = \frac{1}{z-w} + \dots , \tag{4.2}$$

all other products are regular. The two fields can be combined into a two component symplectic fermion

$$\chi^+ \equiv \eta \quad \text{and} \quad \chi^- \equiv \partial\xi , \tag{4.3}$$

which has a bosonic sector being identical to the triplet model. This choice assures that χ^+ and χ^- have the same conformal weight $h = 1$. The symplectic fermion description differs from the ghost system only by the treatment of the zero modes in χ^- and ξ. The fermion modes are defined by the usual power series expansion

$$\chi^\pm(z) = \sum_{m \in \mathbb{Z}+\lambda} \chi^\pm_m z^{-m-1} , \tag{4.4}$$

where $\lambda = 0$ in the untwisted (bosonic) sector and $\lambda = \frac{1}{2}$ in the twisted (fermionic) sector. The modes satisfy the anticommutation relations

$$\{\chi^\alpha_m, \chi^\beta_n\} = m\,\varepsilon^{\alpha\beta}\delta_{m+n,0}\,, \tag{4.5}$$

with the totally antisymmetric tensor $\varepsilon^{\pm\mp} = \pm 1$. The symplectic fermions decompose the Virasoro modes and the W-modes of the $\mathcal{W}(2,3,3,3)$ triplet algebra:

$$\begin{aligned} L_n &= \frac{1}{2}\varepsilon_{\alpha\beta}\sum_{j\in\mathbb{Z}+\lambda} :\chi^\alpha_j\chi^\beta_{n-j}: + \frac{\lambda(\lambda-1)}{2}\delta_{n,0}, \\ W^0_n &= -\frac{1}{2}\sum_{j\in\mathbb{Z}+\lambda} j\cdot\{ :\chi^+_{n-j}\chi^-_j: + :\chi^-_{n-j}\chi^+_j: \}, \\ W^\pm_n &= \sum_{j\in\mathbb{Z}+\lambda} j\cdot\chi^\pm_{n-j}\chi^\pm_j. \end{aligned} \tag{4.6}$$

The highest weight states of the triplet model[1] are now related to each other by the fermion symmetry. In the twisted sector, the doublet states of weight $h = 3/8$ are connected to the singlet at weight $h = -1/8$ by $\nu^\alpha = \chi^\alpha_{-1/2}\mu$. The states of weight 0 in the untwisted sector are related by $\xi^\pm = -\chi^\pm_0\omega$, $\Omega = \chi^-_0\chi^+_0\omega$. Furthermore, one finds $\phi^\alpha = \chi^\alpha_{-1}\Omega$ and $\psi^\alpha = \chi^\alpha_{-1}\omega$. Thus, the symplectic fermion symmetry interconnects and intertwines the representations $\mathcal{R}_0$ with $\mathcal{R}_1$ and on the other hand $\mathcal{V}_{-1/8}$ with $\mathcal{V}_{3/8}$.

4.2 Coherent boundary states

In this section, the results of S. Kawai and J. F. Wheater [35] are briefly desribed.

The starting point is the consistency equation for boundary states under the symplectic fermion symmetry:

$$\left(\chi^\pm_m - \mathrm{e}^{\pm\mathrm{i}\phi}\overline{\chi}^\pm_{-m}\right)\left|B\right\rangle = 0, \tag{4.7}$$

where ϕ is the phase difference between the two boundaries. They showed that this equation is solved by the coherent states

$$\left|B_{0\phi}\right\rangle = N\exp\left(\sum_{k>0}\frac{\mathrm{e}^{\mathrm{i}\phi}}{k}\chi^-_{-k}\overline{\chi}^+_{-k} + \frac{\mathrm{e}^{-\mathrm{i}\phi}}{k}\overline{\chi}^-_{-k}\chi^+_{-k}\right)\left|0_\phi\right\rangle. \tag{4.8}$$

Here, N is a normalisation factor and $\left|0_\phi\right\rangle$ is a non-chiral ground state, which is one of the “invariant vacua” $\{(\Omega\otimes\overline{\Omega}), (\omega\otimes\overline{\omega}), (\mu\otimes\overline{\mu})\}$. Kawai and Wheater designed the boundary states in such a way that they obey the Virasoro boundary state equation and the equation for the modes of the $\mathcal{W}$-algebra. This implies that the phase ϕ can only take the values $\phi = 0$ and $\phi = \pi$. Therefore, there are six possible boundary states[2]:

$$\left|B_{\Omega+}\right\rangle \equiv \left|B_{\Omega,\phi=0}\right\rangle,\ \left|B_{\Omega-}\right\rangle,\ \left|B_{\omega\pm}\right\rangle,\ \text{and}\ \left|B_{\mu\pm}\right\rangle. \tag{4.9}$$

[1] see section 3.1

[2] denoted by (+) if $\phi = 0$ and (−) for $\phi = \pi$

The corresponding cylinder amplitudes $\langle B|\hat{q}|C\rangle$ for the interesting (untwisted) sector are[3]

$$\begin{array}{c} \\ |B_{\Omega+}\rangle \\ |B_{\Omega-}\rangle \\ |B_{\omega+}\rangle \\ |B_{\omega-}\rangle \end{array} \begin{array}{c} \begin{array}{cccc} \quad|B_{\Omega+}\rangle & |B_{\Omega-}\rangle & \qquad|B_{\omega+}\rangle & \qquad|B_{\omega-}\rangle \end{array} \\ \begin{pmatrix} 0 & 0 & \eta(q)^2 & \Theta_{1,2}(q) \\ 0 & 0 & \Theta_{1,2}(q) & \eta(q)^2 \\ \eta(q)^2 & \Theta_{1,2}(q) & d(d+\ln(q))\eta(q)^2 & d(d+\ln(q))\Theta_{1,2}(q) \\ \Theta_{1,2}(q) & \eta(q)^2 & d(d+\ln(q))\Theta_{1,2}(q) & d(d+\ln(q))\eta(q)^2 \end{pmatrix} \end{array}. \tag{4.10}$$

To get rid of the unphysical terms proportional to $\log(q)\Theta_{1,2}(q)$, one of the states $|B_{\omega\pm}\rangle$ was discarded and the physical boundary conditions were derived with this reduced set of states.

Candidates for the Ishibashi states were also given by the condition that they diagonalise the cylinder amplitudes, i. e. $\langle i|\hat{q}|j\rangle = \delta_{ij}\chi_i(q)$. However, it was not possible to express the physical boundary states in terms of this basis. Kawai and Wheater proposed the following five states and five corresponding duals:

$$\begin{aligned} |V_0\rangle &= \tfrac{1}{2}|B_{\Omega+}\rangle + \tfrac{1}{2}|B_{\Omega-}\rangle, & \langle V_0| &= -\tfrac{1}{2}\langle B_{\omega+}| - \tfrac{1}{2}\langle B_{\omega-}|, \\ |V_1\rangle &= \tfrac{1}{2}|B_{\Omega+}\rangle - \tfrac{1}{2}|B_{\Omega-}\rangle, & \langle V_1| &= \tfrac{1}{2}\langle B_{\omega+}| - \tfrac{1}{2}\langle B_{\omega-}|, \\ |V_{-1/8}\rangle &= \tfrac{1}{2}|B_{\mu+}\rangle + \tfrac{1}{2}|B_{\mu-}\rangle, & \langle V_{-1/8}| &= \tfrac{1}{2}\langle B_{\mu+}| + \tfrac{1}{2}\langle B_{\mu-}|, \\ |V_{3/8}\rangle &= \tfrac{1}{2}|B_{\mu+}\rangle - \tfrac{1}{2}|B_{\mu-}\rangle, & \langle V_{3/8}| &= \tfrac{1}{2}\langle B_{\mu+}| - \tfrac{1}{2}\langle B_{\mu-}|, \\ |R\rangle &= \sqrt{2}\,|B_{\Omega+}\rangle, & \langle R| &= -\sqrt{2}\,\langle B_{\omega-}|. \end{aligned} \tag{4.11}$$

The states based on the ground state $(\omega\otimes\overline{\omega})$ are used in exactly the same way as the $|X_\lambda\rangle$ in the previous chapter: They serve as duals to states which otherwise would be null states. The (ket-)states form only a four-dimensional space. Especially, $|R\rangle$ is associated to the indecomposable representations but only built on the subrepresentations. It is evident that the states $|B_{\omega\pm}\rangle$ cannot obey equation (4.7) without further restrictions because they are based on the state $(\omega\otimes\overline{\omega})$ which is obviously not invariant: $(L_0-\overline{L}_0)(\omega\otimes\overline{\omega}) = (\Omega\otimes\overline{\omega}) - (\omega\otimes\overline{\Omega}) \neq 0$, unless the right-hand side state is discarded as in the local theory. This was not mentioned by Kawai and Wheater. It is shown in the next section that their considerations are indeed compatible with the result of chapter 3 and lead to the same results if starting from the correct "vacua".

4.3 Boundary states

Here, the boundary states for the symplectic fermion algebra are derived using the method of chapter 2[4]. The boundary state consistency equation for this symmetry algebra is given

[3]The different factors and signs in contrast to [35] arise due to the different normalisation of the metric that was chosen there.

[4]This is subject of [57].

by (4.7):

$$\left(\chi_m^{\pm} - \mathrm{e}^{\pm \mathrm{i}\phi}\overline{\chi}_{-m}^{\pm}\right)\big|B\big\rangle = 0, \tag{4.12}$$

where ϕ is the spin which can take the values $\phi = 0, \pi$ at the boundary due to conformal invariance. It is clear that the equations (2.1) and (2.2) are automatically fulfilled once (4.12) is satisfied[5]. This implies that the solutions are linear combinations of the boundary states derived for the $\mathcal{W}$-algebra. It is an interesting question whether the fermion symmetry is more restrictive than the triplet model, i. e. if less states are found here than in the latter theory. Using the method of chapter 2 ten proper boundary states show up. Denoting the $\phi = 0$ case by the quantum number $(+)$ and $\phi = \pi$ by $(-)$ as in the previous discussion and giving the first few terms of the infinite sums, these states are:

$$\begin{aligned}
\big|\Omega, \Omega; \pm\big\rangle &= \big|\Omega, \Omega\big\rangle \pm \big|\phi^+, \phi^-\big\rangle \mp \big|\phi^-, \phi^+\big\rangle + \dots\,, \\
\big|\Omega, \omega; \pm\big\rangle &= \big|\Omega, \omega\big\rangle + \big|\omega, \Omega\big\rangle \pm \big|\xi^+, \xi^-\big\rangle \mp \big|\xi^-, \xi^+\big\rangle + \dots\,, \\
\big|\Omega, \xi^a; \pm\big\rangle &= \big|\Omega, \xi^a\big\rangle \pm \big|\xi^a, \Omega\big\rangle + \dots\,, \quad a = +, -, \\
\big|\mu, \mu; \pm\big\rangle &= \big|\mu, \mu\big\rangle \pm \big|\nu^+, \nu^-\big\rangle \mp \big|\nu^-, \nu^+\big\rangle + \dots\,.
\end{aligned} \tag{4.13}$$

This result may be compared to the one for the triplet model. The following identities are obvious:

$$\begin{aligned}
\big|\Omega, \Omega; \pm\big\rangle &= \big|V_0\big\rangle \pm \big|V_1\big\rangle, & \big|\Omega, \omega; \pm\big\rangle &= \left(\big|R_0\big\rangle + d\big|V_0\big\rangle\right) \pm \left(\big|R_1\big\rangle - t\big|V_1\big\rangle\right), \\
\big|\Omega, \xi^a; \pm\big\rangle &= \big|R_{01}^a\big\rangle \pm \big|R_{10}^a\big\rangle, & \big|\mu, \mu; \pm\big\rangle &= \big|V_{3/8}\big\rangle \pm \big|V_{-1/8}\big\rangle.
\end{aligned} \tag{4.14}$$

This identification uses the fact that the boundary states are defined in such a way that they obey (2.1) and (2.2). Thus, having found the contributions at weight $h = 0$ as written down in (4.13) they can be compared to the results of section 3.4. If they match the state is identified. This is compatible with the result (4.11) of Kawai and Wheater. Translated to their notation the states are

$$\big|B_{\Omega\pm}\big\rangle = \big|\Omega, \Omega; \pm\big\rangle \quad \text{and} \quad \big|B_{\mu\pm}\big\rangle = \big|\mu, \mu; \pm\big\rangle, \tag{4.15}$$

up to possible additive contributions from null-states and the fact that they used a different normalisation. It seems contradictive that here, no boundary state based on $\left(\omega \otimes \overline{\omega}\right)$ is found. But reviewing their work quoted in the last section they use these states denoted by $\big|B_{\omega\pm}\big\rangle$ as the duals to $\big|B_{\Omega\pm}\big\rangle$ as already mentioned before. This is in exact correspondence to what was called the weak boundary states $\big|X_\lambda\big\rangle$. The generic procedure of chapter 2 produces a much bigger collection of states. Especially, those leading to logarithmic terms in the cylinder amplitudes were identified. These were not discussed by S. Kawai and J. F. Wheater. Instead, they obtained the Ishibashi boundary state for the module $\mathcal{R}$ by the identification $2\mathcal{V}_0 + 2\mathcal{V}_1 \equiv \mathcal{R}$. Presumably therefore and for referring to the local theory by setting $\left(\Omega \otimes \overline{\omega}\right) - \left(\omega \otimes \overline{\Omega}\right)$ to zero, their physical boundary conditions differs from the set of Ishibashi states.

[5] This is proven in appendix B.

The coherent state method produces exactly the same amount of states when starting from the same "invariant vacua" as in this thesis[6]:

$$\big\{\big(\Omega\otimes\overline{\Omega}\big),\ \big(\Omega\otimes\overline{\omega}\big)+\big(\omega\otimes\overline{\Omega}\big),\ \big(\Omega\otimes\overline{\xi}^{a}\big),\ \big(\mu\otimes\overline{\mu}\big)\big\}. \tag{4.16}$$

The symplectic fermions decompose the L_0 operator in such a way that

$$\chi_0^{\pm}\omega=-\xi^{\pm}\quad\text{and}\quad\chi_0^{\pm}\chi_0^{\mp}\omega=\mp\Omega. \tag{4.17}$$

This provides a possible meaning to the intertwining operators $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}^{\dagger}$ and the corresponding boundary states $\big|R_{01}\big\rangle$ and $\big|R_{10}\big\rangle$: They might be closely related to the fermionic zero modes.

[6] except for possible null-state contributions

Chapter 5

Discussion

In this thesis a mathematically consistent way for the treatment of boundary states in logarithmic conformal field theories was presented and applied to the rational c $= -2$ theory. The advantage of the invented method is it's simplicity: It does only make use of an arbitrary basis for each representation module and the expansion of a given state with respect to this basis. In particular, this basis does not have to be orthonormal. As a side effect the algorithm produces the inverse metric on each representation module. The construction turns out to be finite in the sense that the components of the boundary states can be deduced up to any given finite level in a finite number of steps.

By applying this method to the rational c $= -2$ logarithmic conformal field theory, ten states that obey the Ishibashi boundary state conditions were identified and arranged in a scheme very similar to the embedding scheme of the local theory proposed by M. R. Gaberdiel and H. G. Kausch. Six of these states are null states in the space of boundary states. The remaining four together with the corresponding $\mathcal{S}$ matrix can be treated by the standard Cardy formalism in order to obtain the physical relevant boundary conditions concerning the three-dimensional representation of the modular group. On the other hand, additional states were found in such a way that their pairings with the boundary null-states together with the non-vanishing pairings of boundary states reproduce the five-dimensional representation of the modular group. By referring to these additional states as the dual states corresponding to the boundary null-states a slightly modified version of the Cardy formalism could be applied such that at least in a physical limit the partition function coefficients coincide with the fusion rules of the bulk theory as in the ordinary cases. For the application of this limit, the same problems arise as for the $\mathcal{S}$ matrix transforming the bulk characters. It is remarkable that the Cardy formalism works in both cases. The meaning of this, however, is still unknown but it is worth noting in this context that exactly the same elements that are presumed to form the set of torus amplitudes in the bulk theory were found. The investigation of the deeper meaning of the additional so-called weak boundary states is dedicated to future work.

In ordinary rational conformal field theories, solving Cardy's consistency condition reduces to finding non-negative integer-valued matrix representations (NIM-representations) of the Verlinde algebra [4, 54]. The five-dimensional solution for the c $= -2$ logarithmic conformal field theory does involve negative integers. They occur in a similar way as in the computation of fusion matrices in [28]. This demonstrates that the features coming along with rationality cannot be entirely applied to logarithmic conformal field theories, even though they share many properties with rational conformal field theories. The negative integers reflect precisely the linear dependencies among the boundary states which appear in the above discussed limiting procedure. If these dependencies are taken into account in the correct way, the final solution can be written without negative integers. Unfortunately, this last step has to be done by hand, since the computation via the $\mathcal{S}$ matrix and Cardy's ansatz leads to some negative integer coefficients. It remains an interesting open question, in which sense more general solutions than the NIM-representations should be taken into account for settings slightly more general than ordinary rational conformal field theory.

Y. Ishimoto conjectured that for every indecomposable representation of rank 2, there exists exactly one boundary state [37]. This work shows that this conjecture holds in the $c = -2$ case, even though not strictly. For each indecomposable representations two boundary states were derived where one only refers to the contained subrepresentation. This seems in contradiction to the stated conjecture. On the other hand, one of these two states turns out to be a null state in the space of boundary states.

In one of the first works on this topic, I. I. Kogan and J. F. Wheater tried to fix the zero-norm state problem by a perturbative procedure. By doing this, they introduced a physical limit as well that looks much like ours, namely they multiplied the vacuum Ishibashi boundary state by a factor of $1/\epsilon$ and did the limiting in the calculation of the pairings. The problem was that the characters they arrive at are not the ones that are really observed. Even more severe is that if one introduces a non-vanishing scalar product of the bulk vacuum state with itself, then the L_0 mode does not behave well any longer, i. e. the Shapovalov forms would be non-symmetric. Nevertheless, the principle idea of introducing such a limit is the same as compared to the limiting procedure of chapter 3.

S. Kawai and J. F. Wheater tried to solve the boundary problem by introducing symplectic fermions. They found six boundary states and were able to relate them to the bulk properties in the usual way by defining the bra and ket states completely independent of each other. To connect their work to the present one, the boundary states obeying the symplectic fermion symmetry were also derived here. It could be shown that the two starting points — symplectic fermion description of the $c = -2$ ghost system and $\mathcal{W}(2, 3, 3, 3)$ theory — lead to exactly the same results and that therefore, the results of Kawai and Wheater and those of this work coincide.

None of the cited works, however, mentioned the mixed states intertwining the two different indecomposable bulk representations. These play an important role because they contribute to a probably arising deeper fundamental relation between the boundary and the local theory. Among others, this shows that the logarithmic conformal field theories, especially in the vicinity of a boundary, are not yet completely understood.

Appendix A

Virasoro indecomposable representations and their boundary states

In this appendix, the indecomposable representations $\mathcal{R}_0$ and $\mathcal{R}_1$ are discussed. Especially, the metric as well as the inverse metric are derived and the boundary states are given in a more explicit way. For simplicity this is only done in the Virasoro sector, i. e. omitting the contributions from the $\mathcal{W}$-algebra. It is clear that all the consideration made in this chapter are still valid when considering the extension to complete symmetry algebra. This is possible, because the additional fields are primary with respect to the Virasoro field, i. e. the energy momentum tensor.

A.1 The representation $\mathcal{R}_0$

A basis for the representation $\mathcal{R}_0$ is given by the following states:

$$\begin{array}{c|c|c|c|c} \text{level } 0 & \text{level } 1 & \text{level } 2 & \text{level } 3 & \dots \\ \hline \Omega,\, \omega & \Psi \equiv L_{-1}\omega & L_{-2}\Omega,\, L_{-1}^2\omega,\, L_{-2}\omega & L_{-3}\Omega,\, L_{-2}L_{-1}\omega,\, L_{-3}\omega & \dots \end{array}. \tag{A.1}$$

These states span the representation $\mathcal{R}_0$ up to weight 3. The remaining states in this span are $L_{-1}\Omega$, $L_{-1}^2\Omega$, $L_{-1}L{-2}\Omega$, $L_{-2}L_{-1}\Omega$, $L_{-1}^3\Omega$, $L_{-1}^3\omega$, and $L_{-1}L_{-2}\omega$. In terms of the basis they read:

$$\begin{aligned} L_{-1}\Omega = L_{-1}^2\Omega = L_{-1}^3\Omega = L_{-2}L{-1}\Omega = 0, \\ L_{-1}L_{-2}\Omega = \left(L_{-3} + L_{-2}L_{-1}\right)\Omega = L_{-3}\Omega, \\ L_{-1}L_{-2}\omega = \left(L_{-3} + L_{-2}L_{-1}\right)\omega, \\ L_{-1}^3\omega = 2L_{-2}L_{-1}\omega - L_{-3}\Omega. \end{aligned} \tag{A.2}$$

Those relations appear by means of the Virasoro null states included in the representation. The metric elements are given by the Shapovalov forms (2.19):

$$\delta_{ll'}g_{mn} \equiv \left\langle l, m \middle| l', n \right\rangle \equiv \lim_{z\to\infty}\lim_{w\to 0} z^{2l}\left\langle \phi_{l,m}(z)\phi_{l',n}(w)\right\rangle. \tag{A.3}$$

Exemplarily, the metric element for the two states $L_{-2}\omega$ and $L_{-2}\Omega$ is deduced here with the help of the normalisation given in (3.7):

$$\begin{aligned} \left\langle L_{-2}\Omega \middle| L_{-2}\omega \right\rangle &= \left\langle \Omega \middle| L_2 L_{-2} \middle| \omega \right\rangle = \left\langle \Omega \right| (4L_0 - 1) \left| \omega \right\rangle \\ &= 4\left\langle \Omega \middle| \Omega \right\rangle - \left\langle \Omega \middle| \omega \right\rangle = -1. \end{aligned} \tag{A.4}$$

The metric g for the first levels is:

$$
\begin{array}{c c}
 & \begin{array}{cccccccccc} \Omega & \omega & \Psi & L_{-2}\Omega & L_{-1}\Psi & L_{-2}\omega & L_{-3}\Omega & L_{-2}\Psi & L_{-3}\omega & \cdots \end{array} \\
\begin{array}{c} \Omega \\ \omega \\ \Psi \\ L_{-2}\Omega \\ L_{-1}\Psi \\ L_{-2}\omega \\ L_{-3}\Omega \\ L_{-2}\Psi \\ L_{-3}\omega \\ \vdots \end{array} &
\begin{pmatrix}
0 & 1 & & & & & & & & \\
1 & d & & & & & & & & \\
 & & 2 & & & & & & & \\
 & & & 0 & 0 & -1 & & & & \\
 & & & 0 & 4 & 6 & & & & \\
 & & & -1 & 6 & 4-d & & & & \\
 & & & & & & 0 & 0 & -4 & \\
 & & & & & & 0 & 6 & 10 & \\
 & & & & & & -4 & 10 & 6-4d & \\
 & & & & & & & & & \ddots
\end{pmatrix}
\end{array} . \tag{A.5}
$$

The inverse metric in terms of the same basis and thus, following section 3.3, the coefficient matrix of the generalised Ishibashi state $\big|R_0\big\rangle = \sum_{l,m,n} g^{-1}_{mn}\big(\mathbb{1}\otimes\overline{\mathcal{U}}\big)\big|l,m;l,n\big\rangle$ is:

$$
g^{-1} = \begin{pmatrix}
-d & 1 & & & & & & & & \\
1 & 0 & & & & & & & & \\
 & & \frac{1}{2} & & & & & & & \\
 & & & d+5 & \frac{3}{2} & -1 & & & & \\
 & & & \frac{3}{2} & \frac{1}{4} & 0 & & & & \\
 & & & -1 & 0 & 0 & & & & \\
 & & & & & & \frac{2}{3}+\frac{d}{4} & \frac{5}{12} & -\frac{1}{4} & \\
 & & & & & & \frac{5}{12} & \frac{1}{6} & 0 & \\
 & & & & & & -\frac{1}{4} & 0 & 0 & \\
 & & & & & & & & & \ddots
\end{pmatrix} . \tag{A.6}
$$

The coefficients of $\big|V_0\big\rangle = \sum_{l,m,n}\gamma_{mn}\big(\mathbb{1}\otimes\overline{\mathcal{U}}\big)\big|l,m;l,n\big\rangle$ can directly be calculated from this matrix:

$$
\gamma = \begin{array}{c c}
 & \begin{array}{cccc} \Omega & L_{-2}\Omega & L_{-3}\Omega & \cdots \end{array} \\
\begin{array}{c} \Omega \\ L_{-2}\Omega \\ L_{-3}\Omega \\ \vdots \end{array} &
\begin{pmatrix}
1 & & & \\
 & -1 & & \\
 & & -\frac{1}{4} & \\
 & & & \ddots
\end{pmatrix}
\end{array} . \tag{A.7}
$$

The anti-unitary operator $\mathcal{U}$ can in principle be neglected here. By definition, it commutes with the Virasoro modes and therefore, it has no effect on the states in the Virasoro sector.

A.2 The representation $\mathcal{R}_1$

The second indecomposable representation in the triplet theory, $\mathcal{R}_1$, is more difficult to handle. It contains a spin-degeneracy that cannot be neglected completely when calculating the Virasoro sector. It splits into two disjunct parts $\mathcal{R}_1^+$ and $\mathcal{R}_1^-$. Only combined scalar products are non-vanishing. Therefore, only pairings of the form $\langle \alpha^- | \beta^+ \rangle$ are considered here, i. e. $\mathcal{R}_1^-$ serves as the space for the duals to the states in $\mathcal{R}_1^+$. The complete metric is obtained by tensoring the results with

$$G \equiv \begin{array}{c} \\ \mathcal{R}_1^- \\ \mathcal{R}_1^+ \end{array} \begin{array}{c} \begin{array}{cc} \mathcal{R}_1^- & \mathcal{R}_1^+ \end{array} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array}, \tag{A.8}$$

i. e. the metric g is derived by $G \otimes g^{(-+)}$, where $g^{(-+)}$ is the metric under the above-mentioned restrictions. Since the notation is clarified, the spin indices will be dropped from now on unless any confusions could arise.

A basis is given by

$$\begin{array}{l|l|l|l} \text{level 0} & \text{level 1} & \text{level 2} & \dots \\ \hline \xi & \phi,\ \psi & L_{-1}\phi,\ L_{-2}\xi,\ L_{-1}\psi & \dots \end{array}, \tag{A.9}$$

that obey the additional relations $\phi \equiv L_{-1}\xi$ and $\xi \equiv -\frac{1}{2}L_1\psi$.

With the help of (3.7), the metric g yields

$$\begin{array}{c|c} & \begin{array}{cccccccc} \xi & \phi & \psi & L_{-1}\phi & L_{-2}\xi & L_{-1}\psi & \cdots \end{array} \\ \begin{array}{c} \xi \\ \phi \\ \psi \\ L_{-1}\phi \\ L_{-2}\xi \\ L_{-1}\psi \\ \vdots \end{array} & \begin{pmatrix} 1 & & & & & & \\ & 0 & -1 & & & & \\ & -1 & -t & & & & \\ & & & 0 & 0 & -2 & \\ & & & 0 & -1 & -3 & \\ & & & -2 & -3 & -2t-1 & \\ & & & & & & \ddots \end{pmatrix} \end{array}. \tag{A.10}$$

Thus, the inverse metric, defining the boundary state $\left| R_1^{(+)} \right\rangle$ is derived:

$$g^{-1} = \begin{pmatrix} 1 & & & & & & \\ & t & -1 & & & & \\ & -1 & 0 & & & & \\ & & & -2-\frac{1}{2}t & \frac{3}{2} & -\frac{1}{2} & \\ & & & \frac{3}{2} & -1 & 0 & \\ & & & -\frac{1}{2} & 0 & 0 & \\ & & & & & & \ddots \end{pmatrix}. \tag{A.11}$$

Finally, the boundary state $\left|V_1^{(-)}\right\rangle$ for the irreducible module $\mathcal{V}_1$ is given in terms of its coefficient matrix γ that reads:

$$\gamma = \begin{array}{c} \\ \phi \\ L_{-1}\phi \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} \phi & L_{-1}\phi & \cdots \end{array} \\ \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & \ddots \end{pmatrix} \end{array}. \tag{A.12}$$

Of course, the considerations could have been done the other way round, i. e. taking the $\mathcal{R}_1^+$ space serving as the duals for $\mathcal{R}_1^-$. With this an equivalent and completely independent set of boundary states is derived: $\left|R_1^{(-)}\right\rangle$ and $\left|V_1^{(-)}\right\rangle$ where all the spin indices are flipped from $+$ to $-$. Finally, one could think of a mixture, i. e. taking $\mathcal{R}_1^+$ in the holomorphic part and $\mathcal{R}_1^-$ in the anti-holomorphic of the boundary states. Altogether, one gains a total number of 4 possibilities leading to 8 states. Compared to the results in chapter 3, this is not quite satisfying because it was claimed that there exists exactly two such boundary states. This problem is solved first by taking into account the complete chiral symmetry algebra that combines the two spaces $\mathcal{R}_1^+$ and $\mathcal{R}_1^-$ which are separate in the Virasoro sector.

A.3 Mixed boundary states

With the above defined basis over the representations $\mathcal{R}_0$ and $\mathcal{R}_1$ the mixed boundary states for the Virasoro sector are given by the following coefficient matrix γ:

$$\begin{array}{c} \\ \Omega \\ \Psi \\ L_{-2}\Omega \\ L_{-1}\Psi \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} \xi & \phi & L{-}1\phi & L_{-2}\xi & \cdots \end{array} \\ \begin{pmatrix} 1 & & & & \\ & \frac{1}{2} & & & \\ & & \frac{3}{2} & -1 & \\ & & \frac{1}{4} & 0 & \\ & & & & \ddots \end{pmatrix} \end{array}. \tag{A.13}$$

With the help of this matrix, the states are defined by

$$\left|R_{01}\right\rangle = \sum_{l,m,n} \gamma^l_{mn} \left(\mathbb{1} \otimes \overline{\mathcal{U}}\right) \left|l,m;l,n\right\rangle \quad \text{and} \quad \left|R_{10}\right\rangle = \sum_{l,m,n} \gamma^{\dagger\, l}_{\ mn} \left(\mathbb{1} \otimes \overline{\mathcal{U}}\right) \left|l,m;l,n\right\rangle. \tag{A.14}$$

A.4 Weak boundary states

The derivation of the weak states is more complicated than to obtain the boundary states themselves. Exemplarily, the Virasoro boundary state $\left|R_0\right\rangle$ is treated here to obtain $\left|X_0\right\rangle$

and $\big|Y_0\big\rangle$. In principle, one has to keep all the coefficients and shift the states by the inverse action of $\hat{\mathcal{N}}$, i. e.

$$\begin{aligned} \big(\Omega\otimes\overline{\Omega}\big) &\longrightarrow \frac{1}{2}\big\{\big(\Omega\otimes\overline{\omega}\big)+\big(\omega\otimes\overline{\Omega}\big)\big\}\,, \\ \big(\Omega\otimes\overline{\omega}\big) &\longrightarrow \big(\omega\otimes\overline{\omega}\big). \end{aligned} \tag{A.15}$$

On level 1, some problem arises because the only state contributing to $\big|R_0\big\rangle$ is $\big(\Psi\otimes\Psi\big)$ that is not in the image of $\hat{\mathcal{N}}$. This state has to belong to $\big|Y_0\big\rangle$. These considerations have to be done on all levels. Finally, the coefficient matrix for $\big|X_0\big\rangle$ takes the following form:

$$\begin{array}{c} \\ \Omega \\ \omega \\ \Psi \\ L_{-2}\Omega \\ L_{-1}\Psi \\ L_{-2}\omega \\ L_{-3}\Omega \\ L_{-2}\Psi \\ L_{-3}\omega \end{array}
\begin{array}{c} \begin{array}{ccccccccc} \Omega & \omega & \Psi & L_{-2}\Omega & L_{-1}\Psi & L_{-2}\omega & L_{-3}\Omega & L_{-2}\Psi & L_{-3}\omega \end{array} \\
\left(\begin{array}{ccccccccc}
0 & -\frac{d}{2} & & & & & & & \\
-\frac{d}{2} & 1 & & & & & & & \\
 & & 0 & & & & & & \\
 & & & 0 & 0 & \frac{5+d}{2} & & & \\
 & & & 0 & 0 & \frac{3}{4} & & & \\
 & & & \frac{5+d}{2} & \frac{3}{4} & -1 & & & \\
 & & & & & & 0 & 0 & \frac{1}{3}+\frac{d}{8} \\
 & & & & & & 0 & 0 & \frac{5}{24} \\
 & & & & & & \frac{1}{3}+\frac{d}{8} & \frac{5}{24} & -\frac{1}{4}
\end{array}\right) \end{array} . \tag{A.16}$$

The state $\big|Y_0\big\rangle$ is given by $\big|R_0\big\rangle - \hat{\mathcal{N}}\big|X_0\big\rangle$. Its coefficient matrix reads

$$\begin{array}{c} \\ \Omega \\ \omega \\ \Psi \\ L_{-2}\Omega \\ L_{-1}\Psi \\ L_{-2}\omega \\ L_{-3}\Omega \\ L_{-2}\Psi \\ L_{-3}\omega \end{array}
\begin{array}{c} \begin{array}{ccccccccc} \Omega & \omega & \Psi & L_{-2}\Omega & L_{-1}\Psi & L_{-2}\omega & L_{-3}\Omega & L_{-2}\Psi & L_{-3}\omega \end{array} \\
\left(\begin{array}{ccccccccc}
0 & 0 & & & & & & & \\
0 & 0 & & & & & & & \\
 & & \frac{1}{2} & & & & & & \\
 & & & 0 & \frac{3}{4} & 0 & & & \\
 & & & \frac{3}{4} & \frac{1}{4} & 0 & & & \\
 & & & 0 & 0 & 0 & & & \\
 & & & & & & 0 & \frac{5}{24} & 0 \\
 & & & & & & \frac{5}{24} & \frac{1}{6} & 0 \\
 & & & & & & 0 & 0 & 0
\end{array}\right) \end{array} . \tag{A.17}$$

This choice is not unique, but together with the proper state $|X_1\rangle$ it fulfils the condition $|Y_0\rangle = \hat{\mathcal{P}}\hat{\overline{\mathcal{P}}}|X_1\rangle$. The state $|X_1\rangle$ is derived in the same way as $|X_0\rangle$ and defined by

$$\begin{array}{cc} & \begin{array}{ccccccc} \xi & \phi & \psi & L_{-1}\phi & L_{-2}\xi & L_{-1}\psi & \cdots \end{array} \\ \begin{array}{c} \xi \\ \phi \\ \psi \\ L_{-1}\phi \\ L_{-2}\xi \\ L_{-1}\psi \\ \vdots \end{array} & \begin{pmatrix} 0 & & & & & & \\ & 0 & \frac{t}{2} & & & & \\ & \frac{t}{2} & -1 & & & & \\ & & & 0 & 0 & -\frac{1+t}{2} & \\ & & & 0 & 0 & \frac{3}{4} & \\ & & & -\frac{1+t}{2} & \frac{3}{4} & -\frac{1}{2} & \\ & & & & & & \ddots \end{pmatrix} \end{array}. \tag{A.18}$$

Finally, $|Y_1\rangle$ reads

$$\begin{array}{cc} & \begin{array}{ccccccc} \xi & \phi & \psi & L_{-1}\phi & L_{-2}\xi & L_{-1}\psi & \cdots \end{array} \\ \begin{array}{c} \xi \\ \phi \\ \psi \\ L_{-1}\phi \\ L_{-2}\xi \\ L_{-1}\psi \\ \vdots \end{array} & \begin{pmatrix} 1 & & & & & & \\ & 0 & 0 & & & & \\ & 0 & 0 & & & & \\ & & & 0 & \frac{3}{4} & 0 & \\ & & & \frac{3}{4} & -1 & 0 & \\ & & & 0 & 0 & 0 & \\ & & & & & & \ddots \end{pmatrix} \end{array}. \tag{A.19}$$

The construction of the duals for the mixed boundary states is performed in exactly the same way and yields the coefficient matrix

$$\begin{array}{cc} & \begin{array}{cccccc} \xi & \psi & L_{-1}\phi & L_{-2}\xi & L_{-1}\psi & \cdots \end{array} \\ \begin{array}{c} \omega \\ \Psi \\ L_{-2}\varOmega \\ L_{-1}\Psi \\ L_{-2}\omega \\ \vdots \end{array} & \begin{pmatrix} 1 & & & & & \\ & \frac{1}{2} & & & & \\ & & 0 & 0 & \frac{3}{4} & \\ & & 0 & 0 & \frac{1}{4} & \\ & & \frac{3}{4} & -1 & 0 & \\ & & & & & \ddots \end{pmatrix} \end{array} \tag{A.20}$$

for the state $|Z_{01}\rangle$. $|Z_{10}\rangle$ is based on the transposed matrix. Of course, the choice is again not unique. Null state contributions can as always be added without trouble.

Appendix B

Symplectic fermion boundary equation

It is shown that the boundary conditions for the symplectic fermion symmetry include the $\mathcal{W}$-symmetry as introduced as claimed in chapter 4. This implies that boundary states derived in the fermion case are a subset of those calculated with respect to the $\mathcal{W}$-algebra. The boundary condition in the symplectic fermion case is

$$\left(\chi_n^\alpha - (-1)^a \overline{\chi}_{-n}^\alpha\right) \big|B\big\rangle = 0. \tag{B.1}$$

The $\mathcal{W}$-algebra modes are given by

$$\begin{aligned} L_n &= \frac{1}{2}\epsilon_{\alpha\beta} \sum_{j\in\mathbb{Z}+\lambda} :\chi_j^\alpha \chi_{n-j}^\beta: + \frac{\lambda(\lambda-1)}{2}\delta_{n,0}\,, \\ W_n^0 &= -\frac{1}{2} \sum_{j\in\mathbb{Z}+\lambda} j \cdot \left\{\, :\chi_{n-j}^+ \chi_j^-: + :\chi_{n-j}^- \chi_j^+:\, \right\}, \\ W_n^\pm &= \sum_{j\in\mathbb{Z}+\lambda} j \cdot \chi_{n-j}^\pm \chi_j^\pm . \end{aligned} \tag{B.2}$$

Especially, in the bosonic sector ($\lambda = 0$) L_n reads

$$L_n = \frac{1}{2} d_{\alpha\beta} \Big\{ \underbrace{\sum_{j<n-j} \chi_j^\alpha \chi_{n-j}^\beta}_{(1)} + \underbrace{\sum_{j>n-j} \chi_{n-j}^\beta \chi_j^\alpha}_{(2)} + \underbrace{\frac{1}{2}(1+(-1)^n)\chi_{n/2}^\alpha \chi_{n/2}^\beta}_{(3)} \Big\}. \tag{B.3}$$

Using (B.1), one can transform the first term (1), since χ and $\overline{\chi}$ anticommute:

$$\chi_{n-j}^\beta \big|B\big\rangle = (-1)^a \overline{\chi}_{-n+j}^\beta \big|B\big\rangle$$

$$\begin{aligned} \Longrightarrow \qquad \sum_{j<n-j} \chi_j^\alpha \chi_{n-j}^\beta \big|B\big\rangle &= -(-1)^a \sum_{j<n-j} \overline{\chi}_{-n+j}^\beta \chi_j^\alpha \big|B\big\rangle \\ &= -\sum_{j<n-j} \overline{\chi}_{-n+j}^\beta \overline{\chi}_{-j}^\alpha \big|B\big\rangle \\ &= \sum_{j>-n-j} \overline{\chi}_j^\alpha \overline{\chi}_{-n-j}^\beta \big|B\big\rangle. \end{aligned} \tag{B.4}$$

Analogously, the second term (2) can be treated to obtain

$$\sum_{j>n-j} \chi_{n-j}^\alpha \chi_j^\beta \big|B\big\rangle = \sum_{j<-n-j} \overline{\chi}_{-n-j}^\alpha \overline{\chi}_j^\beta \big|B\big\rangle. \tag{B.5}$$

The term (3) only appears for even n. In this case one has:

$$\begin{aligned}
\chi^\alpha_m \chi^\beta_m \big|B\big\rangle &= \ (-1)^a\, \chi^\alpha_m \overline{\chi}^\beta_{-m} \big|B\big\rangle \ = \ -(-1)^a \overline{\chi}^\beta_{-m} \chi^\alpha_m \big|B\big\rangle \\
&= -\overline{\chi}^\beta_{-m} \overline{\chi}^\alpha_{-m} \big|B\big\rangle \\
&= \ \ \overline{\chi}^\alpha_{-m} \overline{\chi}^\beta_{-m} \big|B\big\rangle .
\end{aligned} \tag{B.6}$$

The three equations (B.4), (B.5), and (B.7) together yield

$$\big(L_n - \overline{L}_{-n}\big)\,\big|B\big\rangle = 0, \tag{B.7}$$

if (B.1) is valid.
For the modes W_n analogous calculations lead to

$$\text{(B.1)} \quad \Longrightarrow \quad \big(W_n + \overline{W}_{-n}\big)\,\big|B\big\rangle = 0 \tag{B.8}$$

and thus, the statement is proven.

Appendix C

The local theory at $c = -2$

This appendix is not intended to give a complete prescription of the local logarithmic conformal field theory with central charge $c = -2$ as proposed by M. R. Gaberdiel and H. G. Kausch. It comprises only a very short overview of this subject. For an extensive discussion, one should refer to [8, 29].

The $c = -2$ triplet model is a chiral rational logarithmic conformal field theory. The local theory is based on the idea of constructing a non-chiral theory by tensoring together a chiral and an anti-chiral realisation of the triplet model.

The idea is the following. The total space of states has essentially the following structure:

$$\mathcal{H} = \bigoplus_{\lambda=-1/8,\,3/8} \big(\mathcal{V}_\lambda \otimes \overline{\mathcal{V}}_\lambda\big) \oplus \bigoplus_{\lambda=0,\,1} \big(\mathcal{R}_\lambda \otimes \overline{\mathcal{R}}_\lambda\big). \tag{C.1}$$

Due to the Möbius symmetry on the complex plane the two-point functions of two operators $\phi_1(z,\bar z)$ and $\phi_2(z,\bar z)$ with conformal weight $(h_1,\bar h_1)$ and $(h_2,\bar h_2)$ have to obey

$$\big\langle \phi_1(\mathrm{e}^{-2\pi\mathrm{i}}z, \mathrm{e}^{2\pi\mathrm{i}}\bar z)\phi_2(0,0)\big\rangle = \mathrm{e}^{2\pi\mathrm{i}(h_1-\bar h_1+h_2-\bar h_2)}\big\langle \mathrm{e}^{2\pi\mathrm{i}S}\phi_1(z,\bar z)\mathrm{e}^{2\pi\mathrm{i}S}\phi_2(0,0)\big\rangle, \tag{C.2}$$

with $S = \hat\delta - \bar{\hat\delta}$. The locality of the correlators requires

$$h - \bar h \in \mathbb{Z} \;\text{ and }\; S\phi = 0 \tag{C.3}$$

for any non-chiral field $\phi(z,\bar z)$ (or state $\big|\phi\big\rangle$) of weight $(h,\bar h)$.

Obviously, for states in $\mathcal{R}_\lambda \otimes \overline{\mathcal{R}}_\lambda$ this condition is not satisfied, since, e. g. in $\mathcal{R}_0 \otimes \overline{\mathcal{R}}_0$ $S\big(\omega\otimes\overline{\omega}\big) = \big(\Omega\otimes\overline{\omega}\big) - \big(\omega\otimes\overline{\Omega}\big) \neq 0$. Therefore, one has to take the quotient space $\mathcal{R}_{\lambda,\bar\lambda} = \mathcal{R}_\lambda\otimes\overline{\mathcal{R}}_\lambda / S[\mathcal{R}_\lambda\otimes\overline{\mathcal{R}}_\lambda]$ as the correct (local) representation. This yields two generalised highest weight representations $\mathcal{R}_{0\bar 0}$ and $\mathcal{R}_{1\bar 1}$. Finally, the locality of higher order correlation functions requires the identification of the states in these representations to each other in such a way that the resulting representation $\mathcal{R}$ has the structure given in figure 3.4.

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