

CFT Interpretation of Merging Multiple SLE Traces

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Abstract

In this paper we give a physical interpretation of the probability of a Stochastic LÖWNER Evolution (SLE) trace approaching a marked point in the upper half plane, e. g. on another trace. Our approach is based on the concept of fusion of boundary with bulk or other boundary fields in Conformal Field Theory (CFT). In this context we also state a proposal how the results can be extended to situations that happen not sufficiently far away from the boundary.

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1 Introduction

1.1 Motivation

Stochastic LÖWNER Evolutions (SLE) as introduced by O. Schramm [28] are one parameter families of conformally invariant measures on curves in the plane. For an elaborate introduction, see e. g. [20]. They are described by continuous sets of mappings that satisfy the LÖWNER differential equation with stochastic driving parameter. This equation has been derived by Löwner in 1923 in a more general context for non-stochastic 1D real continuous driving functions, allowing for the description of a wider class of 2D curves [21].

Taking the driving parameter to be BROWNIAN motion has been proven by Schramm to be the only choice for the measure to be conformally invariant, exhibiting reflection symmetry and implying non-self-intersecting curves. This way, the curves can describe the continuum limit of random cluster interfaces connected to the boundary in lattice models of statistical physics at criticality. Actually, this has been rigorously proven for example for the loop-erased random walk (LERW) and the uniform spanning tree (UST).

However, for years there has been another way to describe these physical models: (boundary) Conformal Field Theories (BCFT)s. Unfortunately, in contrast to some SLE cases this description is only based on a conjecture. In spite of this fact, the question how the two theories are related to each other arises naturally in this context. Additionally it is generally hoped for results to benefit from an explicit correspondence since in some cases such as percolation [29] it turned out that proofs in one or the other picture are strikingly easier to do.

The community, especially [1, 3, 14, 8, 6], has already been successful in answering this question to some extent; e. g. observing that the same differential equations arise both in SLE and minimal BCFT context, heuristically relating the change of boundary condition implicitly fixed by the SLE trace to the appearance of the BCFT-fields $\psi_{(1,2)}$ and $\psi_{(2,1)}$. This nontrivial connection is based on lifting the SLE stochastic differential equation (SDE) of conformal mappings to that of elements of a formal group. Letting these act on appropriate representations of the VIRASORO algebra, martingales that correspond to null vectors in its VERMA module can be found [1]. This correspondence requires a relation between the speed of the BROWNIAN motion and the central charge of the minimal BCFT, matching two so-called dual SLEs to a minimal BCFT model.

Recently, there have been several attempts to relate more than the two boundary fields to objects in (modified versions of) SLE [24, 22] and multiple SLE [4, 7, 11]. This has mostly been achieved via the comparison of differential

equations or typical scaling behavior, e. g. of the probabilities of visiting small sections on the real or imaginary axis and points in the upper half plane [2, 5, 16], differential equations arising due to the presence of the stress-energy tensor [10] and scaling exponents in the Q -states Potts model [25].

1.2 Outline

In this work we want to present another contribution to the SLE-BCFT relation. Therefore we will review important concepts in the second section and fix the notations. We will start with a short introduction to single and multiple SLE (sections 2.1 and 2.2), BCFT (section 2.3), their relationship (section 2.4) and the short distance expansion in CFT (section 2.5).

The third section is dedicated to BCFT and SLE probabilities that we want to focus on in this paper. In BCFT this addresses the probability of a certain arch configuration occurring in a theory while in SLE we are interested in the probability of an SLE trace intersecting a small disc of a certain radius.

Afterwards, we will arrive at the key point of this paper in section 4: the behavior of the differential operators of the differential equations that were part of the first step to establish a connection between the two theories. In this part of our paper, we will argue how additional exponents for the small distance behavior can be found by combining the dependency emerging from the small distance expansion of two primary fields with the respective changes in the differential operators.

In the fifth section, we will make use of these results by relating the SLE probability to fusion of boundary with bulk fields in section 5.1 and of boundary fields only in section 5.2. This will lead us to an SLE interpretation of correlation functions of other fields than the $\psi_{(1,2)}$ or $\psi_{(2,1)}$ -type. Afterwards, we will address the obstacle that the SLE probability in question also exhibits an angular dependency for points near the boundary in section 5.3.

The last section will be dedicated to another point of view on multiple SLE. Briefly reviewing concepts of probability theory in section 6.1, we will remind the reader that multiple SLE can be viewed as ordinary SLE weighted by a suitable martingale. This martingale will then be given a probability interpretation in section 6.2, extending the concept of fusion to this picture in section 6.3

The appendix contains the derivation of the limiting behavior of the differential operators stated in section 4. It is straightforward but quite lengthy, hence we restricted ourselves to a heuristic argument in the text, leaving the details of the computations to the appendix.

2 Basic Definitions and Notations

2.1 Single SLE

Now let us fix some notation. For chordal SLEs, the family of LÖWNER mappings shall be denoted by $(g_t)_{t \in \mathbb{R}_0^+}$ which are defined through the so-called LÖWNER equation:

$$dg_t = \frac{2dt}{g_t - \xi_t}, \quad g_0(z) = z, \quad g_t(z) = z + \frac{2t}{z} + \mathcal{O}(z^{-2}) \text{ for } z \rightarrow \infty. \quad (1)$$

Here, $\xi_t = \sqrt{\kappa}B_t$ is BROWNIAN motion of speed κ with $0 \leq \kappa \leq 8$. Additionally we have chosen the standard time parameterization to fix the $SL(2, \mathbb{R})$ invariance such that the half plane capacity is $\text{HCAP}_{\gamma_t} = 2t$. Obviously, $g_t : \mathbb{H}/K_t \rightarrow \mathbb{H}$ where the K_t are the hulls of the so-called traces $\gamma_t := \lim_{\epsilon \rightarrow 0} g_t^{-1}(\xi_t + i\epsilon)$ that describe the cluster interfaces. As physical cluster interfaces these traces are continuous and non self-crossing, exhibiting three phases: for $\kappa \leq 4$ they are a.s. simple, for $4 < \kappa < 8$ they are self-touching and for $\kappa \geq 8$ space filling with HAUSDORFF dimension $d_{\gamma_t} = \min\{1 + \frac{\kappa}{8}, 2\}$. The dimension of the SLE hull K_t is given by that of its trace for $\kappa < 4$ and by that of the so-called dual SLE with parameter $16/\kappa$ for the other cases.

To illustrate the connection to CFT we define $h_t(z) := g_t(z) - \xi_t$, satisfying the stochastic differential equation

$$dh_t = \frac{2dt}{h_t} - d\xi_t. \quad (2)$$

For any time t we associate an element \mathfrak{g}_{h_t} of the germs of holomorphic functions at infinity, N_- , of the form $z + \sum_{m \leq -1} h_m z^{m-1}$. This satisfies according to ITÔs formula:

$$\mathfrak{g}_{h_t}^{-1} \cdot d\mathfrak{g}_{h_t} = dt \left(-2l_{-2} + \frac{\kappa}{2} l_{-1}^2 \right) + d\xi_t l_{-1}, \quad (3)$$

with $l_n = -z^{n+1} \partial_z$. In CFT, the l_n correspond to the generators L_n of the VIRASORO algebra \mathfrak{vir} [1]:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m(m^2 - 1) \delta_{m+n, 0}, \quad (4)$$

and it can be shown that there exists a homomorphism $\mathfrak{g}_h \rightarrow G_h$ such that G_h is an operator acting on appropriate representations of \mathfrak{vir} , satisfying an equation analogous to (3). Now it is easy to see that $G_{h_t} |\psi_{(1,2)}\rangle$ and $G_{h_t} |\psi_{(2,1)}\rangle$ are local martingales if the ψ s are primary fields of weights $h_{(1,2)} = \frac{6-\kappa}{2\kappa}$ or $h_{(2,1)} = \frac{3\kappa-8}{16}$. Indeed, this is equivalent[‡] to saying that these fields have a degenerate descendent on level two:

$$\left(-2L_{-2} + \frac{\kappa}{2} L_{-1}^2 \right) |\psi_h\rangle = 0, \quad (5)$$

if the following relation between the central charge of the CFT c and the speed of the BROWNIAN motion κ holds [1]:

$$c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa} \leq 1. \quad (6)$$

2.2 Multiple SLE

Multiple SLE following [18], based on [4, 11], can be regarded as m single chordal SLEs in the same domain. For consistency, some requirements such as conformal invariance, reparameterization invariance and absolute continuity have to be made. Allowing local growth at m tips in the upper half plane results

[‡]Note that in [26] it has been shown that this identification is only true modulo a phase that accounts for the branch points of CFT correlation functions. However, to keep things simple, we will not concentrate on this subtlety in this paper.

in a modified LÖWNER mapping G_t that describes m single SLEs in a single equation:

$$dg_t^i(z) = \frac{2c_t^i}{g_t^i - w_t^i} dt \quad \text{for } i = 1, \dots, m \quad \rightarrow \quad dG_t(z) = \sum_{i=1}^m \frac{2a_t^i dt}{G_t(z) - x_t^i}. \quad (7)$$

Defining $G_t =: H_t^i \circ g_t^i$, we can specify the relationship between the old and the new driving parameters $x_t^i = H_t^i(w_t^i)$ and time parameterisations $a_t^i = H_t^{i'}(w_t^i)^2 c_t^i$. Loosely speaking, H_t^i is the mapping that removes the remaining $n - 1$ SLE traces from the setting after the action of g_t^i .

The conditions lead us to the κ -relation: all involved speeds κ_i of the m SLEs should correspond to the same central charge. From (6) we can deduce that for all $i, j \in \{1, \dots, m\}$ we have $\kappa_i = \kappa_j$ or $\kappa_i = 16/\kappa_j$.

Another implication is a change of the measure that leads to a change of the drift term from purely BROWNIAN motion to:

$$dw_t^i = \sqrt{\kappa} dB_t \rightarrow dx_t^i = \sqrt{\kappa_i} dB_t^i + \kappa_i a_t^i \partial_{x_t^i} \log Z[x_t] dt + \sum_{k \neq i} \frac{2a_t^i}{x_t^i - x_t^k} dt, \quad (8)$$

where $Z[x_t]$ is usually being interpreted as the partition function of the corresponding BCFT.

Enforcing conformal invariance and absolute continuity yields an exponential change of the measure given by the RADON-NIKODYN derivative according to GIRSANOV's theorem. Hence, similarly to the interpretation of SLE($\kappa, \vec{\rho}$) [30], we can interpret this SLE again as "usual" SLE only weighted by another local bounded martingale [18]. Here, the martingale is given by

$$M_t := \frac{Z[x_t]}{Z[x_0]} \prod_{i=1}^m (H_t^{i'}(w_t^i))^{h_i} \exp\left(\frac{c_i}{6} \int_0^t \mathcal{S}H_s^i(w_s^i) c_s^i ds\right) \exp\left(-\int_0^t \frac{1}{Z[x_s]} \mathcal{D}_{-2}^m(x_s^i) Z[x_s] a_s^i ds\right), \quad (9)$$

where \mathcal{S} denotes the SCHWARZIAN derivative and $\mathcal{D}_{-2}^m(x_s^i)$ is given by

$$\mathcal{D}_{-2}^m(x_s^i) = \frac{\kappa_i}{2} \partial_{x_s^i}^2 - 2 \sum_{k \neq i} \left(\frac{h_k}{(x_s^k - x_s^i)^2} - \frac{1}{(x_s^k - x_s^i)} \partial_{x_s^i} \right). \quad (10)$$

2.3 Boundary CFT Revisited

The goal of this side trip is to point out the subtleties of BCFT on the upper half plane since it is usually dealt with as a chiral theory on the full complex plane which forces us to think about how to extend the LÖWNER equation from \mathbb{H} to \mathbb{C} , too. Additionally, the boundary behavior of bulk fields gets affected by the formalism.

In BCFT, due to the boundary conditions that have to be imposed one way or the other, the behavior of the primary fields under conformal transformation changes. Hence, the way we have to treat observables that are given by products of primary fields $\phi(z, \bar{z})$ with holomorphic and antiholomorphic coordinate

dependencies differs from that of the boundary fields $\psi(x)$;

$$\phi(z, \bar{z}) \rightarrow \phi(f(z), f(\bar{z})) = (f'(z))^{-h} (f'(\bar{z}))^{-\bar{h}} f \phi(z, \bar{z}), \quad (11)$$

$$\psi(x) \rightarrow \psi(f(x)) = (f'(x))^{-h} f \psi(x). \quad (12)$$

wherein f denotes the functional change of the field due to the conformal transformation.

In conformal field theory on the full complex plane, we can regard the holomorphic and antiholomorphic coordinates, z and \bar{z} as independent variables since any conformal transformation factors into the two parts due to the CAUCHY-RIEMANN differential equations. Any variation of a correlation function with respect to a conformal transformation given by $z \mapsto z + \epsilon(z)$ and $\bar{z} \mapsto \bar{\epsilon}(\bar{z})$ can be written as

$$\delta_{\epsilon, \bar{\epsilon}} \langle X \rangle_{\mathcal{C}} = -\frac{1}{2\pi i} \oint_{\mathcal{C}} dz \epsilon(z) \langle T(z) X \rangle + \frac{1}{2\pi i} \oint_{\mathcal{C}} d\bar{z} \bar{\epsilon}(\bar{z}) \langle \bar{T}(\bar{z}) X \rangle. \quad (13)$$

with the counterclockwise contour \mathcal{C} including all the (anti-)holomorphic positions of the primary fields in X .

However, in boundary CFT on the upper half plane $\overline{\mathbb{H}}$ including the real axis, we are confined to those conformal transformations that leave the real axis, i.e. the boundary, invariant. This gives us the constraint $\epsilon(x) = \bar{\epsilon}(x)$ and $T(x) = \bar{T}(x)$ for any $x \in \mathbb{R}$. Taking a look at a correlation function trying to separate the holomorphic and antiholomorphic parts again, we see that we have a non-vanishing contribution from the boundary:

$$\delta_{\epsilon} \langle X \rangle_{\overline{\mathbb{H}}} = -\frac{1}{2\pi i} \oint_{\mathcal{C}^+} dz \epsilon(z) \langle T(z) X \rangle - \frac{1}{2\pi i} \int_{-\infty}^{\infty} dx \epsilon(x) \langle T(x) X \rangle, \quad (14)$$

$$\delta_{\bar{\epsilon}} \langle X \rangle_{\overline{\mathbb{H}}} = \frac{1}{2\pi i} \oint_{\mathcal{C}^+} d\bar{z} \bar{\epsilon}(\bar{z}) \langle \bar{T}(\bar{z}) X \rangle + \frac{1}{2\pi i} \int_{-\infty}^{\infty} dx \bar{\epsilon}(x) \langle \bar{T}(x) X \rangle. \quad (15)$$

where \mathcal{C}^+ indicates a counterclockwise contour in the upper half plane $\overline{\mathbb{H}}$ (including the real axis) encircling all (anti-)holomorphic positions of the primary fields in X .

Obviously, due to the boundary conditions, both boundary terms give exactly the same contribution – the two terms are no longer independent. From now on we will set $\bar{z} = z^*$ and consider the antiholomorphic quantities, e.g. $\bar{T}(\bar{z})$, as being the analytic continuation of the holomorphic quantities, e.g. $T(z)$ in the lower half plane[‡]. Thus we arrive at only one set of VIRASORO generators by taking advantage of $\delta_{\bar{\epsilon}} \langle X \rangle_{\overline{\mathbb{H}}} = 0$ and reexpressing the boundary integral in one of the equations (14) and (15). In this picture, the bulk fields depending on holomorphic and antiholomorphic coordinates become two separate fields, one being the “mirror image” of the other $\phi(w, \bar{w})_{\overline{\mathbb{H}}} = \phi(w)_{\overline{\mathbb{H}}} \otimes \phi(\bar{w})_{\mathbb{C}/\overline{\mathbb{H}}} = \phi(w)_{\overline{\mathbb{H}}} \phi(w^*)_{\mathbb{C}/\overline{\mathbb{H}}}$. Effectively, we are using $2n$ holomorphic degrees of freedom in this picture instead of n holomorphic and n antiholomorphic ones with a boundary condition:

$$\delta_{\epsilon} \langle X \rangle_{\mathbb{C}_b} = -\frac{1}{2\pi i} \oint_{\mathcal{C}^+} dz \epsilon(z) \langle T(z) X \rangle + \frac{1}{2\pi i} \oint_{\mathcal{C}^-} dz \epsilon(z) \langle \bar{T}(z) X \rangle. \quad (16)$$

[‡]This is possible since $\bar{T}(z^*)$ for $\Im m(z) < 0$ is holomorphic and $T(z) = \bar{T}(\bar{z})$ on the real axis.

This point of view suggests that by only considering conformal transformations $f(z)$, $f : D \subset \overline{\mathbb{H}} \rightarrow \overline{\mathbb{H}}$ that preserve the boundary, we have to modify the behaviour of the primary bulk and boundary fields under these transformations:

$$\phi(z, \bar{z}) \rightarrow \phi(f(z), f(\bar{z})) = (f'(z))^{-h} (f'^*(z^*))^{-\bar{h}} f \phi(z, \bar{z}), \quad (17)$$

$$\psi(x) \rightarrow \psi(f(x)) = (f'(x))^{-h} f \psi(x). \quad (18)$$

Equation (17) is precisely the point which is usually ignored in the literature, as well as the doubling of points when going from the usual picture with chiral and antichiral halves on the upper half plane to the only-chiral theory on the full complex plane.

Hence, any differential equation resulting from degenerate fields involves holomorphic as well as antiholomorphic coordinates, e.g. the level two differential equation arising in minimal CFTs that imposes constraints on correlation functions including $\psi_{(2,1)}$ or $\psi_{(1,2)}$ boundary field.

If we choose the LÖWNER mapping $f(z) = g_t(z)$, we will have to take terms including the antiholomorphic coordinates into account if we want to include a truly local observable or want to study boundary behavior which is usually done by taking a look at the limit

$$\lim_{z \rightarrow \bar{z}} \phi(z) \phi(\bar{z}), \quad (19)$$

and inserting the OPE (see section 2.5) for the product of the chiral and antichiral half of the local field $\phi(z, \bar{z})$. In that case, we have to extend the LÖWNER mapping to the lower half plane which should be done by mirroring as well. Hence we define for $z, \bar{z} \in \overline{\mathbb{H}}$ or, equivalently $w \in \mathbb{C}$:

$$z \mapsto g_t(z) \quad \text{and} \quad \bar{z} \mapsto g_t(\bar{z}) \quad \text{with b.c. or} \quad (20)$$

$$w \mapsto g_t(w) \quad := \quad \begin{cases} g_t(w) & \text{for } \Im(w) > 0, \\ g_t^*(w^*) & \text{for } \Im(w) < 0. \end{cases} \quad (21)$$

2.4 SLE Martingales and Physical Quantities in BCFT

Following the nice introductions [4, 27, 6, 26] to the connection between SLE and minimal models of BCFT, we will give a short overview on how martingales in multiple SLE are connected to correlation functions in BCFT.

On the one hand, it is quite obvious from the definition of martingales in the theory of probability that a quantity M_t which satisfies all conditions for martingales can be referred to as “conserved”, i. e. not changing its expectation value in time, or, more mathematically: $\mathbb{E}(M_t) = \mathbb{E}(M_0)$ for all times t which means nothing else than $\frac{d}{dt} \mathbb{E}(M_t) = 0$. Hence, if we assume that observables $O_t^{\text{STAT.P.}}$ should be conserved quantities, we can also say that $\mathbb{E}(M_t) = \mathbb{E}(O_t^{\text{SLE}})$.

On the other hand, in CFT it is common believe that all physically interesting quantities are given in terms of correlation functions of the fields in the theory – i. e. the possible amplitudes of the scattering matrix. Within the theories of minimal models, these correlation functions \mathcal{O}^{CFT} are known to obey certain differential equations $\mathcal{D}\mathcal{O}^{\text{CFT}} = 0$, e.g. due to the existence of singular states or WARD identities.

Hence, the key point is basically to realize that there are correlation functions in BCFT and martingales in SLE that describe the same observables in

statistical physics. Once we accept this, we can show that they are actually described by the same differential equation, i. e. $\frac{d}{dt}\mathbb{E}(O_t^{\text{SLE}})$ with $O_t^{\text{SLE}} \rightarrow O_t^{\text{CFT}}$ will turn out to be the null vector equation of BCFT: $\mathcal{D}O^{\text{CFT}} = 0$.

In the following we will illustrate how the probabilistic description of SLE martingales is connected to the statistical physics language. Unfortunately, we can not be rigorous here since the connection between the respective statistical physics models and BCFT is still only based on a conjecture. Hence the point where we will take the continuum limit is nly based on an assumption which is quite widely believed but has never been proven.

Nevertheless let us have a look at a statistical mechanics model with boundary, e. g. defined on \mathbb{H} , whose continuum limit at the critical point can be described by a BCFT[§]. Hence, at least for most observables $O^{\text{STAT.P.}}$ we assume that their expectation values will be described by BCFT correlation functions. We will allow for a finite but large set of possible states S to whose elements normalized BOLTZMANN weights $w(s)$ with probability $\mathbf{P} = w(s)/Z$ are attached where $Z = \sum_{s \in S} w(s)$ denotes the partition function. Hence we can state the expectation value of observables described by random variables $O^{\text{STAT.P.}} : S \rightarrow \mathbb{H}$ as:

$$\mathbb{E}(O^{\text{STAT.P.}}) = \mathcal{O}^{\text{STAT.P.}} = \frac{1}{Z} \sum_{s \in S} O^{\text{STAT.P.}}(s)w(s). \quad (22)$$

Now we want to restrict ourselves to a special situation, leading to a conditioned expectation value, i. e. an expectation value with respect to the knowledge of the situation up to some time t . This will allow us to relate the statistical physics expectation value to the one of SLE by taking subsets of S that correspond to interfaces γ_t between the boundary and an interior point. Therefore we define $(S_\alpha)_{\alpha \in I}$ to be a collection of disjoint subsets of S whose union is again S to get a sigma algebra $\mathcal{F} = \{\bigcup_{\alpha \in I'} S_\alpha : I' \subset I\}$. This induces a filtration, i. e. an increasing family $(\mathcal{F}_t)_{t \geq 0}$ of sigma algebras, $\{\emptyset, S\} \subset \mathcal{F}_s \subset \mathcal{F}_t$ for all $0 \leq s < t$. Hence the partition function for this special situation is given by

$$Z_\alpha^{(t)} = \sum_{s \in S_\alpha^{(t)}} w(s). \quad (23)$$

and the conditional expectation value is:

$$\mathcal{O}_t^{\text{SLE}} = \mathbb{E}(O^{\text{SLE}}|\mathcal{F}_t) = \sum_{\alpha \in I_t} \left(\frac{1}{Z_\alpha^{(t)}} \sum_{s \in S_\alpha^{(t)}} O^{\text{STAT.P.}}(s)w(s) \right) \mathbf{1}_{S_\alpha^{(t)}}. \quad (24)$$

Obviously, by definition, this is a martingale. In the following we will use the shorthand notation $O_t^{\text{SLE}} = O^{\text{SLE}}|\mathcal{F}_t$.

In our case, the special situation shall be that of m interfaces γ_t^i emerging at the points x_0^i on the boundary and growing (not intersecting each other) into the upper half plane. Our filtration therefore is given by $\mathcal{F}_t = \sigma(\gamma_{t'}^i : 0 \leq t' \leq t, i = 1, \dots, m)$. As always, we define $G_t : \mathbb{H}/\bigcup_{i=1}^m K_t^i \rightarrow \mathbb{H}$.

Hence, in the continuum limit we follow the general assumption that we can identify the statistical mechanics observable that we have already shown to be

[§]As said above, we are aware of the fact that his part still lacks a proper proof but we will nevertheless assume its correctness here.

an SLE martingale with a BCFT correlation function:

$$\mathcal{O}_t^{\text{SLE}} \rightarrow \frac{\mathcal{O}_{\mathbb{H}_t}^{\text{CFT}}}{\langle \mathbf{1} \rangle_{\mathbb{H}_t}^{\text{CFT}}} = \frac{\langle \psi(\infty), {}^{G_t} O(\{(z_t^k, \bar{z}_t^k)\}) \prod_{l=1}^m \psi(x_t^l) \rangle_{\mathbb{H}}^{\text{CFT}}}{\langle \psi(\infty), \prod_{l=1}^m \psi(x_t^l) \rangle_{\mathbb{H}}^{\text{CFT}}}, \quad (25)$$

for some observable $O(\{(z_t^k, \bar{z}_t^k)\}) = \prod_{k=1}^n \phi(z_t^k, \bar{z}_t^k)$ with the upper index G_t denoting the action of the LÖWNER mapping on the observable. Note that we did not write out the JACOBIAN factors for the also transformed boundary fields – they cancel in the numerator and denominator.

Now we will explicitly compute the ITÔ derivative to arrive at the relationship with null vector conditions of BCFT. Therefore we will state some intermediate results.

In minimal BCFT, the fields $\phi(z, \bar{z})$ are primary fields and thus behave under a conformal map G_t such as the multiple LÖWNER mapping as follows:

$${}^{G_t} \phi_j(z_j, \bar{z}_j) = (G'_t(z_j))^{h_j} (G'_t(z_j)^*)^{\bar{h}_j} \phi_j(G_t(z_j), G_t(z_j)^*), \quad (26)$$

where h_j is the weight of the field ϕ_j and $z_j = z_0^j$ the coordinate in the initial domain $\mathbb{H}_0 = \mathbb{H}$. Thus we can use this to compute the variation of the bulk fields $d({}^{G_t} \phi_j(z_j, \bar{z}_j)) = {}^{G_t+dt} \phi_j(z_j, \bar{z}_j) - {}^{G_t} \phi_j(z_j, \bar{z}_j)$ introducing the simplified notation $G_t(z_j) = z_t^j$:

$$\begin{aligned} d({}^{G_t} \phi_j(z_j, \bar{z}_j)) &= d\left((G'_t)^{h_j} (G_t^{*'})^{\bar{h}_j} \phi_j\left(z_t^j, z_t^{*j}\right)\right) \\ &= h_j (G'_t)^{h_j-1} dG'_t (G_t^{*'})^{\bar{h}_j} \phi_j\left(z_t^j, z_t^{*j}\right) \\ &\quad + \bar{h}_j (G_t^{*'})^{\bar{h}_j-1} dG_t^{*'} (G'_t)^{h_j} \phi_j\left(z_t^j, z_t^{*j}\right) \\ &\quad + (G'_t)^{h_j} (G_t^{*'})^{\bar{h}_j} d\phi_j\left(z_t^j, z_t^{*j}\right) \end{aligned} \quad (27)$$

Switching the order of the derivatives with respect to z_j and t , we can use the LÖWNER differential equation to compute dG'_t and $dG_t^{*'}$. Additionally, we use the chain rule for $d\phi_j\left(z_t^j, z_t^{*j}\right)$:

$$\begin{aligned} d({}^{G_t} \phi_j(z_j, \bar{z}_j)) &= -\sum_{i=1}^m 2 \left\{ \left(\frac{h_j}{(z_t^j - x_t^i)^2} - \frac{1}{(z_t^j - x_t^i)} \partial_{z_t^j} \right) \right. \\ &\quad \left. - \left(\frac{\bar{h}_j}{(z_t^{*j} - x_t^i)^2} - \frac{1}{(z_t^{*j} - x_t^i)} \partial_{z_t^{*j}} \right) \right\} dt {}^{G_t} \phi_j(z_j, z_j^*). \end{aligned} \quad (28)$$

As the boundary fields depend on stochastic variables, we have to apply ITÔ's rule. Note that the off-diagonal contributions vanish due to $dt^2 = dB_t^i dt = 0$ and $dB_t^i dB_t^j = \delta_{ij} dt$:

$$d\psi_i(x_t^i) = \partial_{x_t^i} \psi(x_t^i) dx_t^i + \frac{\kappa}{2} \partial_{x_t^i}^2 \psi(x_t^i) dt. \quad (29)$$

With the help of equations (28) and (29), it can be shown explicitly that

$$\mathcal{O}_t^{\text{CFT}} = \frac{\langle \psi(\infty), O(\{(G_t(z_j), G_t^*(z_j^*))\}) \prod_{l=1}^m \psi(x_t^l) \rangle}{\langle \psi(\infty), \prod_{l=1}^m \psi(x_t^l) \rangle} \prod_{k=1}^n G'_t(z_k)^{h_k} G_t^{*'}(z_k^*)^{\bar{h}_k} \quad (30)$$

is (not only by definition) a closed martingale, obeying m differential equations for $i = 1, \dots, m$. This can be shown via taking the time derivative of $\mathcal{O}_t^{\text{SLE}}$, using the LÖWNER mapping and the quotient rule of the ITÔ calculus to get:

$$\begin{aligned}
\frac{d}{dt} \mathbb{E} (O_t^{\text{SLE}}) &= \left[\frac{\kappa}{2} \partial_{x_t^i}^2 - 2 \sum_{l \neq i=1}^m \left(\frac{h_l}{(x_t^l - x_t^i)^2} - \frac{1}{(x_t^l - x_t^i)} \partial_{x_t^i} \right) \right. \\
&\quad \left. - 2 \sum_{j=1}^n \left\{ \left(\frac{h_j}{(z_t^j - x_t^i)^2} - \frac{1}{(z_t^j - x_t^i)} \partial_{z_t^j} \right) \right. \right. \\
&\quad \quad \left. \left. - \left(\frac{\bar{h}_j}{(z_t^{*j} - x_t^i)^2} - \frac{1}{(z_t^{*j} - x_t^i)} \partial_{z_t^{*j}} \right) \right\} \right] \mathcal{O}_t^{\text{CFT}} \\
&=: \mathcal{D}_{-2}^{m,n} \mathcal{O}_t^{\text{CFT}} (x_t^i, \{x_t^l\}_{l \neq i}, \{z_t^k, z_t^{*k}\}) \\
&= 0. \tag{31}
\end{aligned}$$

Now, following [4], we will introduce m functions x^l and n functions z^k, z^{*k} , given by $t \mapsto x_t^l$ and $t \mapsto z_t^k, t \mapsto z_t^{*k}$, respectively. In this notation, equation (31) becomes

$$\frac{d}{dt} \mathbb{E} (O_t^{\text{SLE}}) = \mathcal{D}_{-2}^{m,n} \mathcal{O}_t^{\text{CFT}} (x^i, \{x^l\}_{l \neq i}, \{z^k, z^{*k}\}) \tag{32}$$

which shall be interpreted as a functional expression: the R.H.S. is true for any coordinate that the LÖWNER images of the traces γ can assume, hence it does not depend on the specific point in time we choose on the L.H.S. This is not surprising at all, since $\mathcal{O}_t^{\text{SLE}}$ is an SLE martingale, hence we expect the time derivative of its expectation value to vanish anyway. In words this relation shows that the martingale condition in SLE for statistical physics observables translates to null vector differential equations obeyed by the same observables expressed via the CFT correlation functions. Heuristically, this shows that, indeed, in the continuum limit, both models describe the same physics and hence some quantities may be related to each other.

The inclined reader may also find another point of view on the relationship between the SLE and CFT form of the differential equation in [26].

Note that if we chose $m = 1$, we would get the single SLE case back since there are only two boundary fields in that case – $\psi_{(2,1)}(0)$ and $\psi_{(2,1)}(\infty)$. Hence the partition function would reduce to the free case since (naively)

$$\langle \psi_{(2,1)}(\infty), \psi_{(2,1)}(\xi_t) \rangle \propto \lim_{x_\infty \rightarrow \infty} (\xi_t - x_\infty)^{-2h_{(2,1)}} \tag{33}$$

is usually set to a constant. Hence we would have no differential equations for the correlation function of the boundary fields alone.

2.5 Fusion and the OPE in CFT

In order to be able to study and describe the behavior of quantities approaching each other properly, we have to use the short distance expansion of products of primary fields in CFT.

In minimal CFT, the primary fields correspond to highest weights in the KAC table, characterized by their KAC labels (r, s) . The so-called fusion rules

tell us which primaries and descendants are involved in the short distance product (“operator product expansion” (OPE)) of two given fields[9]:

$$\phi_0(z)\phi_1(w) = \sum_h g_{(r',s')} \sum_Y (z-w)^{h-h_0-h_1+|Y|} \beta_{Y,(r',s')} L_{-Y} \phi_h(w), \quad (34)$$

where $Y = \{k_1, k_2, \dots, k_n\}$, $k_1 \geq \dots \geq k_n$, $|Y| = \sum_{i=1}^n k_i$ and $g_{(r',s')}$ the coefficient of the three-point function of the involved fields of weights $h_0 = h_{(r_0,s_0)}$, $h_1 = h_{(r_1,s_1)}$ and $h = h_{(r',s')}$.

In principle, if the two fundamental fields $\phi_{(2,1)}$ and $\phi_{(1,2)}$ are present in a theory, all other fields may be generated by consecutive fusion of a suitable number of copies. For our purposes, it suffices to know that

$$\phi_{(1,2)} \times \phi_{(1,2)} = \phi_{(1,1)} + \phi_{(1,3)}, \quad (35)$$

$$\phi_{(2,1)} \times \phi_{(2,1)} = \phi_{(1,1)} + \phi_{(3,1)}. \quad (36)$$

Thus if the SLE trace tips correspond to boundary fields of type $\psi_{(2,1)}$ or $\psi_{(1,2)}$, it should in principle be possible to find quantities that correspond to the other entries of the KAČ table by investigating merging traces in multiple SLE. Hence, we will focus on the behavior of SLE and CFT quantities in which two coordinates approach each other.

3 Probabilities in BCFT and SLE

In the first part of this section we will give an heuristic approach to probabilities related to the partition function in BCFT.

In the second part we will give a short review on an important probability of a special situation in SLE: the answer to the question asking how probable it is for the SLE trace to hit a small ball in the upper half plane, hence being in the vicinity of a marked point. Of course, this probability has been derived for single SLEs. However, multiple SLE can always be viewed as m single SLEs if considering small time steps or large distances between the curves. Thus we think that it is justified to use the formula originally derived only for single SLEs for our purposes, too.

3.1 Partition Functions and Probabilities in BCFT

In boundary CFT, the partition function of a system with boundary conditions $\alpha_1, \dots, \alpha_m$ changing at the positions $x_1, \dots, x_m \in \mathbb{R}$ is given by [9]

$$Z_{\alpha_1, \dots, \alpha_m} = \left\langle \psi(\infty), \prod_l^m \psi_{\alpha_l, \alpha_{l+1}}(x_l) \right\rangle Z_{\text{free}}. \quad (37)$$

Interpreting the partition function of the free system, Z_{free} , as a measure of the total number of states and the partition function of the system with fixed boundary conditions, $Z_{\alpha_1, \dots, \alpha_m}$, as a measure for the interfaces due to the changes in the boundary conditions, we see that the fraction

$$\frac{Z_{\alpha_1, \dots, \alpha_m}}{Z_{\text{free}}} =: \mathbf{P}_{\gamma_1, \dots, \gamma_m} \quad (38)$$

is the fraction of interfaces $\gamma_1, \dots, \gamma_m$ fulfilling the given boundary conditions among all possible interfaces of the full system [15]. Therefore switching to the special situation introduced in section 2.4, we will consider $\mathbf{P}_{\gamma_t^1, \dots, \gamma_t^m}$ as the probability that m given paths emerge from the boundary points x_t^1, \dots, x_t^m , i. e.:

$$\mathbf{P}_{\gamma_t^1, \dots, \gamma_t^m} = \left\langle \psi(\infty), \prod_l^m \psi_{\alpha_l, \alpha_{l+1}}(x_t^l) \right\rangle, \quad (39)$$

which is the same probability now conditioned on already having m interfaces $\gamma_{(0,t]}$ in \mathbb{H} , growing further to infinity.

In [4] this probability has been shown to be a martingale with respect to the multiple SLE measure obeying m differential equations. Thus for $i = 1, \dots, m$ we have:

$$\left(\frac{\kappa_i}{2} \partial_{x_t^i}^2 - 2 \sum_{l \neq i=1}^m \left(\frac{h_l}{(x_t^l - x_t^i)^2} - \frac{1}{x_t^l - x_t^i} \partial_{x_t^l} \right) \right) \mathbf{P}_{\gamma_t^1, \dots, \gamma_t^m} = 0, \quad (40)$$

with $h_i, h_l \in \{h_{(1,2)}, h_{(2,1)}\}$ for the setting to be described by SLE, too.

From this follows, that

$$\frac{Z[x_t]}{Z[x_0]} = \frac{\mathbf{P}_{\gamma_t^1, \dots, \gamma_t^m}}{\mathbf{P}_{\gamma_0^1, \dots, \gamma_0^m}}, \quad (41)$$

where $\mathbf{P}_{\gamma_0^1, \dots, \gamma_0^m}$ is just the probability to chose m specific starting points for our SLE. Hence the ratio of the partition function of time t to time 0 gives us the probability that we have m SLE traces up to time t that started somewhere on the boundary, i. e. the arch configuration where we have m archs not having paired up to time t .

3.2 The SLE Probability of Intersecting a Disc

In [5], the probability of an SLE_κ trace γ , $0 < \kappa < 8$, intersecting a disc $\mathcal{B}_\epsilon(z_0)$ of radius ϵ centered at a point z_0 in the upper half plane is derived based on an idea of Oded Schramm. It extends the result of chapter 7.4 of [20]:

$$\mathbf{P}_{\epsilon, z_0} := \mathbf{P}(\gamma_{(0, \infty]} \cap \mathcal{B}_\epsilon(z_0) \neq \emptyset) \asymp \left(\frac{\epsilon}{\Im(z_0)} \right)^{2-d_\gamma} (\sin \alpha(z_0))^{8/\kappa-1}, \quad (42)$$

where $d_\gamma = \max\{2, 1 + \kappa/8\}$ is the dimension of the SLE path and $\alpha(z_0)$ the angle between the real axis and the vector pointing to z_0 .

However, in the following we want to study the merging of SLE traces. Since they are conditioned not to intersect with each other, they can only come close to each other on their “outer parts”, i. e. the hulls K_t . Thus for $4 < \kappa < 8$ we will have to replace d_γ by d_K , the dimension of the SLE hull K_t , which is given by $1 + \frac{2}{\kappa}$ (d_γ for an SLE with speed $16/\kappa$).

Note that the angular dependency on $\alpha(z_0)$ becomes dominant for points near the boundary but can be neglected for those sufficiently far away. Therefore we will split the discussion of the dependency on the distance ϵ and the angle $\alpha(z_0)$.

4 The Effect of Fusion on the Scaling Properties

This section is dedicated to the behavior of the differential operators acting on the correlation functions in BCFT or martingales in SLE. A short heuristic introduction to the quite exhausting computational part can be found in the appendix A.

From here on we will make use of the interpretation introduced at the end of section 2.4: we will replace the dependency on the coordinates x_t^l, z_t^k by the functional dependency on x^l, z^k , respectively.

4.1 The Löwner Equation Differential Operators

For any $i = 1, 2, \dots, m$ and $h_j, \bar{h}_j \neq 0$ (if $h_j = 0$ or $\bar{h}_j = 0$, the corresponding term in the sum just vanishes), let us define

$$\begin{aligned} \mathcal{D}_{-2}^{m,n}(x^i, \{x^l\}, \{(z^k, \bar{z}^k)\}) := & \frac{\kappa}{2} \partial_{x^i}^2 - 2 \sum_{l \neq i=1}^m \left(\frac{h_l}{(x^l - x^i)^2} - \frac{1}{(x^l - x^i)} \partial_{x^l} \right) \\ & - 2 \sum_{k=1}^n \left(\frac{h_k}{(z^k - x^i)^2} - \frac{1}{(z^k - x^i)} \partial_{z^k} \right) + a.h. \end{aligned} \quad (43)$$

where *a.h.* denotes the antiholomorphic contributions which, in the following, we will drop for simplicity. Additionally, we do not want to spend time on the case that two fields fuse to the identity, hence we will leave that part out in the following considerations, too.

In a situation with $x^i \rightarrow z^j$, (i. e. the LÖWNER differential equation drives the j^{th} field at z^j to the real axis where the tip of the i^{th} SLE trace is growing), we have two effects if $h_j = h_i \in \{h_{(1,2)}, h_{(2,1)}\}$: For two degenerate fields corresponding to the representation $\phi_{(r,s)}$ of the same weight on level $r \cdot s = 2$ in the KAC table, we know that the outcome of fusing them is (apart from the identity) a degenerate field corresponding to $\phi_{(r',s')}$ with $r' = 2r - 1$ and $s' = 2s - 1$ on level $r' \cdot s' = 3$. When acting on the appropriate correlation function...

- (a) the i -th differential operator changes ($z^j - x^i = \epsilon$, $2z = z^j + x^i$) :

$$\mathcal{D}_{-2}^{m,n}(x^i; \{x^l\}, \{z^k\}) \rightarrow \epsilon \mathcal{D}_{-3}^{m,n-1}(z; \{x^l\}, \{(z^k, \bar{z}^k)\}_{k \neq j}) \quad (44)$$

where $\mathcal{D}_{-3}^{m,n-1}(z; \{x^l\}, \{(z^k, \bar{z}^k)\}_{k \neq j})$ is the differential operator imposing the null vector differential equation on a correlation function including a primary field of weight $h = h_{(r',s')}$:

$$\begin{aligned} & \mathcal{D}_{-3}^{m,n-1}(z; \{x^l\}, \{(z^k, \bar{z}^k)\}_{k \neq j}) \\ & := \frac{\kappa}{2} \partial_{x^i}^3 - 4 \sum_{l \neq i=1}^m \left(\frac{h_l}{(x^l - x^i)^2} - \frac{1}{(x^l - x^i)} \partial_{x^l} \right) \partial_{x^i} \\ & \quad - 4 \sum_{k \neq j=1}^n \left(\frac{h_k}{(z^k - x^i)^2} - \frac{1}{(z^k - x^i)} \partial_{z^k} \right) \partial_{x^i} + a.h. \end{aligned} \quad (45)$$

$$\begin{aligned}
& + 4 \left(\frac{8}{\kappa} - 1 \right) \sum_{l \neq i=1}^m \left(\frac{2h_l}{(x^l - x^i)^3} - \frac{1}{(x^l - x^i)^2} \partial_{x^l} \right) \\
& + 4 \left(\frac{8}{\kappa} - 1 \right) \sum_{k \neq j=1}^n \left(\frac{2h_k}{(z^k - x^i)^3} - \frac{1}{(z^k - x^i)^2} \partial_{z^k} \partial_{x^i} \right) + a.h.,
\end{aligned}$$

where *a.h.* denotes the antiholomorphic contributions.

- (b) for $\epsilon \rightarrow 0$, the weight h_j changes to $h_{(r',s')}$ in all of the other $m - 1$ equations.

In the next part we will repeat the same considerations for the arch configuration observable. Of course, we could also investigate the situation where $x_t^i \rightarrow x_t^k$. However, all interesting features of that case will be contained in the one discussed in the next part so that we did not want to include it here. Additionally, $z_t^j \rightarrow z_t^l$ will not be discussed since it would only lead to a different observable and no new differential equations or changes of the SLE quantities.

4.2 The Purely Boundary Differential Operators

In multiple SLE as investigated in [18], we do not only get the differential operators (43) but also others relating only the boundary fields. They show up when we consider the observable that tells us which arch configuration is present in the theory. It corresponds to the correlator of the boundary CFT fields [4] and hence vanishes under the action of:

$$\mathcal{D}_{-2}^m(x^i; \{x^l\}_{l \neq i}) := \frac{\kappa}{2} \partial_{x^i}^2 - 2 \sum_{l \neq i=1}^m \left(\frac{h_l}{(x^l - x^i)^2} - \frac{1}{(x^l - x^i)} \partial_{x^l} \right). \quad (46)$$

To investigate the different arch configurations that can show up in a theory with m traces starting at the boundary, it is natural to look at the effects when $x^i \rightarrow x^{i+1}$. Obviously, the traces could either pair up or just come close to each other without touching. The first situation has already been identified with fusion to the identity [4] so that we do not want to consider that part here. Note that only neighboring SLE traces are able to fuse when choosing $a_t^k = 1$ [18]; in any case other situations might lead to intersection problems of the SLE traces so that we do not want to consider them here.

Similarly to the situation of bulk-boundary fusion, in this limit the differential operator $\mathcal{D}_{-2}^m(x^i)$ becomes dependent on $\delta := x^{i+1} - x^i$ and $x := (x^i + x^{i+1})/2$. However, the situation is slightly different since the differential operators $\mathcal{D}_{-2}^m(x^k)$ for $k \neq i, i + 1$ also depend on δ and x , so that we have to change all differential operators. Fortunately, this is an easy work which can be solved by a simple expansion in the small parameter δ . Thus we get (again dropping the part where the fields fuse into the identity) when acting on the appropriate correlation function:

$$\mathcal{D}_{-2}^m(x^i; \{x^l\}_{l \neq i}) \rightarrow \delta \mathcal{D}_{-3}^{m-1}(x; \{x^l\}_{l \neq i, i+1}), \quad (47)$$

where $\mathcal{D}_{-3}^{m-1}(x; \{x^l\}_{l \neq i, i+1})$ is analogously defined to (45) and for $j \neq i, i + 1$:

$$\mathcal{D}_{-2}^m(x^j; \{x^l\}_{l \neq j}) \rightarrow \mathcal{D}_{-2}^{m-1}(x^j; x, \{x^l\}_{l \neq j, i, i+1}), \quad (48)$$

with $h = h_{(r',s')}$ as introduced in the previous section showing up as the weight of the field depending on x .

As said above, the case where two SLE paths not only come close to each other but actually meet at t_0 has already been discussed in [4, 18]. This situation corresponds to a kind of annihilation process of two SLE tips; the traces vanish from the upper half plane for $t > t_0$, hence being identified with the fusion branch that results in the identity field. However, it is quite more interesting to study the other part of the fusion process – although the probability of such an event gets small for small distances, the question which power law can be assigned to it may reveal the correct correspondence to BCFT quantities.

Together with the results in section 2.4, we will show in the following how we can relate a boundary field approaching a field of an observable \mathcal{O} or another boundary field to the probability of intersecting a ball of radius $r \asymp \epsilon$. Note that these events in the SLE picture are an SLE trace approaching a point in the upper half plane or two SLE traces approaching each other, respectively.

5 Interpretation of Merging multiple SLE traces

In this section we will try to relate the results of the previous two sections via identifying the scaling behavior of the SLE probability $\mathbf{P}_{\epsilon, z_0}$ and that of correlation functions after the fusion of fields in BCFT.

5.1 SLE Traces Visiting a Point in the Upper Half Plane

Let us consider the following situation: we have a multiple SLE with m interfaces γ^k starting from the boundary and going up to ∞ . Additionally, we have an observable given by the correlation function of n primary fields, located at the points z_0^l in the upper half plane: $O(\{z_0^l, \bar{z}_0^l\}) = \prod_{l=1}^n \phi_l(z_0^l, \bar{z}_0^l)$. We will only consider the case where one of them, say the j^{th} has the same weight as the BCFT field corresponding to the i^{th} interface – $h_j = h_i = h_{(r,s)}$ with $(r, s) \in \{(1, 2), (2, 1)\}$. For this situation, we have m SLE martingales as shown in section 2.4.

Going back to what we learned before, we take a look at the differential operators given by the LÖWNER equation (43) and their limits for $x^i \rightarrow z^j$ (44):

$$\mathcal{D}_{-2}^{m,n}(x^i; \{x^l\}, \{z^k\}) \rightarrow \epsilon \mathcal{D}_{-3}^{m,n-1}(z; \{x^l\}, \{(z^k, \bar{z}^k)\}_{k \neq j}) \quad (49)$$

leaving out the identity part as said above.

Additionally, we remember that the SLE martingale $O_t^{\text{SLE}}(x_t^i, \{x_t^l\}, \{z_t^k\})$ can be expressed via the CFT expectation value of our observable O

$$\mathcal{O}^{\text{CFT}}(x^i, \{x^l\}, \{z^k\}) := \frac{\left\langle \psi(\infty), \psi_{(r,s)}(x^i) \phi_{(r,s)}(z^j) \prod_{k \neq j=1}^n \phi(z^l) \prod_{l \neq i=1}^m \psi(x^l) \right\rangle}{\left\langle \psi(\infty), \psi_{(r,s)}(x^i) \phi_{(r,s)}(z^j) \prod_{l \neq i=1}^m \psi(x^l) \right\rangle}. \quad (50)$$

Inserting the OPE for $\psi_{(r,s)}(x^i) \phi_{(r,s)}(z^j)$, the differential equation

$$\mathcal{D}_{-2}^{m,n} \mathcal{O}^{\text{CFT}}(x^i, \{x^l\}, \{z^k\}) = 0 \quad (51)$$

shows the following behavior as $\epsilon := x^i - z^j \rightarrow 0$:

$$\epsilon \mathcal{D}_{-3}^{m,n-1} \epsilon^{-\mu} \tilde{\mathcal{O}}^{\text{CFT}}(x, \{x^l\}_{l \neq i}, \{z^k\}_{k \neq j}) = 0, \quad (52)$$

introducing $-\mu = h_{(r',s')} - 2h_{(r,s)}$, $2x = x^i + z^j$ and $\tilde{\mathcal{O}}^{\text{CFT}}$ as the expectation value with $\psi_{(r,s)}(x^i) \phi_{(r,s)}(z^j)$ fused to $\psi_{(r',s')}(x)$.

Remembering equation (31), i. e.

$$\frac{d}{dt} \mathbb{E}(\mathcal{O}_t^{\text{SLE}}) = \mathcal{D}_{-2}^{m,n} \mathcal{O}^{\text{CFT}}(x^i, \{x^l\}, \{(z^k, z^{*k})\}), \quad (53)$$

we have just shown that the R.H.S behaves like

$$\epsilon^{1-\mu} \quad (54)$$

for small values of ϵ . Note that this should be interpreted as a functional relation: since the dependency on the distance ϵ is the same for all times, as viewn as a functional dependency it is time independent in some sense.

The outcome on the L.H.S. is more easily to find: The expectation value of a martingale conditioned on a certain event is given by the expectation value of that specific martingale times the probability of the event. Obviously, in this case the event is that of an SLE trace coming at least ϵ close to the point z_j in the upper half plane. Hence we have:

$$\frac{d}{dt} \mathbb{E}(\mathcal{O}_t^{\text{SLE}}) \rightarrow \frac{d}{dt} \mathbf{P}_{\epsilon, z_j} \mathbb{E}(\tilde{\mathcal{O}}_t^{\text{SLE}}). \quad (55)$$

Now we will again assume, that the SLE expectation value of the martingale corresponds via our statistical physics picture to the respective correlation function in BCFT. Hence we expect the scaling behavior of $\mathbf{P}_{\epsilon, z_j}$ to be the same as that of the R.H.S.. Remembering from section 3.2 that the probability of an SLE trace intersecting a ball of radius ϵ located at z_j in the upper half plane sufficiently far away from the boundary:

$$\mathbf{P}_{\epsilon, z_j} \asymp \epsilon^{2-d_K}. \quad (56)$$

In order to see that $1 - \mu$ is indeed equal to $2 - d_K$, we have to insert some knowledge about the weights of the primary fields and the dimension of the SLE hull. Via the formula for the central charge we can deduce that

$$h_{(r,s)} = \frac{(r\kappa - 4s)^2 - (\kappa - 4)^2}{16\kappa} \quad (57)$$

which means that

$$\begin{aligned} h_{(1,2)} &= \frac{6-\kappa}{2\kappa} & h_{(1,3)} &= \frac{8-\kappa}{\kappa} & \rightarrow & \mu = \frac{2}{\kappa}, \\ h_{(2,1)} &= \frac{3\kappa-8}{16} & h_{(3,1)} &= \frac{\kappa-2}{2} & \rightarrow & \mu = \frac{\kappa}{8}. \end{aligned} \quad (58)$$

Hence we have $1 - \mu \in \{1 - \frac{\kappa}{2}, 1 - \frac{8}{\kappa}\}$ which is the same as $2 - d_K$ as $d_K \in \{1 + \frac{\kappa}{2}, 1 + \frac{8}{\kappa}\}$.

Therefore the event of the outer hull of the SLE coming close to a point can be identified[‡] with fusion of the BCFT boundary field with a bulk field of the same dimension.

[‡]Of course, here “identifying” is not meant in a mathematically rigorous sense!

Of course, the other $m - 1$ martingales M_t^k and correlation functions M^k , $k \neq i$ are also affected as discussed in section 4.1. However, this does not lead to any interesting behavior since the only changing quantity is the observable, but at a point sufficiently far away from any of the other $m - 1$ SLE traces and $n - 1$ points with primary fields attached.

5.2 SLE Traces Merging

Certainly, the point z_j we have been looking at could also be lying on one of the other SLE traces. Hence, in complete analogy to the considerations above, we will treat the situation of two SLE traces approaching each other. Obviously, this corresponds to a specific arch configuration. Thus we take the results from section 3.1, where we investigated the correlation function of boundary fields

$$\langle \psi_{\alpha_\infty, \alpha_1}(\infty), \psi_{\alpha_1, \alpha_2}(x^1) \dots \psi_{\alpha_m, \alpha_\infty}(x^m) \rangle. \quad (59)$$

As we already argued, in BCFT this is the probability $\mathbf{P}_{\gamma_t^1, \dots, \gamma_t^m}$ that this special configuration emerges by chance. Additionally, as a byproduct of the calculations including an observable, it has been proven in [4] to be a martingale, too. As the boundary changing operators are taken to be primary fields with weight $h_i = h_{i+1} = h_{(r,s)}$ with $(r, s) \in \{(1, 2), (2, 1)\}$, the correlator vanishes if we let the operators $\mathcal{D}_{-2}^m(x^i, \{x^k\}_{k \neq i})$ act on it.

In the limit $x^i \rightarrow x^{i+1}$, we know that the CFT quantity

$$\mathcal{D}_{-2}^m(x^i, \{x^k\}_{k \neq i}) \langle \dots \psi_{\alpha_i, \alpha_{i+1}}(x^i) \psi_{\alpha_{i+1}, \alpha_{i+2}}(x^{i+1}) \dots \rangle \quad (60)$$

splits up into two parts of which the identity part will be left out – as already stated above, it is uninteresting since the situation becomes essentially the same as for $m - 2$ SLEs starting at a later point in time [4]. The other part is proportional to

$$\mathcal{D}_{-3}^{m-1}(x, \{x^k\}_{k \neq i, i+1}) \delta^{1-\mu} \langle \dots \psi_{\alpha_i, \alpha_{i+2}}(x) \dots \rangle. \quad (61)$$

with $1 - \mu = 2 - d_K$ as already shown in the previous section. Hence we can deduce that the probability that two BCFT fields $\psi_{(r,s)}$ are fusing to $\psi_{(r',s')}$ can be expressed as the corresponding SLE traces coming close to each other, since if we look at the upper half plane at time t before applying G_t , we see that if the i^{th} SLE trace approaches the $(i + 1)^{\text{st}}$ near the point $z_0 \in \{G_t^{-1}(x_t^{i+1})\}^\dagger$, the SLE probability is given by

$$\mathbf{P}_{\delta, z_0^{i+1}} \propto \delta^{2-d_K} = \delta^{1-\mu}, \quad (62)$$

thus exhibiting the same functional δ dependence as $\mathcal{D}_{-3}^{m-1} \mathbf{P}_{\gamma^1, \gamma^{i-1}, \gamma^{\text{merged}}, \gamma^{i+2}, \gamma^m}$. Thus fusion of boundary fields in CFT can be considered as merging of SLE hulls. More precisely: merging SLE traces of speed...

- (i) $\dots 4 < \kappa < 8$ corresponds to fusing two $h_{(1,2)}$ to an $h_{(1,3)}$ boundary field,
- (ii) $\dots 0 < \kappa < 4$ corresponds to fusing two $h_{(2,1)}$ to an $h_{(3,1)}$ boundary field.

[†]Of course $z_0 \notin \mathbb{H}/K_t$, hence this formula should rather be interpreted as the limit $z_0^{i+1} \in \{\lim_{\epsilon \rightarrow 0} G_t^{-1}(x_t^{i+1} + i\epsilon)\}$.

5.3 The Angular Dependency

Unfortunately, we are not able to give a complete satisfactory interpretation to the angular dependency of $\mathbf{P}_{\epsilon, z_0}$. Hence this section will be quite sketchy but we are convinced that the basic recipe how to achieve this goal works in principle. Our proposal is based on the connection of the absolute value $|z_0|$, the imaginary part $\Im(z_0)$ and the angle $\alpha(z_0)$ between the real axis and the vector pointing to a point z_0 :

$$|z_0| = \Im(z_0) \cdot \sin \alpha(z_0)^{-1}, \quad (63)$$

as has been attempted in [2]. Thus we will give a short handwavy description on our thoughts on this subject.

Equation (63) could be used after applying three known properties of n -point functions:

- (i) behavior under conformal transformations for a fixed time t :

$$\begin{aligned} \langle \dots \psi(x_t) \dots \rangle &= \langle \dots \psi(G_t(z_0)) \dots \rangle \\ &= (G'_t(z_0))^{-h} \langle \dots \psi(z_0) \dots \rangle, \end{aligned} \quad (64)$$

$$\begin{aligned} \langle \dots \phi(z_t, z_t^*) \dots \rangle &= \langle \dots \phi(G_t(z_t), G_t^*(z_t^*)) \dots \rangle \\ &= |G'_t(z_0)|^{-2h} \langle \dots \phi(z_0, z_0^*) \dots \rangle, \end{aligned} \quad (65)$$

for local bulk fields $\phi(z_0, z_0^*)$ with weights $h_0 = \bar{h}_0 = h$ and boundary fields $\psi(x_t)$ of weight h . Note that the interpretation of (64), (65) is difficult in the standard CFT context since we only have the formalism to deal with the whole upper half plane – not \mathbb{H}/K_t . Additionally, it is not clear what G_t might be in the limit of two traces coming close to each other. These equations should therefore just be used to illustrate how the angular dependence on $\alpha(z_0)$ could in principle be recovered from the general correlation function.

- (ii) cluster decomposition: if we separate the variables of a correlation function into two sets and scale one set towards ∞ , the behavior of the correlation function is dominated by the product of two correlation functions, corresponding to the two sets of variables, multiplied by an appropriate power of the separation [17]:

$$\left\langle \prod_i \phi(\lambda w_i) \prod_j \phi(z_j) \right\rangle \sim \left\langle \prod_i \phi(w_i) \right\rangle \left\langle \prod_j \phi(z_j) \right\rangle \lambda^{-\sum_i h_i}. \quad (66)$$

For our purposes, we will separate the two fusing fields from the rest which leaves us with a two point function of two $h_{(r,s)}$ fields with $(r, s) \in \{(1, 2), (2, 1)\}$ which is nonzero and the correlation function of the rest. Afterwards we insert the OPE, choose the non-identity part including the $h_{(r',s')}$ field with $r' = 2r - 1$ and $s' = 2s - 1$ as well as its descendants. Here it is of utmost importance that we are dealing with BCFT where the one point functions of boundary fields does not vanish.

In the theory of probability, this corresponds to independent events X, Y :

$$\mathbb{E}(XY) = \mathbb{E}(X) \cdot \mathbb{E}(Y). \quad (67)$$

This assumption is valid if the other SLE traces and the rest of the observable is far away from the interesting point.

(iii) conformal invariance then leaves us with the following prefactor:

$$\epsilon^{-\mu} |z|^{-h_{(r',s')}}. \quad (68)$$

where ϵ is the distance between the two fusing fields and z their mean coordinate.

Note that for the bulk field case, we have to compare it to the product of two probabilities since this implies that the mirrored LÖWNER trace approaches the mirrored point z_0^* – hence we are looking for a correspondence to $\mathbf{P}_{\epsilon, z_0}^2$.

The behavior near the boundary is only important in the case of $4 < \kappa < 8$ where we have $h_{(1,3)}$ type fields since otherwise the SLE trace is simple. Thus for $t > 0$ it neither touches the boundary nor itself and therefore we can assume that it does not come near the boundary again. Hence, we have $h_0 = h_{(1,3)} = (8 - \kappa)/\kappa$ and the factor $\sin \alpha(z_0)^{8/\kappa-1}$ shows up squared for $h_0 = \bar{h}_0$ in $\mathbf{P}_{\epsilon, z_0}^2$ and to power one in \mathbf{P}_{δ, x_0} . Note that our sine exponent differs from the one mentioned in [2] but is indeed the same as given in [5] (cf. (42)).

Nevertheless, the situation is not that easy to be solved since – apart from the problems already listed above – it is still an open problem how to translate the probability $\mathbf{P}_{\epsilon, z_0}$ to the multiple SLE properly. It has been derived for a situation where a single SLE starts at 0, exhibiting an angle $\alpha(z_0)$ which is measured at the origin. Hence what has to be done is to extend the formalism to any reference point ξ_t which should be an easy problem using translation invariance of the problem. As a second step, more than just one SLE trace should be considered which will surely lead to modifications, too, and probably requires a definition of multiple SLE on the disk. However, this would go beyond the scope of our paper which is dedicated to the SLE-BCFT correspondence and not intended to focus on new SLE results. Hence we just argue that if we assume that all other SLE traces and observables are sufficiently far away from the z_0 , we can apply the cluster decomposition theorem for local quantum field theories. Hence, equation (68) gives us an independent prefactor of a correlation function of the rest of the other fields involved in the theory. This way, the asymptotic behavior stays the same and our considerations should be valid in general m multiple SLE with any observable O given by a product of local primary fields.

6 Multiple SLE as Martingale Weighted SLE

Following the idea in [30] for $\text{SLE}(\kappa, \vec{\rho})$, we will argue that multiple SLE can be interpreted as m single SLEs moving in the same upper half plane weighted by the martingale M_t given in (9). Additionally, we give a probability interpretation of this martingale following [11, 12] and discuss the case of collapsing starting points.

6.1 The Associated Martingale M_t

Suppose we have a random process (X_t) , $t \geq 0$, taking values in some space X . Now let us define a function $f : X \rightarrow [0, \infty)$ such that $f(X_t)$ is measurable with

bounded expectation value. In that case, we are able to weight the probability measure P_{old} by $f(X_t)$ and hence consider a new probability measure P_{new} which can be achieved by the RADON-NIKODYM derivative:

$$\frac{dP_{\text{new}}}{dP_{\text{old}}} = \frac{f(X_t)}{\mathbb{E}(f(X_t))}. \quad (69)$$

In our case, the new law of X_t under the new measure can be determined explicitly. Let \mathcal{F}_t be the filtration to which X_t is adapted. Then $M_t := \mathbb{E}(f(X_t)|\mathcal{F}_t)$ is a martingale by construction and for every $A \in \mathbb{F}_t$ we have $P_{\text{new}}(A) = \mathbb{E}(1_A M_t)/M_0$. With the help of GIRSANOV's theorem, we can compute the change of the drift term due to the weighting with the martingale M_t explicitly. For any local martingale N_t^{old} under the old measure, we get a new local martingale N_t^{new} via:

$$N_t^{\text{new}} = N_t^{\text{old}} - \int_0^t \frac{d\langle N^{\text{old}}, M \rangle_{t'}}{M_{t'}}. \quad (70)$$

In our case, it has been derived in [18] that the weight martingale M_t is given by (9) and thus leads to the change in the drift terms stated in (8).

Hence we can say that multiple SLE is just ordinary SLE under a new measure.

6.2 Probability Interpretation for M_t

Analogously to the ideas presented in [11, 12, 13], we will give a probability interpretation to the martingale found for multiple SLE processes (9). However, we have to point out the crucial difference between our ansatz and the considerations made in [11, 12]: in our opinion, multiple SLE does not correspond to $\text{SLE}(\kappa, \vec{\rho})$ with $\vec{\rho} = (2, 2, \dots, 2)$. In terms of the formalisms typically used for the description of $\text{SLE}(\kappa, \vec{\rho})$, the stochastic nature of the coordinates of the force points is only due to the BROWNIAN motion of the SLE coordinate although for $\rho_i = 2$ the i^{th} field would be of the correct weight. However, in multiple SLE, every field that corresponds to an interface exhibits its own BROWNIAN motion. However, if we take $\text{SLE}(\kappa, \vec{\rho})$ as a special case of multiple SLE with $\kappa_i = a_i^i = 0$ for all i except one and the partition function to be that of the COULOMB gas formalism as described in [8, 19], the two SLE types become the same. We will come back to that interpretation in a forthcoming paper [23].

For $h_i = \frac{6-\kappa_i}{2\kappa_i} = h_{(1,2)} \geq \frac{5}{8}$, i. e. $\kappa_i \leq \frac{8}{3}$, let us consider the single SLE picture with g_t^i as the i^{th} of m LÖWNER mappings of a multiple SLE setting with driving function $w_t^i = \sqrt{\kappa_i} B_t^i$ as in (1). For $i = 1, \dots, m$, let \mathcal{D}_{c_i} be independent realizations of the BROWNIAN loop soup of intensity $-c_i$. Let K_i be the i^{th} hull generated by $\gamma_i[0, \infty)$ and the loops that intersect it. For each i we know from the properties of the BROWNIAN loop soup that the probability that K_i intersects some hull A_i is

$$\mathbf{P}_i := \mathbf{P}\{K_i \cap A_i = \emptyset\} = \Phi'_{A_i}{}^{h_i} \exp\left(-\frac{c_i}{6} \int_0^t \mathcal{S}\Phi_s(w_s^i) ds\right), \quad (71)$$

where $\Phi_{A_i} : \mathbb{H}/A_i \rightarrow \mathbb{H}$ and, again, \mathcal{S} the SCHWARZIAN derivative.

Now let us consider multiple SLE with the multiple LÖWNER mapping as stated in (7). Let A_i be the hull created by $\bigcup_{j \neq i} K_j$. Then $H_t^i(w_t^i) = x_t^i$

as defined in section 2.2 plays the role of Φ_{A_i} in this context. Following the argumentation in [20] we define

$$M_t := \frac{Z[x_t]}{Z[x_0]} \prod_{i=1}^m (H_t^{i'}(w_t^i))^{h_i} \exp\left(-\frac{c_i}{6} \int_0^t \mathcal{S}H_s^i(w_s^i) c_s^i ds\right) \exp\left(-\int_0^t \frac{1}{Z[x_s]} \mathcal{D}_{-2}^m(x_s^i) Z[x_s] a_s^i ds\right) \quad (72)$$

on $\{t < t_A\}$ with t_A being the intersection time. M_t is a martingale and $0 \leq M_t \leq (H_t^{i'}(w_t^i))^{h_i} \leq H_t^{i'}(w_t^i) \leq 1$ for $\alpha_i \geq \frac{5}{8}$ as shown in [18].

Knowing that the level two null vector differential equation imposed by the action of $\mathcal{D}_{-2}^m(x_s^i)$ is satisfied since $h_i = h_{(1,2)}$, we can drop the last factor. Taking a look at the expectation value of M_t and using (71), it is easy to see that

$$\mathbb{E}(M_t) = \mathbb{E}\left(\frac{Z[x_t]}{Z[x_0]}\right) \prod_{i=1}^m \mathbf{P}_i. \quad (73)$$

As already argued in section 3.1, the ratio of the partition functions just gives us the probability, that we have an arch configuration starting at arbitrary points without traces pairing up. Hence, the expectation value of our martingale M_t equals the probability $\mathbf{P}^{m\gamma}$ that we have m traces that do neither intersect each other nor pair up to form archs.

6.3 Fusion in the Associated Martingale

Now we will consider two points collapsing, i. e. $x_t^{i+1} \rightarrow x_t^i$, with $h_i = h_{i+1} = h_{(1,2)}$. Note that we still restrict ourselves to the case where the tips are not meeting at their endpoints to pair up. Using $x_t^k = H_t^k(w_t^k)$, we get for the first factor:

$$\lim_{x_t^i \rightarrow x_t^{i+1}} \frac{Z[x_t]}{Z[x_0]} = \lim_{w_t^i \rightarrow w_t^{i+1}} (H_t^{i+1}(w_t^{i+1}) - H_t^i(w_t^i))^{-\mu} \frac{\tilde{Z}[x_t]}{Z[x_0]} \propto (H_t^i(w))^{-\mu} \frac{\tilde{Z}[x_t]}{Z[x_0]} \quad (74)$$

with $\mu = h_i + h_{i+1} - h$ and \tilde{Z} denoting the partition function wherein we fused the i^{th} and $(i+1)^{\text{st}}$ boundary field of weight h at coordinate $w_t = (w_t^i + w_t^{i+1})/2$. From CFT fusion rules we know that $h = h_{(1,1)}$ or $h = h_{(1,3)}$. As we agreed to restrict ourselves to the second case (see above), we have in this limit

$$\mathbf{P}_{x_t^i \rightarrow x_t^{i+1}}^{m\gamma} \propto (\phi'_{A_i})^{h_{(1,3)}} \prod_{j \neq i, i+1} (\phi'_{A_j})^{h_j}, \quad (75)$$

switching back to the notation used in (71).

This can be easily extended to the case of $m \geq 2$ curves and hence in the limit of all points collapsing to x_t , we get:

$$\mathbf{P}_{\{x_t^i\} \rightarrow x_t}^{m\gamma} \rightarrow (\phi'_{A_i})^{h_{(1, m+1)}}. \quad (76)$$

in complete agreement with CFT fusion rules. Hence we will take this as another hint at the picture of SLE tips corresponding to boundary CFT fields of weight

$h_{(1,2)}$. Note that this behavior has also been observed in [11, 13] for a different picture of multiple SLE; a similar case is studied for $\text{SLE}(\kappa, \bar{\rho})$ in [12]. Of course, we should not be suprised that this result is similar to that of $\text{SLE}(\kappa, \bar{\rho})$ as the latter is only a special case of multiple SLE which we will show in an forthcoming paper [23].

Moreover, if we let $h_i = \frac{3\kappa_i - 8}{16} \geq \frac{5}{8}$, i. e. $\kappa_i \rightarrow 16/\kappa_i \leq \frac{8}{3}$ which means that $\kappa_i \leq 6$ and $c_i \leq 0$, we cover the $h_i = h_{(2,1)}$ fusion to $h_{(m+1,1)}$ case.

Additionally, in complete agreement to the results of the appendix of [18], we could also set $a_t^i = 1 \quad \forall i$. It has been shown that this corresponds to a setting without mutual intersections of the SLE traces. Inserting that in the notation used in [18], $c_t^i = 1$, we have $(H_t^{i'}(w_t^i))^2 = 1$ and hence $\mathbf{P}_i = 1$ which also means nothing else than no intersections occur w.p.1.

7 Discussion

In this paper we have shown how to interpret merging SLE traces and traces approaching a marked point z_j as CFT quantities via their scaling behavior. The outcome is quite natural: locally, these events can be interpreted as the fusion of two $h_{(r,s)}$ to an $h_{(r',s')}$ field with $(r, s) \in \{(1, 2), (2, 1)\}$ and $r' = 2r - 1$, $s' = 2s - 1$.

However, this interpretation is only valid in a small region around z_j due the nature of the SLE mapping. It erases any event from the scene so that for later times, the event is gone and leaves no hint at all that it ever happened.

The two events – merging traces and traces approaching a marked point – may be distinguished by also taking the antichiral part of the bulk field into account. We have proposed a way how this can be done by considering the SLE probability of visiting a point near the boundary where the initial behavior takes over in terms of an angular dependency. In contrast to previous papers [2], we are able to give an interpretation of the sine exponent $h_{(1,3)}$ initially derived in [5] and have been able to give a hint how to extend the interpretation to the general case of n point functions.

Interestingly, the assignment of the different SLE phases – $0 < \kappa < 4$ and $4 < \kappa < 8$ – to different types of fields – $h_{(3,1)} = \frac{\kappa-2}{2}$ and $h_{(1,3)} = \frac{8-\kappa}{\kappa}$ – has already been observed in another context by Riva and Cardy [25] while investigating the Q -states POTTS model and its relation to SLE.

In order to achieve our results, we have tried a novel approach to the view on the SLE-BCFT correspondence: in our opinion, the mapping between the two theories goes via the identification of whole differential equations. Hence we can not close our eyes upon the behavior of the differential operators. This inevitably leads us to additional exponents that enable us to give a quite natural interpretation while staying completely within the KAČ table of minimal BCFT models for candidates for interpretation.

Additionally we picked up the viewpoint of all SLEs being just ordinary SLEs under different measures to give a probability interpretation to the weighting martingale. We pointed out how, again, other KAČ table weights show up as scaling exponents in multiple SLE settings in complete agreement with what we would expect from CFT fusion rules.

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A Appendix: How to Compute “Fused” Differential Operators

In this appendix, we want to show how the additional exponent of the “fused” differential operators emerges from general considerations. The whole idea is based on appendix 8.A of [9] where the interested reader may find more information on the subject, too.

The Operator Product Expansion

As already said above, the operator product expansion of two primary fields at x_0, x_1 of weights h_0 and h_1 is given by

$$\phi_0(x_0)\phi_1(x_1) \sim \sum_{h(r',s')} \frac{g(r',s')}{\epsilon^{h_0+h_1-h}} \sum_Y \epsilon^{|Y|} \beta_{Y,(r',s')} L_{-Y} \phi_h(x) \quad (77)$$

where $\mu = h_0 + h_1 - h_{(r',s')}$ and $g_{(r',s')}$ the coefficient of the three-point function of $\phi_0(x_0), \phi_1(x_1)$ and $\phi_{h_{(r',s')}}(x)$ (setting $h_{(r',s')} = h$):

$$\frac{g_{(r',s')}}{(x_0 - x_1)^{h_0+h_1-h} (x_0 - x)^{h_0+h-h_1} (x_1 - x)^{h+h_1-h_0}} = \langle \phi_0(x_0)\phi_1(x_1)\phi_{h_{(r',s')}}(x) \rangle. \quad (78)$$

In our case, we have $h_0 = h_1 = h_{(2,1)} = \frac{3}{16}\kappa - \frac{1}{2}$, so that $h_{(r',s')} = h_{(1,1)} = 0$ or $h_{(r',s')} = h_{(3,1)} = \frac{\kappa}{2} - 1$ since $g_{(r',s')} = 0$ for all other values $h_{(r',s')}$. Of course, we can compute the case for $h_0 = h_1 = h_{(1,2)} = \frac{6-\kappa}{2\kappa}$ as well (leading to $h_{(r',s')} = h_{(1,1)} = 0$ or $h_{(r',s')} = h_{(1,3)} = \frac{8-\kappa}{\kappa}$), just change $\kappa \rightarrow \frac{16}{\kappa}$ in the computations.

In the following we will concentrate on the $h_{(3,1)}$ branch of fusion since that will be the more interesting case for SLE purposes; the identity branch has already been identified with meeting tips of two SLE curves that “annihilate” and vanish from the scene for all later times – leaving a setting with two curves less but no perviously unknown features whatsoever [4]. Hence, any result we are stating here is always up to the coefficient $g_{(r',s')}$ and the part coming from the identity branch of the fusion process. But since different representations do not mix, the results will nevertheless be exact enough for our purposes.

The Differential Operators

In the theory of minimal models, the primary fields are degenerate, thus satisfying differential equations. Hence the left hand side of our OPE inserted

in a correlation function satisfies a so-called null vector differential equation which should be still valid after fusion meaning that we should get differential equations on the RHS, too. Hence, for the LHS, we have:

$$\langle \phi_0(x_0) [(\mathcal{L}_{-1}^n)^2(x_1) - t\mathcal{L}_{-2}^n(x_1)] \phi_1(x_1)\phi_3(x_3)\dots\phi_n(x_n) \rangle = 0 \quad (79)$$

with

$$\mathcal{L}_{-k}^n(x_i) = \sum_{j \neq i=0}^{n-1} \frac{h(k-1)}{(x_j - x_i)^k} - \frac{1}{(x_j - x_i)^{k-1}} \partial_{x_j}. \quad (80)$$

and $\mathcal{L}_{-1}^n(x_i) = \partial_{x_i}$. As said above, there should be a corresponding equation for the RHS after fusion. Therefore we introduce $\mathcal{L}'_{-r}(x, \epsilon)$, i. e. the around $x = x_1$ expanded differential operators, with $\epsilon = x_0 - x_1$ being the deviation of x_0 from x . Note that we could expand around any other (from ϵ linearly independent) coordinate since the problem is translational invariant. We just take this special choice here for the computations to be as simple as possible. With some effort we get:

$$\begin{aligned} \mathcal{L}'_{-r}(x, \epsilon) &= \sum_{j \neq 1}^{n-1} \frac{h(k-1)}{(x_j - x_1)^k} - \frac{1}{(x_j - x_1)^{k-1}} \partial_{x_j} \Big|_{x_1=x, x_0-x_1=\epsilon} \\ &= \frac{h(k-1)}{(x_0 - x_1)^k} - \frac{1}{(x_0 - x_1)^{k-1}} \partial_{x_0} + \sum_{j=2}^{n-1} \frac{h(k-1)}{(x_j - x_1)^k} - \frac{1}{(x_j - x_1)^{k-1}} \partial_{x_j} \\ &= \frac{h(k-1)}{\epsilon^k} - \frac{1}{\epsilon^{k-1}} \partial_\epsilon + \sum_{j=2}^{n-1} \frac{h(k-1)}{(x_j - x)^k} - \frac{1}{(x_j - x)^{k-1}} \partial_{x_j} \\ &= \epsilon^{-r} [h_0(r-1) - \epsilon \partial_\epsilon] + \mathcal{L}_{-r}^{n-1}(x), \end{aligned} \quad (81)$$

and hence

$$\mathcal{L}_{-r}^n(x_1) = \mathcal{L}'_{-r}(x, \epsilon) = \epsilon^{-r} \left[h_0(r-1) - \epsilon \frac{d}{d\epsilon} \right] + \mathcal{L}_{-r}^{n-1}(x). \quad (82)$$

Thus, the differential equation on the RHS becomes:

$$\begin{aligned} \left\langle \left[(\mathcal{L}'_{-1}(x, \epsilon))^2 - t\mathcal{L}'_{-2}(x, \epsilon) \right] \left[\sum_h \frac{g}{\epsilon^\mu} \sum_Y \epsilon^Y \beta_Y (L_{-Y} \phi_h(x)) \right] \right. \\ \left. \times \phi_3(x_3) \dots \phi_{n-1}(x_{n-1}) \right\rangle = 0 \end{aligned} \quad (83)$$

The Computation of the Coefficients

Now we will show how to compute $\beta_{Y, (r', s')}$ the from global conformal invariance. It is known that a primary field $\phi_h(z)$ of weight h behaves as follows under conformal transformations $z \rightarrow f(z) = z' = z + \varepsilon(z)$:

$$\phi_h(z) = \left(\frac{\partial f(z)}{\partial z} \right)^h \phi_h(f(z)) \Leftrightarrow \phi_h(z')(dz')^h = \phi_h(z)(dz)^h. \quad (84)$$

Thus, for an infinitesimal transformation, we have:

$$(\phi_h(z) + \varepsilon(z) \partial \phi_h(z)) (dz)^h (1 + h\varepsilon'(z)) = \phi_h(z)(dz)^h. \quad (85)$$

From this follows that for $\varepsilon(z) = \sum_k \varepsilon_k z^{k+1}$ and $\delta_\varepsilon L_{-Y} \phi_h(x) = \varepsilon_k L_k L_{-Y} \phi_h(x)$ after comparing coefficients of powers of z , we have for our special situation:

$$\epsilon^{-\mu} \sum_Y \epsilon^{|Y|} \beta_Y L_k L_{-Y} \phi_h(x) = (h_0(k+1)\epsilon^k + \epsilon^{k+1} \partial_\epsilon) \epsilon^{-\mu} \sum_Y \epsilon^{|Y|} \beta_Y L_{-Y} \phi_h(x). \quad (86)$$

In other words, all this implies the following rule of thumb:

- For any level j we take the j equations

$$L_k \phi_h^{(j)}(x) = [h_0(k+1) + j - k - \mu] \phi_h^{(j-k)}(x) \quad (87)$$

with $k = 1, \dots, j$ from the covariance conditions with

$$\phi_h^{(j)}(x) = \sum_{(|Y|=j)} \beta_Y L_{-Y} \phi_h(x). \quad (88)$$

- Moreover, we have to compute the following commutators algebraically:

$$L_k \sum_{|Y|=j} \beta_Y L_{-Y} \phi_h(x) = \sum_{|Y|=j} \beta_Y [L_k, L_{-Y}] \phi_h(x), \quad (89)$$

(since $\phi_h(x)$ is a primary field, the action of positive generators of the VIRASORO-algebra vanishes).

Afterwards we compare the respective coefficients of $\phi_h^{(j-k)}(x)$ and hence get the β_Y that depend on the choice of coordinate change $(x_0, x_1) \rightarrow (x, \epsilon)$.

The New Differential Equation

Hence the new differential operators are:

$$\begin{aligned} \mathcal{L}'_{-1} &= \epsilon^{-1} \left[h_0(1-1) - \epsilon \frac{d}{d\epsilon} \right] + \mathcal{L}_{-1}(x) \\ &= -\frac{d}{d\epsilon} + \mathcal{L}_{-1}(x) \end{aligned} \quad (90)$$

$$\begin{aligned} \mathcal{L}'_{-2} &= \epsilon^{-2} \left[h_0(2-1) - \epsilon \frac{d}{d\epsilon} \right] + \mathcal{L}_{-2}(x) \\ &= \frac{h_0}{\epsilon^2} - \frac{1}{\epsilon} \frac{d}{d\epsilon} + \mathcal{L}_{-2}(x) \end{aligned} \quad (91)$$

with which we can state the differential equation for the fused correlation function:

$$\begin{aligned} &\left\langle \left[\left(\frac{d^2}{d\epsilon^2} - 2 \frac{d}{d\epsilon} \mathcal{L}_{-1}(x) + \mathcal{L}_{-1}^2(x) \right) \right. \right. \\ &\left. \left. - t \left(\frac{h_0}{\epsilon^2} - \frac{1}{\epsilon} \frac{d}{d\epsilon} + \mathcal{L}_{-2}(x) \right) \right] \epsilon^{-\mu} \sum_j \epsilon^j \phi_h^{(j)} \times \phi_2(x_2) \dots \phi_h(x_n) \right\rangle = 0 \end{aligned}$$

In the following, we set $h_{(3,1)} = h$. After executing the derivative with respect to ϵ and ordering the coefficients of the various powers of ϵ , we get:

$$\begin{aligned} 0 &= [(j+2-\mu)(j+2-\mu-1) - t(h_0 - (j+2-\mu))] \phi_h^{(j+2)} \\ &\quad - 2(j+1-\mu) L_{-1} \phi_h^{(j+1)} + (L_{-1}^2 - t L_{-2}) \phi_h^{(j)} \end{aligned} \quad (92)$$

The Reason for $\epsilon^{1-\mu}$

Now we want to get a level three null vector condition. For dimensional reasons, it is quite obvious that we have to take the $j = 1$ term which gives us an additional factor of ϵ : after finishing the computations, the RHS will be proportional to $\epsilon^{j-\mu}$ times the differential operator on level three times a correlation function involving ϕ_h which has the same transformational properties under conformal transformations as the level two differential operator acting on a correlation function involving ϕ_{h_0} and ϕ_{h_1} . This is the crucial point why we have an additional distance factor in our equations in the main part of this paper.

The interested reader may question why it is sufficient to just compute the $j = 1$ part of the OPE. The shortest argument possible can be stated as follows: since we already know the outcome of fusion – for the fields as well as for the differential operator, we know, too, that the other terms have to vanish as we know that the differential equations have to be fulfilled before and after fusion. A more elaborate reasoning may be found in section appendix 8.A.3 of [9].

Computing the Coefficients

Setting $j = 1$, we get:

$$0 = \underbrace{[(3 - 2h_0 + h)(2 - 2h_0 + h) - t(3h_0 - 3 - h)]}_{V_1^1} \phi_h^{(3)} \quad (93)$$

$$\underbrace{-2(2 - 2h_0 + h)L_{-1}\phi_h^{(2)}}_{V_2^1} + L_{-1}^2\phi_h^{(1)} - \underbrace{t}_{V_3^1} L_{-2}\phi_h^{(1)}$$

$$= [V_1^1 (\beta_{111}^3 L_{-1}^3 + \beta_{12}^3 L_{-1} L_{-2} + \beta_{21}^3 L_{-2} L_{-1} + \beta_3^3 L_{-3}) + V_2^1 L_{-1} (\beta_{11}^3 L_{-1}^2 + \beta_2^3 L_{-2}) + L_{-1}^2 \beta_1^3 L_{-1} + V_3^1 L_{-2} \beta_1^3 L_{-1}] \phi_h(x) \quad (94)$$

$$= K_1^3 L_{-1}^3 + K_{12}^3 L_{-1} L_{-2} + K_{21}^3 L_{-2} L_{-1} + K_3^3 L_{-3} \quad (95)$$

with

$$K_1^1 := V_1^3 \beta_{111}^3 + V_2^3 \beta_{11}^3 + \beta_1^3 \quad (96)$$

$$K_{12}^1 := V_1^3 \beta_{12}^3 + V_2^3 \beta_2^1 \quad (97)$$

$$K_{21}^1 := V_1^3 \beta_{21}^3 + V_3^3 \beta_1^1 \quad (98)$$

$$K_3^1 := V_1^3 \beta_3^3 \quad (99)$$

using the commutator $L_{-1} L_{-2} = L_{-2} L_{-1} + L_{-3}$.

Knowing the form of the algebraic level three null operator:

$$L_{-1}^3 - (h+1)(L_{-2} L_{-1} + L_{-1} L_{-2}) + (h+1)^2 L_{-3}, \quad (100)$$

we only have to compare the coefficients:

$$\begin{aligned} 0 &= K_1^3 L_{-1}^3 + K_{12}^3 L_{-1} L_{-2} + K_{21}^3 L_{-2} L_{-1} + K_3^3 L_{-3} \\ &\propto L_{-1}^3 - (h+1)L_{-2} L_{-1} - (h+1)L_{-1} L_{-2} + (h+2)(h+1)L_{-3} \end{aligned} \quad (101)$$

wherein the proportionality factor K_1^3 can in principle depend on h . Hence all that is left to be shown is:

$$-(h+1) = \frac{K_{12}^3}{K_1^3} \quad (102)$$

$$-(h+1) = \frac{K_{21}^3}{K_1^3} \quad (103)$$

$$(h+1)^2 = \frac{K_3^3}{K_1^3} \quad (104)$$

for which we have to compute the exact values of the β_Y . Therefore we will use the following reexpressions:

$$t = \frac{h+1}{2} \quad (105)$$

$$c = -\frac{(3h-1)(h-2)}{h+1} \quad (106)$$

$$h_0 = h_1 = \frac{1}{8}(3h-1). \quad (107)$$

Now we will use our rule of thumb stated above:

$j = 1$ from covariance:

$$\begin{aligned} L_1\phi_h^{(1)}(x) &= [h_0(1+1) + 1 - 1 - (2h_0 - h)] \phi_h^{(1-1)}(x) \\ &= h\phi_h(x) \end{aligned} \quad (108)$$

algebraically:

$$\begin{aligned} L_1\phi_h^{(1)}(x) &= L_1L_{-1}\beta_1^3\phi_h(x) \\ &= 2h\beta_1^3\phi_h(x) \end{aligned} \quad (109)$$

Hence it follows $\beta_1^3 = 1/2$.

$j = 2$ from covariance:

$$\begin{aligned} L_1\phi_h^{(2)}(x) &= [h_0(1+1) + 2 - 1 - (2h_0 - h)] \phi_h^{(2-1)}(x) \\ &= (h+1)\phi_h^{(1)}(x) \\ &= \frac{h+1}{2}L_{-1}\phi_h(x) \end{aligned} \quad (110)$$

$$\begin{aligned} L_2\phi_h^{(2)}(x) &= [h_0(2+1) + 2 - 2 - (2h_0 - h)] \phi_h^{(2-2)}(x) \\ &= (h+h_0)\phi_h(x) \end{aligned} \quad (111)$$

algebraically:

$$\begin{aligned} L_1\phi_h^{(2)}(x) &= L_1(\beta_{11}^3L_{-1}^2 + \beta_2^3L_{-2})\phi_h(x) \\ &= (2(2h+1)\beta_{11}^3 + 3\beta_2^3)L_{-1}\phi_h(x) \end{aligned} \quad (112)$$

$$\begin{aligned} L_2\phi_h^{(2)}(x) &= L_2(\beta_{11}^3L_{-1}^2 + \beta_2^3L_{-2})\phi_h(x) \\ &= (6h\beta_{11}^3 + (4h+c/2)\beta_2^3)\phi_h(x) \end{aligned} \quad (113)$$

comparing the coefficients, it follows:

$$\beta_{11}^3 = \frac{1}{8} \frac{h+1}{h+2} \quad (114)$$

$$\beta_2^3 = \frac{1}{4} \frac{h+1}{h+2} \quad (115)$$

$j = 3$ from covariance:

$$\begin{aligned} L_1\phi_h^{(3)}(x) &= [h_0(1+1) + 3 - 1 - (2h_0 - h)] \phi_h^{(3-1)}(x) \\ &= (h+2)\phi_h^{(2)}(x) \\ &= (h+2)(\beta_{11}^3 L_{-1}^2 + \beta_2^3 L_{-2}) \phi_h(x)(x) \end{aligned} \quad (116)$$

$$\begin{aligned} L_2\phi_h^{(3)}(x) &= [h_0(2+1) + 3 - 2 - (2h_0 - h)] \phi_h^{(3-2)}(x) \\ &= (h+1+h_0)\phi_h^{(1)} \\ &= \frac{h+1+h_0}{2} L_{-1}\phi_h(x) \end{aligned} \quad (117)$$

$$\begin{aligned} L_3\phi_h^{(3)}(x) &= [h_0(3+1) + 3 - 3 - (2h_0 - h)] \phi_h^{(3-3)}(x) \\ &= (h+2h_0)\phi_h(x) \end{aligned} \quad (118)$$

algebraically:

$$\begin{aligned} L_1\phi_h^{(3)}(x) &= L_1(L_{-1}^3\beta_{111}^3 + L_{-1}L_{-2}\beta_{12}^3 + L_{-2}L_{-1}\beta_{21}^3 + L_{-3}\beta_3^3) \phi_h(x) \\ &= ([6(h+1)L_{-1}^2] \beta_{111}^3 + [3L_{-1}^2 + 2(h+2)L_{-2}] \beta_{12}^3 \\ &\quad + [2hL_{-2} + 3L_{-1}^2] \beta_{21}^3 + [4L_{-2}] \beta_3^3) \phi_h(x) \\ &= ([6(h+1)\beta_{111}^3 + 3\beta_{12}^3 + 3\beta_{21}^3] L_{-1}^2 \\ &\quad + [2(h+2)\beta_{12}^3 + 2h\beta_{21}^3 + 4\beta_3^3] L_{-2}) \phi_h(x) \end{aligned} \quad (119)$$

$$\begin{aligned} L_2\phi_h^{(3)}(x) &= L_2(L_{-1}^3\beta_{111}^3 + L_{-1}L_{-2}\beta_{12}^3 + L_{-2}L_{-1}\beta_{21}^3 + L_{-3}\beta_3^3) \phi_h(x) \\ &= ([6(3h+1)] \beta_{111}^3 + [5 + 4(h+1) + \frac{c}{2}] \beta_{12}^3 \\ &\quad + [4(h+1) + c/2] \beta_{21}^3 + [5] \beta_3) L_{-1}\phi_h(x) \end{aligned} \quad (120)$$

$$\begin{aligned} L_3\phi_h^{(3)}(x) &= L_3(L_{-1}^3\beta_{111}^3 + L_{-1}L_{-2}\beta_{12}^3 + L_{-2}L_{-1}\beta_{21}^3 + L_{-3}\beta_3^3) \phi_h(x) \\ &= (24h\beta_{111}^3 + (16h+2c)\beta_{12}^3 + 10h\beta_{21}^3 + (6h+2c)\beta_3^3) \phi_h(x) \end{aligned} \quad (121)$$

comparing the coefficients, it follows:

$$\beta_{111}^3 = \text{free} \quad (122)$$

$$\beta_{12}^3 = h(h+1)\beta_{111}^3 - \beta_3^3 - \frac{(h-3)(h-1)}{48} \quad (123)$$

$$\beta_{21}^3 = -(h+2)(h+1)\beta_{111}^3 + \beta_3^3 + \frac{(h-3)(h-1)}{48} \quad (124)$$

$$\beta_3^3 = \text{fre} \quad (125)$$

Introducing

$$I_2^3 := \frac{K_{12}^3 + K_{21}^3}{K_1^3} =! = -2(h+1) \quad (126)$$

$$I_3^3 := \frac{K_{12}^3 + K_3^3}{K_1^3} =! = h(h+1) \quad (127)$$

we get

$$K_1^3 = \frac{1}{16(h+2)} [48\beta_{111}^3(h+2)(h+3) - (h^2 + 2h - 7)] \quad (128)$$

$$K_{12}^3 = h(h+1)K_1^3 - 3\beta_3^3(h+3) \quad (129)$$

$$K_{21}^3 = -(h+2)(h+1)K_1^3 + 3(h+3)\beta_3^3 \quad (130)$$

$$K_3^3 = 3(h+3)\beta_3^3 \quad (131)$$

Thus the prefactors become:

$$H_{12}^3 = h(h+1) - \frac{48\beta_3^3(h+2)(h+3)}{48\beta_{111}^3(h+2)(h+3) - (h^2 + 2h - 7)} \quad (132)$$

$$H_{21}^3 = -(h+2)(h+1) + \frac{48\beta_3^3(h+2)(h+3)}{48\beta_{111}^3(h+2)(h+3) - (h^2 + 2h - 7)} \quad (133)$$

$$H_3^3 = \frac{48\beta_3^3(h+2)(h+3)}{48\beta_{111}^3(h+2)(h+3) - (h^2 + 2h - 7)} \quad (134)$$

which leads us obviously to

$$I_2^3 = H_{12}^3 + H_{12}^3 = -2(h+1) \quad (135)$$

$$I_3^3 = H_{12}^3 + H_3^3 = h(h+1) \quad (136)$$

hence the level two null vector condition on a primary $\phi_{(2,1)}$ field translates to a level three null vector equation on a primary $\phi_{(3,1)}$ field after fusion with another $\phi_{(2,1)}$ field.

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