Logarithmic Conformal Field Theory with Supersymmetry

Diplomarbeit

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Chapter 1

Introduction

Conformal field theories (CFTs) are the class of quantum field theories which in addition to the Poincaré group are invariant under conformal transformations. Interestingly, in two dimensions, the group of conformal transformations becomes infinite-dimensional. Since the seminal paper of Belavin, Polyakov and Zamolodchikov [1] CFT's were recognized as a very rich and exciting field of research with a wide range of applications, especially in statistical physics and string theory.

Another influential paradigm of theoretical research is supersymmetry. An ordinary transformation group acting on a representation preserves particle statistics. It is a common belief among physicists that there might be a symmetry transforming particles of different statistics among each other. One of the most significant reasons for the importance of supersymmetry is the Coleman-Mandula theorem [3], which is seen as one of the most important no-go theorems in theoretical physics. It states that the symmetry group of quantum fields in theories with a mass gap is exhausted with the Poincaré group tensored with the internal symmetry group. However, it allows supersymmetry which extends the internal symmetry by introducing a gradation of the Lie bracket. Generators of the supersymmetry group transform fermionic representations into bosonic ones and vice versa. That way, every particle has a superpartner of same mass. This is obviously not the case as far as we have observed, which indicates that if supersymmetry exists, it must be broken. As of February 2010, no experimental evidence of particle supersymmetry in nature has been found. With first particle beams being injected in the Large Hadron Collider, there is hope and excitement in the community to obtain experimental data which indicates that broken particle supersymmetry is more than a mere theoretical construct.

In theories without a mass gap, the symmetry group is limited to the conformal transformations and internal degrees of freedom. The conformal algebra, which is given by the Virasoro algebra, can be easily extended to a graded algebra by introducing supersymmetry generators. The N=1 and N=2 algebras and their representations where studied quite extensively in the late 1980s and early 1990s [4], [5], [6]. The interest somewhat faded away later, until it was realized later that the N=2 superconformal theory is more interesting than assumed, and that a lot of "structure" of N=0 and N=1 theories does not generalize to N=2. One impressive example is the existence of subsingular vectors in N=2 theory [82]. In supersymmetric conformal field theory, there is an additional gauge symmetry group for N>1. The larger N, the “bigger“ is the gauge symmetry. Super Virasoro theories with N>2 were treated rather scarcely in the literature [72], [71], but it is already obvious that their representation theory might bring surprises as well.

Meanwhile, another discovery in CFT drew a lot of attention: logarithms in correlation functions for some non-unitary representations with all their implications. Although the appearance of logarithmic divergences was noticed before, Gurarie [7] initiated a more intensive research on what became logarithmic conformal field theory (LCFT) by pointing out in that in the four-point function of $c = -2$, $h = -\frac{1}{8}$ fields of the non-unitary minimal models, the usual approach of calculating correlation functions by expanding operator products in Laurent series does not work. A Laurent series does not exist, since there are logarithmic divergences at $x = 0$, where x is the anharmonic ratio of field coordinates. He showed that this happens if there are at least two operators of the same conformal dimension that transform as a reducible, but indecomposable representation of the Virasoro algebra. In this case, the dilation operator is non-diagonalizable and has a Jordan cell structure. This feature makes logarithmic CFTs a very unique theory among field theories.

Jordan cells appear in a CFT if the fusion product of two fields contains at least two fields of the same conformal dimension or if the fields involved differ by one in their conformal dimensions. Although most of the known LCFT's are indecomposable with respect to the Virasoro algebra, other operators of the symmetry algebra, like generators of gauge symmetries can lead to logarithmic singularities as well. In general, representations of the extended Virasoro algebra where logarithms in correlation functions occur posses an infinite number of irreducible representations, although one counterexample is known [79].

LCFT was soon applied to and found in many theories some of which include WZNW models [8] [9], fractional quantum Hall effect [10], 2D turbulence [11] [12] [13] [15] , D-brane recoil [62], gravitationally dressed CFTs [16] [17], critically disordered models [18] [19], unifying W algebra [20], [21], normalizable zero-modes in string backgrounds [22] [23] and $c_{p,1}$ non-unitary minimal models [24] [25].

From technical point of view, it is obvious that non-diagonalizable representations must be treated very differently from to the diagonalizable ones. Most of the mathematical “tool set” previously applied to CFT's has to be generalized to apply to the logarithmic case, although the generalization is not always well-understood and sometimes the results require some additional interpretation. Some of the rather unusual features of LCFT include appearance of negative multiplicities in the application of the Verlinde formula or logarithms in character expressions. Some of the (with varying success) generalized methods include logarithmic null vectors [26], [27], fusion rules [29] [30] [31], character expressions [32] [33] and partition functions [34] [35]. Logarithmic stress-energy tensors and Sugawara construction are treated in [36]. Some progress was achieved using the nilpotent variable formalism to describe Jordan cell structure [38].

Despite much effort, the level of understanding of LCFT's is currently by far not as general and complete as that of “conventional“ CFT's consisting of completely reducible representations. Since Gurarie's work, the best understood LCFT's remain $c_{1,p}$ models with extended conformal grid.

Of particular interest in the context of this thesis are supersymmetric extensions of the Virasoro algebra. The literature treating LSCFT is rather sparse.

Some aspects of the $N = 1$ Neveau-Schwarz sector of logarithmic supersymmetric conformal field theory (LSCFT) were treated in [41], among them general transformation properties and logarithmic correlation functions. These findings were generalized to the Ramond sector in [43], [44]. In [45], a very general, geometric method was developed to calculate correlators for logarithmic $N = 0$, $N = 1$ and $N = 2$ theories and $N = 0$ and $N = 2$ two-point functions were calculated.
In this thesis, the Neveu-Schwarz sectors of $N = 1, 2, 3$ extended super Virasoro algebras with indecomposable representations are studied. We answer a series of questions related to the structure of LSCFT's. In section 2, the necessary background information on CFT's is presented in a very condensed form. Section 3 is an (again, very brief) overview of logarithmic CFT's and indecomposable representations in general, only in as far as it is necessary for the understanding of the presented work. Section 4 is a treatment of $N = 1, 2, 3$-extended super Virasoro theories in general and logarithmic theories in particular. The main results can be found in this part of the thesis. We find that the logarithmic $N = 1$ theories are probably just supersymmetric extensions of already known $N = 0$ models. Contrary to previous conjectures made in the literature we conclude that this fact holds for theories with $N > 1$. This implies that there are no logarithmic fields with respect to the supersymmetric current. A treatment of indecomposable su(2) representations is presented, and we find that, although constructible, this structure does not appear in $N = 3$ theories. We find the two-point functions of the $N = 3$ theories by solving the superconformal Ward identities. We find that, surprisingly, only su(2)-singlets and doublets have nontrivial correlation functions. Using this fact, we are able to obtain the general n-point function of the N=3 super Virasoro theory. Furthermore, we find no "hidden connection" between logarithmic theories and supersymmetric theories previously conjectured in the literature on the basis of apparent similarity between supersymmetric fields and logarithmic fields in the nilpotent variable formalism. The thesis is concluded with a few final remarks on our findings.

Chapter 2

Preliminaries about CFT

In this section, conformal transformations of coordinates and quantum fields are discussed. Representation theory of the Virasoro algebra is introduced. Free theories containing real bosons and fermions are considered.

2.1 Conformal Transformations

To begin the discussion of conformal field theory, consider conformal transformations of coordinates. Classically, fields are real functions on a manifold. In a quantum field theory, fields are "promoted" to operators acting on a Fock space, satisfying certain commutation relations with observables of the theory. We will restrict ourselves to flat spaces and begin naturally on a Minkowski metric of dimension $d = p + q$, with p negative and q positive eigenvalues.

General global and infinitesimal transformations of coordinates are of the form:

$$x \to f(x)$$
$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$$

with a not yet specified function $\epsilon(x)$. We are interested in special cases of general coordinate transformations. These transformations, by definition, transform the metric tensor in a way that is given in its global and infinitesimal form as:

$$g_{\mu\nu}(x) \to \Omega(x) g_{\mu\nu}$$
$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \omega(x) g_{\mu\nu}(x).$$

Thus, we multiply the metric tensor by a real function, leaving angles between vectors invariant, but not preserving lengths of vectors. Invariance of the action can be reformulated in terms of the stress-energy tensor. Field theories with a conserved ($\partial_\mu T^{\mu\nu} = 0$) and traceless ($T^\mu_\mu = 0$) stress-energy tensor are invariant under Poincaré and general conformal transformations, respectively. Invariance under conformal transformations implies that no mass terms are allowed in the Lagrangian of the theory (that is, if a Lagrangian can be formulated at all).

From transformation of the metric tensor we are able to derive constraints on $\epsilon(x)$. Since a tensor of rank 2 must transform as:

$$g_{\mu\nu}(x) \to \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x).$$

After a short calculation we arrive at an equation imposing constraints on $\epsilon(x)$ for the case that it generates a conformal transformation:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \partial_\lambda \epsilon^\lambda g_{\mu\nu}$$

from which follows that:

$$\left(1 - \frac{2}{d}\right) \partial_\rho \partial_\mu \partial_\lambda \epsilon^\lambda = 0. \tag{2.1}$$

Surprisingly, for $d = 2$ an arbitrary $\epsilon(x)$ is allowed, whereas in $d > 2$ it can be dependent on x only in at most second order, generating the whole Poincaré group and two new elements, namely scale transformations and special conformal transformations. Together, they form the conformal group.

For $d = 2$, (2.1) tells us that $\partial_1\epsilon_1 = \partial_2\epsilon_2$ and $\partial_1\epsilon_2 = -\partial_2\epsilon_1$. These equations are known as the Cauchy-Riemann differential equations and play a prominent role in complex analysis. They are automatically fulfilled by real and imaginary parts of holomorphic functions on $\mathbb{C}$. Complexifying ϵ and the coordinates leads to:

$$\begin{aligned} \epsilon &= \epsilon_1 - i\epsilon_2 & \bar{\epsilon} &= \epsilon_1 + i\epsilon_2 \\ z &= x^1 - ix^2 & \bar{z} &= x^1 + ix^2 \\ \partial_z \bar{\epsilon}(z, \bar{z}) &= 0 & \partial_{\bar{z}} \epsilon(z, \bar{z}) &= 0. \end{aligned}$$

That way, $\epsilon(z, \bar{z})$ and $\bar{\epsilon}(z, \bar{z})$ are arbitrary functions of z and $\bar{z}$, respectively, so we may as well write $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$.

To formulate a transformation law on the space of quantum fields one follows the standard approach of deriving a symmetry algebra by studying infinitesimal generators of continuous symmetries. The infinitesimal transformation is represented by some operator L_n, which is a Noether charge. We expect charges to generate conformal transformations on the space of functions:

$$[L_n, \phi(x)] = \delta\phi(x).$$

If we transform a coordinate via:

$$z \to z^{'} = z - z^{n+1},$$

then the generator of the corresponding conformal transformation in complex coordinates on the space of functions reads:

$$L_n = -z^{n+1} \partial_z.$$

We are interested in the $d = 2$ case, since the infinite dimensional symmetry algebra leads to a much greater solvability of a theory, imposing additional constraints on fields and their correlation functions. Luckily, we can map the underlying two-dimensional space from the cylinder to the complex plane and use methods developed in complex analysis to simplify calculations significantly. First, the space coordinate x^1 is compactified by imposing a periodic boundary condition on functions living on this so called worldsheet parametrized by (x^0, x^1). Compactification removes the eventual problem of infrared divergences since now we have an upper limit of the “wavelength” corresponding to a field mode. Now, we perform a Wick rotation of the time coordinate $x^2 = ix^0$. At last, we map the obtained cylinder coordinates on the complex plane via exponentiation:

$$w = e^{i(x^0+x^1)}. \tag{2.2}$$

The conformal current in complex coordinates equals $T(z)\epsilon(z)$. Expanding $\epsilon(z)$ in modes we obtain the charges:

$$L_n = \frac{1}{2\pi i}\oint dz z^{n+1} T(z). \tag{2.3}$$

Inverted, this relation reads:

$$T(z) = \sum_n z^{-n-2} L_n.$$

One has to keep in mind that the fields $\phi(x)$ are classically real. Our description on the complex plane is somewhat redundant. There are two copies of generators: one holomorphic L_n, and one anti-holomorphic $\bar{L}_n$. They correspond to the left-moving (chiral) and right-moving (anti-chiral) fields on the cylinder. There are two commuting copies of the Virasoro algebra, each acting on a two-dimensional space. The Hilbert space is a tensor product of two copies of Virasoro representations:

$$\mathcal{H} = Vir \otimes \overline{Vir}. \tag{2.4}$$

The dimensional redundancy seems unnecessary but it simplifies calculations. The reality conditions have to be imposed in the end when one is calculating measurable quantities.
Calculating $[L_n, \phi]$ at first seems problematic since we have to calculate an operator product at the same coordinate point, which corresponds to an equal-time operator product. The solution to this is to take different points z and w and expand the operator product for $z \to w$.

In ordinary quantum field theory, we introduce a time ordering of the operators, since the Hamiltonian of a theory can only be bounded from below if the operator on the right is taken at a later time then the one on the left. On the complex plane, the analogous ordering is a radial one, since points of "equal time" lie on circles around the origin:

$$R\left(A(z,\bar{z})B(w,\bar{w})\right) := A(z,\bar{z})B(w,\bar{w})\Theta(|z|-|w|) + B(w,\bar{w})A(z,\bar{z})\Theta(|w|-|z|)$$

were Θ is the Heaviside step function. Using radial ordering, it can be seen that in the commutator $[L_n, \phi(w,\bar{w})]$, the integral contours are two circles, one with a radius smaller and one with a radius greater than $|w|$:

$$[L_n, \phi(w,\bar{w})] = \frac{1}{2\pi i}\left(\oint_{|z|>|w|} - \oint_{|z|<|w|}\right) dz z^{n+1} R\left(T(z)\phi(w,\bar{w})\right).$$

We can simply deform the contours to one curve running around w with an arbitrarily small radius. Keeping that in mind, we write:

$$[L_n, \phi(w,\bar{w})] = \frac{1}{2\pi i}\oint dz z^{n+1} R\left(T(z)\phi(w,\bar{w})\right).$$

This integral is solvable if we know the radially ordered operator product expansion of the stress-energy tensor with the field. One has to find possible constraints on the OPE.

A key feature of conformal field theory (CFT) is the conformal bootstrap, an idea which was first expressed in the pioneering work of Belavin, Polyakov and Zamolodchikov [1]. In particular, this means that we do not rely on a Lagrangian or Hamiltonian formulation as a starting point of the theory, but try to solve all correlation functions using symmetries and associativity of operators. The main assumption is the idea that we can always express an operator product as a linear combination of local operators:

$$\phi_i(x)\phi_j(y) = \sum_k C_{ij}^k(x-y)\phi_k(y) \tag{2.5}$$

where C_{ij}^k are complex-valued coefficients. The exact solvability is only possible if there are finitely many fields ϕ^k. This makes the so-called rational conformal field theories (RCFTs) particularly "solvable", since they are the ones with finitely many representations. A field, defined to be a primary field, transforms as:

$$w \to f(w)$$
$$\phi(w,\bar{w}) \to \phi'(w,\bar{w}) = \left(\frac{\partial f(w)}{\partial w}\right)^h \phi(f(w),\bar{w}).$$

The primary fields are the starting points of the theory. They correspond to "lowest-energy"-representations which are eigenstates to the dilatation operator L_0 with eigenvalues h. It will be shown that the remaining states of the theory, the so-called descendant states can be generated from the highest-weight vectors $|h\rangle$. It is often preferable to work with the state formalism since infinitesimal transformation properties of the descendants are given by complicated formulae.

Infinitisemally, the transformation of the primary fields amounts to:

$$\delta_\epsilon\phi(w,\bar{w}) = h\left(\partial_w\epsilon(w)\right)\phi(w,\bar{w}) + \epsilon(w)\partial_w\phi(w,\bar{w}).$$

This must equal the commutator of an arbitrary conformal charge with the field:

$$[L_n,\phi(z)] = (z^{n+1}\partial_z + h(n+1)z^n)\phi(z). \tag{2.6}$$

In [72] it was shown that the infinitesimal transformations of the primary field integrate to global transformations:

$$e^{\lambda L_n}\phi(z)e^{-\lambda L_n} = e^{\lambda(z^{n+1}\partial_z + h(n+1)z^n)}\phi(z). \tag{2.7}$$

Which amounts to:

$$e^{\lambda L_n}\phi(z)e^{-\lambda L_n} = \frac{1}{(1-n\lambda z^n)^{\frac{h(n+1)}{n}}}\phi\left(\frac{z}{(1-n\lambda z^n)^{\frac{1}{n}}}\right) \qquad n \neq 0.$$
$$e^{\lambda L_0}\phi(z)e^{-\lambda L_0} = e^{\lambda h}\phi\left(e^{\lambda h}z\right).$$

From (2.6), we can recover the operator product expansion (OPE) of $R(T(z)\phi(w,\bar{w}))$ (omitting the radial ordering symbol from now on for convenience):

$$T(z)\phi(w,\bar{w}) = \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) + \text{power series}. \tag{2.8}$$

We observe that the Laurent series terminates after the second term. There are only two interesting terms, since the Taylor part of the expansion disappears under closed integrals around w and will be omitted without further comment.

2.2 Free Bosonic Theory

To see how this is supposed to work we start from a simple toy model: N free real bosons on the cylinder. In this theory the Lagrangian (density) is well-known:

$$S = \frac{1}{8\pi} \int d^2x \sum_i \partial_\alpha \phi^i(x_0, x_1) \partial^\alpha \phi_i(x_0, x_1).$$

By minimization of the action we obtain the classical equation of motion:

$$(\partial_0^2 - \partial_1^2)\phi^i(x^0, x^1).$$

It is obvious that the fields transform as functions, that is they are eigenfunctions to the scaling operator L_0 with scaling dimension $h = 1$. The compactification of the "space" direction imposes periodic boundary conditions on the space of fields on the cylinder:

$$\phi^i(x^0, 0) = \phi^i(x^0, 2\pi).$$

Periodicity in turn implies that a Fourier series exists for these fields:

$$\phi^i(x^0, x^1) = \sum_{n=-\infty}^{n=\infty} e^{inx^1} f_n^i(x^0).$$

Since every mode satisfies the equation of motion, we have:

$$\begin{aligned} \partial_0^2 f_n^i(x^0) &= -n^2 f_n^i(x^0) \\ f_n^i(x^0) &= a_n^i e^{inx^0} + b_n^i e^{-inx^0} \quad n \neq 0 \\ f_0^i(x^0) &= p^i x^0 + q^i. \end{aligned} \tag{2.9}$$

Then the classical real field can be written, recombining mode coefficients, as:

$$\phi^i(x^0, x^1) = q^i + 2p^i x^0 + i \sum_{n=1}^{\infty} \left(\frac{1}{n} (\alpha_n^i e^{-in(x^0+x^1)} + \tilde{a}_n^i e^{-in(x^0-x^1)} \right).$$

The usual procedure, by which we turn the classical field into a quantum field and which is often called the "second quantization" is straightforward. We promote the mode coefficients (which we then call modes) to operators by imposing following equal-time commutation relations between fields and their canonical momenta $\pi = \partial_0 \phi^i / 4\pi$:

$$\begin{aligned} [\phi^i(x^0, x^1), \pi^j(x^0, y^1)] &= i\delta^{ij}\delta(x^1 - y^1) \\ [\phi^i(x^0, x^1), \phi^j(x^0, y^1)] &= 0 \\ [\pi^i(x^0, x^1), \pi^j(x^0, y^1)] &= 0. \end{aligned}$$

The same commutation relations expressed in terms of modes are:

$$[\alpha_k^i, \alpha_l^j] = [\tilde{\alpha}_k^i, \tilde{\alpha}_l^j] = k\delta^{ij}\delta_{k+l,0}$$

$$[\alpha_k^i, \tilde{\alpha}_l^j] = 0.$$

Transformation to the conformal plane via (2.2) yields:

$$\phi^i(z,\bar{z}) = q^i - i(p^i \log(z) + p^i \log(\bar{z})) + i\sum_{n\neq 0} \frac{1}{n}\left(\alpha_n^i z^{-n} + \tilde{\alpha}_n^i \bar{z}^{-n}\right).$$

The somewhat unusual zero mode leads to a field which can not be factorized into holomorphic and anti-holomorphic parts and logarithmic singularities (however, this is not the case for its derivatives $\partial_z\phi^i$ and $\partial_{\bar{z}}\phi^i$). Fortunately, we can deal with this by subtracting an infinite constant from the stress-energy tensor of the theory, which turns the quantized field into an infinite set of harmonic oscillators.

The holomorphic part of the energy-momentum tensor is classically the Legendre transform of the Langrangian:

$$T(z) = -\frac{1}{2}\sum_i \partial_z\phi^i(z)\partial_z\phi^i(z).$$

To write a quantum version of this equation, one uses the standard technique of normal ordering which precisely amounts to radial ordering of an operator product. Normal ordering amounts to moving all positive modes, which annihilate the vacuum:

$$\alpha_k^i|0\rangle = 0, \quad k > 0$$

to the right. That way, the expectation value of the normally ordered product vanishes but the terms containing the commutators which appear in the process of normal ordering remain. We also commute p^i to the right with respect to q^i. The radial ordering, written in terms of harmonic oscillators reads:

$$R(\phi^i(z,\bar{z})\phi^j(w,\bar{w})) =: \phi^i(z,\bar{z})\phi^j(w,\bar{w}) :=$$

$$-i[p^i, q^j](\log(z) + \log(\bar{z})) + \text{power series}(\bar{z}, \bar{w}).$$

Which, inserting commutation relations and using the Taylor expansion of the logarithm amounts to:

$$R(\phi^i(z,\bar{z})\phi^j(w,\bar{w})) =: \phi^i(z,\bar{z})\phi^j(w,\bar{w}) : -\delta^{ij}(\log(z-w) + \log(\bar{z}-\bar{w})).$$

Since the radial ordering implies $z < w$, the Taylor series is convergent and the above expression well-defined. The product of derivatives yields:

$$R(\partial_z\phi^i(z,\bar{z})\partial_w\phi^j(w,\bar{w})) =: \partial_z\phi^i(z,\bar{z})\partial_w\phi^i(w,\bar{w}) : -\frac{\delta^{ij}}{(z-w)^2}.$$

Omitting the term containing the normal ordering, the radial ordering sign and the arguments of the derivatives, we are left with:

$$\partial\phi^i(z)\partial\phi^j(w) = -\frac{\delta^{ij}}{(z-w)^2}.$$

Now we have the term which has to be subtracted in the definition of the non-singular quantum version of the energy-momentum tensor:

$$T(z) \equiv -\frac{1}{2}\sum_i \left(: \partial\phi^i(z)\partial\phi^i(z) : -\frac{\delta^{ij}}{(z-w)^2} \right).$$

Using this definition, one can easily derive the operator product expansion of $T(z)$ with $\partial\phi^i(w)$. We just have to expand the latter around z, which is the usual Taylor series. We end up with:

$$T(z)\partial\phi^j(w) = \frac{\partial\phi^i(w)}{(z-w)^2} + \frac{\partial^2\phi^j(w)}{z-w}.$$

With some more effort, the operator product expansion between the stress-energy tensor with itself can be calculated:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w).$$

The first term in this expansion prevents $T(w)$ from being a conformal field. In CFT literature, this term is called the conformal anomaly. It contains a number, c. In a free bosonic theory, c is the number of boson fields contained in the theory, therefore it has to be a natural number. However, we consider a larger set of possible theories and assume c to be any real number, even if we can not write down Lagrangians of the theory for most of our c. The quantity c is called the central charge. At first glance at the bosonic theory where the conformal weight is the tensor rank of the fields and c the number of bosons in the Lagrangian, taking c and h to be non-integer numbers seems somewhat artificial, but it was shown in a multitude of cases that theories with non-integer c and h have a wide range of applications, instead of being mere mathematical curiosities.

2.3 The Virasoro Algebra

Since now we have computed the OPE of the stress-energy tensor with itself and we know that its modes act on the space of fields as (2.3), we are now able to compute the commutation relation between modes L_n which yield the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{n,-m}. \tag{2.10}$$

Without the last term we would just get the classical Witt algebra, which is a Lie algebra:

$$[l_n, l_m] = (n-m)l_{m+n}. \tag{2.11}$$

Thus, the Virasoro algebra is a Lie algebra with a central extension. The term containing c seems to be highly unusual. If c is just a real number, it can not appear on the right-hand side of commutator. Algebraically, we are forced to treat c as an operator commuting with every other element of the Virasoro algebra. The chiral sector of a CFT is just a collection of representations of the Virasoro algebra, spanned by $\mathbb{I}, L_n$, with $\mathbb{I}$ being the central element. A representation is characterized by a pair of numbers (h, c). For practical reasons, we usually obtain states of the CFT using the so-called highest weight representations, which are, in fact, ground states with respect to the Hamiltonian and therefore lowest energy states. They are found by diagonalizing L_0:

$$L_0|h\rangle = h|h\rangle.$$

The highest-weight state can be created from the vacuum by acting on it with a corresponding primary field at $z = 0$:

$$|h\rangle = \lim_{z\to 0} \phi_h(z)|0\rangle.$$

We can easily derive the fact that any highest-weight state $|h\rangle$ is annihilated by any L_n with $n > 0$ directly from the Virasoro algebra. The vacuum state $|0\rangle$ is somewhat unusual compared to ordinary QFT, since it is annihilated only by Virasoro modes L_n with $n \geq -1$. This means that it is only possible to have a state which is annihilated by a maximal number of Virasoro modes, which is defined to be the vacuum, but not all of them. Acting on the vacuum or a highest-weight state with a negative mode $L_{-n}, n > 0$ creates a new state, a descendant with conformal weight $h - n$:

$$L_0 L_n|h\rangle = (-nL_{-n} + L_{-n}L_0)|h\rangle = (h - n)L_{-n}|h\rangle, n > 0.$$

Acting on a descendant again produces a new state. Acting on a highest weight state with a composition of modes produces a state at level n, the descendant of $|h\rangle$:

$$L_{-n_i}...L_{-n_k}|h\rangle, \qquad \sum_i n_i = n.$$

The space spanned by this states is called a Verma module $V_{h,c}$. The Virasoro algebra can be applied to rearrange the indices n_i in decreasing order. A compact definition of a Verma module is [61]:

$$V_h = \mathrm{span}\{\prod_{i\in I} L_{-n_i}|h\rangle : \mathbb{N} \supset I = \{n_1, ...n_k\}, n_{i+1} \geq n_i\}. \tag{2.12}$$

The space of states built by applying negative-moded Virasoro generators on a highest-weight space is called a conformal family. It might happen that in a Verma module on a certain level there is a linear combination of states, which is again a state $|\chi\rangle$ with the property that it is orthogonal to all other states in the theory. Acting with positively moded Virasoro generators on such a state, we arrive at a singular vector $|\chi_s\rangle$, a special null state. If one considers χ_s to be a "primary field", then its Verma module consists of null vectors only. We set $\chi_0 \equiv 0$. What is now meant by Verma module is (2.12) with singular vectors removed, or expressed in a more mathematical way, the quotient space of (2.12) by the subspace generated from singular vectors. Now the Hilbert space picture (2.4) can be refined since it decomposes in a direct sum over the Verma modules built on highest-weight states:

$$\mathcal{H} = \oplus_{h,\bar{h}} V_h \otimes V_{\bar{h}}.$$

A conformal family containing singular vectors is called degenerate. We also call the primary states which contain null vectors in their respective Verma modules degenerate. A natural question arising at some point is the classification of CFT's. This is a very difficult question, and it is doubtful if all CFT's can be classified at all. First, we turn our attention to unitary theories, where the scalar product between two states is positive-definite and the Virasoro generators satisfy the hermiticity condition $L_n^\dagger = L_{-n}$. The existence of a non-negative norm imposes strong constraints on the possible values of c and h. The problem is that given a highest weight vector $|h\rangle$ the number of excitations grows with each level n, since one can decompose n as a sum of integers in various ways. Consider the first few levels (with $p(n)$ the number of possible partitions of n):

n Level	p(n)	States
0	0	$\|h\rangle$
1	1	$L_{-1}\|h\rangle$
2	2	$L_{-1}^2\|h\rangle$, $L_{-2}\|h\rangle$
3	3	$L_{-1}^3\|h\rangle$, $L_{-2}L_{-1}\|h\rangle$, $L_{-3}\|h\rangle$
4	5	$L_{-1}^4\|h\rangle$, $L_{-1}^2L_{-2}\|h\rangle$, $L_{-2}^2\|h\rangle$, $L_{-3}L_{-1}\|h\rangle$, $L_{-4}\|h\rangle$.

For the next levels, $p(n)$ is 7, 11, 15, 22, 30, 42 and so on. At level 1 the norm of the vector is:

$$\langle h|L_1 L_{-1}|h\rangle = \langle h|L_{-1}L_1 + 2L_0|h\rangle = 2h.$$

That means that for unitary representations, and $|h\rangle \neq |0\rangle$, h must satisfy $h > 0$. For $|h\rangle = |0\rangle$ the first descendant is already a null vector. The condition $c \geq 0$ follows immediately from the action of a negative mode on $|0\rangle$:

$$\langle 0|L_n L_{-n}|0\rangle = \langle 0|\frac{1}{12}c(n^3 - n)|0\rangle.$$

We continue our analysis at level two. Since there are two excitations on this level one has to consider all possible linear combinations in the two-dimensional state space. To do that, we introduce the matrix K_n of dimension $p(n) \times p(n)$ with scalar products of the possible vectors at level n. At level two, this matrix reads:

$$K_2 = \begin{pmatrix} \langle h|L_{-2}^\dagger L_{-2}|h\rangle & \langle h|L_{-2}^\dagger L_{-1}^2|h\rangle \\ \langle h|(L_{-1}^\dagger)^2 L_{-2}|h\rangle & \langle h(L_{-1}^\dagger)^2 L_{-1}^2|h\rangle \end{pmatrix}.$$

This matrix acts on the two-dimensional coefficient space. Using the Virasoro algebra and $\langle h|h\rangle = 1$ one can express the components of K_n as polynomials in c and h:

$$K_2 = \begin{pmatrix} 4h + \frac{1}{2}c & 6h \\ 6h & 4h + 8h^2 \end{pmatrix}.$$

In [64], [65] it was shown that for a CFT to be rational, that means to have a finite number of representations, the values of c and h have to be rational.

For $0 < c < 1$, $h > 0$, there exists a family of rational unitary conformal field theories, the unitary minimal models. They are called minimal because they have a finite number of primary fields, are invariant with respect to conformal transformations only and, additionally, are non-degenerate in their conformal dimensions [2]. They are obtained by imposing non-negativity conditions on scalar products between states in a Verma module.

For minimal models, the possible values of c are parametrized by an integer $m \geq 3$, the possible values of the conformal weight h are parametrized by integers r and s with limited range:

$$c = 1 - \frac{6}{m(m+1)} \qquad m \geq 3 \tag{2.13}$$

$$h_{r,s}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)} \qquad 1 \leq r < m,\ 1 \leq s \leq r.$$

This series can be generalized to the series of (not necessarily unitary) minimal models [53]:

$$c_{(p,q)} = 1 - \frac{6(p-q)^2}{pq}$$

$$h_{(r,s)} = \frac{(pr - qs)^2 - (p-q)^2}{4pq} \qquad 0 < r < q,\ 0 < s < p,$$

with p, q coprime positive integers. which means they do not have non-trivial common divisors. To recover (2.13), simply set $m = p - 1 = q$.

Until now, we were discussing conformal invariance generated by elements of the Virasoro algebra. The set of symmetry generators can be extended, demanding additional invariance. The commutation relations between the Virasoro algebra and the additional generators is fixed in that case. This can be seen as follows. Assume we have a conformal field $J(z)$ with conformal weight h. We can regard $J(z)$ as a current with modes and mode expansion defined to be:

$$J_r = \frac{1}{2\pi i} \oint dz z^{r+h-1} J(z)$$

$$J(z) = \sum_r z^{-r-h} J_r.$$

We know the action of the Virasoro modes on the current from the fact that the current is a conformal field:

$$[L_n, J(z)] = (z^{n+1}\partial + h(n+1)z^n)J(z).$$

Expanding $J(z)$ in modes and dividing by z^{-h} yields:

$$\sum_r z^{-r}[L_n, J_r] = \sum_r z^{n-r}(-r + hn)J_r.$$

By comparing the coefficients of powers of z we recognize that r has to be shifted by n on the right-hand side, leaving us with the commutator:

$$[L_n, J_r] = (n(h-1) - r)J_{n+r}.$$

If we expect the fields in the theory to be invariant with respect to the generators $\{L_n, J_r\}$, we get the simplest example of an extension of the conformal group, although we have not specified the commutation relations between the mode currents J_r yet.
In this thesis, we will be dealing with representations of extended algebras, namely the N=0, 1, 2, 3 superconformal algebras.

2.4 Free Fermions

In ordinary quantum field theory, the fermionic fields are implemented in the Feynman path integral formalism via Grassmannian (anticommuting) variables. In the quantized theory, this leads to anticommuting modes. This mathematical trick emulates the Pauli exclusion principle: one cannot create two fermionic field modes corresponding to the same state from the vacuum. It is desirable to be able to describe fermions in conformal field theory as well, since there is no reason to assume that conformally invariant fields have to be bosonic.
In Lagrangian formalism, the action of free fermions in two dimensions on the complex plane is:

$$S = \frac{1}{8\pi}\int d^2x(\psi\partial_{\bar{z}}\psi + \bar{\psi}\partial_z\bar{\psi})$$

which leads to equations of motion

$$\partial_{\bar{z}}\psi(z,\bar{z}) = 0, \qquad \partial_z\bar{\psi}(z,\bar{z}) = 0,$$

which in turn implies that the fermionic fields can be factorized into a holomorphic and an anti-holomorphic part. If periodic boundary conditions are imposed on the space of fields, the representations span the *Ramond* sector. In contrast to the bosonic theory, the boundary conditions are allowed to be anti-periodic, with representations spanning the *Neveu-Schwarz* sector. Going through the quantization procedure, we end up with the following mode expansion:

$$\psi(z) = \sum_n b_n z^{-n-\frac{1}{2}}.$$

Here b_n are anticommuting and the n are half-integer numbers for the Neveau-Schwarz and integer numbers for the Ramond sector. It is interesting to see that periodicity changes if a transformation to the cylinder is applied, so purely non-periodic or periodic boundary conditions are not possible for fermions on both the cylinder and the plane. The energy-momentum tensor is:

$$T(z) = -\frac{1}{2} : \psi(z)\partial_z\psi(z) :,$$

from which, taking the operator product expansion with itself and $\psi(w)$ we conclude that both the central charge and the conformal weight equal $\frac{1}{2}$. In this thesis, we are interested in the Neveau-Schwarz sector. The propagator in this case is obviously antisymmetric under exchange of z and w and reads:

$$\psi(z)\psi(w) = \frac{1}{z-w}.$$

It is remarkable that for $c = \frac{1}{2}$ the representations contained in the minimal models (2.13) are the ones with $h \in \{0, \frac{1}{16}, \frac{1}{2}\}$. The representations with $h = \frac{1}{16}$ are spin fields, which appear in the case of aperiodic boundary conditions of fermionic representations.

2.5 Conformal Ward Identities

The measurable quantities in a QFT are the correlation functions, also called the n-point functions:

$$F_n(z_1, z_2, ...z_n) \equiv \langle\phi_1(z_1)\phi_2(z_2)...\phi_n(z_n)\rangle$$

of some fields $\phi_i(z_i)$. The correlation function of primary fields can be obtained using the conformal Ward identities. We demand that the n-point function is invariant under variation corresponding to the three global conformal transformations. Then we can write the variation of the function in terms of variations of fields:

$$\delta F_n = \sum_{i=1}^{n} \langle \phi_1(z_1)...\delta_i\phi_i(z_i)...\phi_n \rangle.$$

Since we know the variation of primary fields (2.6), the three differential equations obtained are:

$$\begin{aligned} \textstyle\sum_i \partial_i F_n &= 0 \\ \textstyle\sum_i (h_i + z_i\partial_i) F_n &= 0 \\ \textstyle\sum_i (2hz_i + z^2\partial_i) F_n &= 0. \end{aligned} \tag{2.14}$$

This can be done using the invariance of the vacuum with respect to $SL(2,\mathbb{C})$, generated by $\{L_{-1}, L_0, L_1\}$. The conformal transformations alone do not give us any further restrictions on n-point functions. The second equation is redundant, since all functions satisfying the first and third equation satisfy the second one. This a consequence of $[L_1, L_{-1}] = 2L_0$.

It is obvious that the one-point functions (vacuum expectation values of a field) disappear in ordinary CFT.

The two-point functions of two primary fields in the holomorphic sector are restricted by:

$$(\partial_{z_1} + \partial_{z_2}) F_2(z_1, z_2) = 0.$$

Changing coordinates to $w = z_1 - z_2$, $v = z_1 + z_2$ leads to:

$$\partial_v F_2(w, v) = 0 \Rightarrow F_2 = F_2(w) = F_2(z_1 - z_2).$$

In the new coordinate, the second conformal Ward identity reads:

$$(h_1 + h_2 + w\partial_w) F(w) = 0.$$

Using the ansatz w^a we obtain the correlator:

$$\langle \phi_{h_i}(z_i)\phi_{h_j}(z_j) \rangle = \frac{\delta_{ij}}{(z_i - z_j)^{2h_i}}. \tag{2.15}$$

For the three-point function find the solution is:

$$\langle \phi_{h_i}(z_i)\phi_{h_j}(z_j)\phi_{h_k}(z_k) \rangle = C_{ijk} z_{ij}^{h_k - h_i - h_j} z_{ik}^{h_j - h_i - h_k} z_{jk}^{h_i - h_j - h_k}. \tag{2.16}$$

The three-point functions are fixed up to structure constants C_{ijk}. Here, one of the z_{ij} is a redundant variable. It is included here to give the solution a symmetric form.

Using $SL(2,\mathbb{C})$, the four-point function can be determined only up to an arbitrary function of two Möbius-invariant crossing ratios $F(x_1, x_2)$. Usually, the four-point correlator is written in the symmetric form:

$$\langle \phi_{h_1}(z_1)\phi_{h_2}(z_2)\phi_{h_3}(z_3)\phi_{h_4}(z_4) \rangle = \prod_{i>j} z_{ij}^{\frac{1}{3}h - h_i - h_j} F(x_1, x_2) \tag{2.17}$$

$$h \equiv \sum_k h_k, \qquad x_1 \equiv \frac{z_{12}z_{34}}{z_{13}z_{24}}, \qquad \frac{z_{12}z_{34}}{z_{14}z_{23}}.$$

Chapter 3

Logarithmic CFT

3.1 Logarithmic Divergencies and Jordan Cell Structure

Existence of logarithmic divergencies in correlation functions got their first proper treatment by Gurarie in [7] for the $c = -2$, $h = -\frac{1}{8}$ model. Logarithmic conformal theories have a variety of interesting properties, the most obvious and far-reaching one is the presence of indecomposable irreducible representations. The L_0-mode of the stress-energy tensor turns out to have Jordan block structure. This feature leads to the necessity of generalization of the tool set (characters, null vectors, partition functions) used in the framework of the ordinary conformal field theotry in which the Hamiltonian $L_0 - \overline{L}_0$ can be diagonalized on the space of states [62], [7].

The appearance of indecomposable representations is due to unavoidable logarithms appearing in some four-point functions. Given any two conformal fields ν_i, ν_j (which are not necessarily primary), it is assumed that the conformal bootstrap is possible and leads to the operator product expansion:

$$\nu_i(z)\nu_j(w) = \sum_n \frac{C_n}{(z-w)^{h_j+h_j-h_n}}\nu_n(w). \tag{3.1}$$

We will see that by making this assumption we are missing a (possibly) large class of conformal field theories.
Assuming there are two fields in the theory with the same conformal dimension, their two-point function is given by:

$$\langle \mu(z)\mu(0)\rangle = z^{-2h}.$$

Let us follow Gurarie and consider the $c = -2$, $h = -\frac{1}{8}$ model. Due to the action of the group $SL(2,\mathbb{C})$, the four-point function of the field μ is given, up to a function $F(x)$ of one anharmonic ratio:

$$\langle \mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4)\rangle = ((z_1-z_2)(z_3-z_4))^{\frac{1}{2}}F(x).$$

Equation (3.1) inserted in the four-point function should give us the Laurent series expansion of $F(x)$. This approach does not work here. The function $F(x)$

is more general then an OPE of this form can describe. This can be seen by using the second-level null vector of the $h = -\frac{1}{8}$, $c = -2$ theory, which leads to the differential equation:

$$x(1-x)\frac{d^2}{dx^2}F(x) + (1-2x)\frac{d}{dx}F(x) - \frac{1}{4}F(x) = 0,$$

which is a special case of the hypergeometric equation:

$$x(1-x)\frac{d^2}{dx^2}F(x) + (d-(a+b+1)x)\frac{d}{dx}F(x) - abF(x) = 0. \tag{3.2}$$

The hypergeometric equation is known to have two linearly independent solutions which are given by the hypergeometric function:

$$F_1 = F(a, b, d; x) \tag{3.3}$$

$$F_2 = x^{1-d}F(a-d+1, b-d+1, 2-d; x).$$

This corresponds to the operator product expansion of the fields $\mu(z)$:

$$\mu(z)\mu(0) = \frac{1}{z^{2h}}(\mathbb{I} + z^{d-1}\mathbb{I}').$$

As we will see, logarithms emerge iff $\mathbb{I}$ has the same conformal dimension as $\mathbb{I}$ (then $d = 1$) or $\mathbb{I}'$ has the same conformal dimension as one of the descendants of $\mathbb{I}$. An ordinary second-order differential equation can be solved by applying the Frobenius method. It involves an ansatz of the form:

$$x^\alpha \sum_n a_n x^n,$$

and solving the resulting equation for α putting $n = 0$. This is known as the indicial equation, which for the hypergeometric function reads:

$$\alpha(\alpha - 1 + d) = 0. \tag{3.4}$$

Usually, one obtains the coefficients a_n one after another. In the case that the solutions to the indicial equation differ by an integer, we get terms proportional to $x^n \ln x$ in the second solution to the differential equation. In he case $c = -2$, $h = -\frac{1}{8}$, d equals one and both α_i are the same. The solutions to the hypergeometric equation then read [83]:

$$F(x) = F(\frac{1}{2}, \frac{1}{2}, 1; x)$$

$$F(x-1) = \log(x)F(\frac{1}{2}, \frac{1}{2}, 1; x) + H(x).$$

Here, $F(x)$ the hypergeometric function of second order and $H(x)$ some logarithm-free function. Both of the solutions have logarithmic singularities for some x, therefore one cannot simply discard one of them as unphysical. This is a direct contradiction to (3.1). If $d = 1$ in (3.4) in the case of a Jordan cell of dimension two the properties of the operator $\mathbb{I}'$ can be summarized as:

$$L_0|\phi\rangle = h|\phi\rangle$$

$$L_0|\phi'\rangle = h|\phi'\rangle + |\phi\rangle. \tag{3.5}$$

The action of the Virasoro modes on the corresponding fields is:

$$[L_n, \phi] = (z^{n+1}\partial_z + h(n+1)z^n)\phi$$
$$[L_n, \phi'] = (z^{n+1}\partial_z + h(n+1)z^n)\phi' + (n+1)z^n\phi.$$

Here, ϕ is an ordinary primary field and ϕ' is its logarithmic partner. It is noteworthy that the second equation can be obtained from the first one by a formal derivation with respect to h and the identification $\frac{\partial\phi}{\partial h} \to \phi'$. This will be referred to as the "derivation trick" throughout the thesis.

In the case that the solutions of the indicial equation differ by an integer, the OPE of two fields becomes logarithmic as well. The Jordan structure of L_0 in this case is quite different. The solutions of the indicial equation provide the dimensions of the fields appearing in the OPE of two fields, which means that $\mathbb{I}'$ has an integer conformal dimension. In (3.2), that means that $d = m+1$ for $m \in \mathbb{Z}$. If $m > 0$ then $\mathbb{I}'$ has a negative conformal dimension $-m$ and one of its descendants on level m degenerates with $\mathbb{I}$. For negative m the conformal dimension of $\mathbb{I}'$ is positive and the degeneration occurs with one of the descendants of $\mathbb{I}$. As an example, in the $c = -2$, $h = 1$ theory with $m < 0$ the action of the Virasoro modes on states is [61]:

$$\begin{aligned} L_0|\phi\rangle &= |\phi\rangle \\ L_0|\phi'\rangle &= |\phi'\rangle + |\phi\rangle \\ L_1|\phi'\rangle &= |\xi\rangle. \end{aligned} \tag{3.6}$$

If $|\phi'\rangle$ is annihilated by all positive Virasoro modes, we call this state quasi-primary. Calculations involving fields where the corresponding states not quasi-primary are more intricated and were treated in the literature to a much lesser extent.

In the most general case, the differential equation satisfied by $F(x)$ is a higher-order Fuchsian differential equation. Most of the findings obtained from the $c = -2$ theory can be generalized in a straightforward way. We differentiate between the two cases of Jordan block structure of different dimensions and designate them by a pair of integers (m, n). Here, $m = d - 1$ with d from (3.3) and n the dimension of Jordan block.

In the first case, one has $m = 0, n \in \mathbb{N}$. All logarithmic partners of primary fields are quasi-primary. The most general behaviour of the function $F(x)$ near $x = 0$ obtained from a higher-order Fuchsian differential equation with coinciding roots of the indicial equation is:

$$F(x) = \sum_{n=0}^{\infty} a_n x^n + \log(x)\sum_{n=0}^{\infty} b_n x^n + \log(x)^2 \sum_{n=0}^{\infty} c_n x^n + \dots. \tag{3.7}$$

In this case, the Jordan cell contained in the operator L_0 is more then two-dimensional. The transformation properties of logarithmic fields can be obtained from following simple arguments. Considering the action of L_0 on a two-dimensional logarithmic state (3.5), infinitesimal transformations of fields are fixed by:

$$[L_n, \phi(z)] = z^{n+1}\partial_z\phi(z) + (n+1)^n h\phi(z) \tag{3.8}$$

$$[L_n, \phi'(z)] = z^{n+1}\partial_z\phi'(z) + (n+1)z^n h\phi'(z) + (n+1)^n\phi(z). \tag{3.9}$$

In the case that (3.7) does not terminate after the second term, one obtains a higher-dimensional (n-dimensional) Jordan block. We write:

$$\phi = \begin{pmatrix} \phi_0 \\ \vdots \\ \phi_n \end{pmatrix}$$

$$I = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

Then the action of $L_0 = \mathbb{I}h + I$ is given by:

$$[L_n, \phi(z)] = z^{n+1}\partial_z\phi(z) + (n+1)z^n h\phi(z) + I(n+1)^n\phi(z). \tag{3.10}$$

Same as for higher-dimensional Jordan cells, one can indeed regard h in (2.6) as a matrix acting on a column vector with field components. The infinitesimal transformations for a two-dimensional Jordan cell integrate up to:

$$\phi(z) \to \left(\frac{\partial f^{-1}}{\partial z}\right)^h \phi\left(f^{-1}(z)\right)$$

$$\phi'(z) \to \left(\frac{\partial f^{-1}}{\partial z}\right)^h \left(\phi'\left(f^{-1}(z)\right) + \log\left(\frac{\partial f^{-1}(z)}{\partial z}\right)\phi\left(f^{-1}(z)\right)\right).$$

The transformation properties of the logarithmic field can be obtained from its logarithmic partner by a formal differentiation with respect to h (derivation trick) and the identification:

$$\phi' = \frac{\partial\phi}{\partial h}. \tag{3.11}$$

This holds in infinitesimal as well as in the global form (which is a non-trivial fact). This fact was used, for example, in [27] to find some logarithmic null vectors.
In the second case one has $m \in \mathbb{Z}, n \in \mathbb{N}$. The roots of the indicial equation differ by an integer. The logarithmic partner is not a proper primary field as in (3.6). The general construction for the rank two Jordan cell for this case is:

$$\begin{aligned} L_0|\phi\rangle &= h|\phi\rangle \\ L_0|\phi'\rangle &= h|\phi'\rangle + |\phi\rangle \\ L_k|\phi'\rangle &= |\xi\rangle. \end{aligned} \tag{3.12}$$

This is a class of the less-understood logarithmic theories. Some progress in understanding these theories was achieved in [28]. We will not deal with representations of this kind in the presented thesis.

We have to clarify here what we mean by a rational CFT, since there are two meanings used in mathematics and physics literature. First of all, we demand that there are only finitely many irreducible representations in the theory. The mathematicians' definition further requires all representations to be completely reducible, while the physicists' definition requites a finite dimensional highest weight space. In [58] it was shown that the first definition implies the second one. The other way does not work since there are certain logarithmic CFTs which are rational only according to the second definition.
Logarithmic CFTs corresponding to $c_{p,1}$ models with maximally extended triplet algebra [79] are called rational since they possess a finite-dimensional pace of irreducible representations, a finite dimensional highest-weight space but some of their representations are not completely reducible. This is the reason why the two notions of rationality are different and one should be aware of the difference. However, this is the only known example for a rational LCFT. The other known theories containing indecomposable representations also contain an infinite, countable space of irreducible representations. They are only quasi-rational and have the property that the fusion product of two irreducible representations is finite-dimensional and contains representations with Jordan cell structure.

3.2 Correlation functions in LCFT

In a well-defined logarithmic CFT one should be able to calculate corresponding correlation functions between fields of the theory. If the transformation properties of logarithmic fields, generated by elements of the Virasoro algebra are known, one can apply conformal Ward identities and solve the obtained differential equations [37]. It is also well-known that correlation functions in logarithmic conformal field theory do not factorize, but it in LSCFT, the proper correlation functions can be obtained by one assuming that Jordan cells exist in the chiral representation of a field. Correlation functions are calculated using conformal Ward identities and their proper, not factorizable counterparts are obtained via the substitution:

$$z^h \to z^h \bar{z}^{\bar{h}}$$

$$\log(z) \to \log(|z|^2).$$

Consider the simplest case, a two-dimensional Jordan block containing fields ϕ and ϕ' of same conformal dimension. Then the correlator of ϕ is fixed by ordinary conformal Ward identities:

$$\langle \phi(z)\phi(w) \rangle = \frac{C}{(z-w)^{-2h}}.$$

On the other hand, the correlator $\langle \phi(z)\phi'(w) \rangle$ satisfies:

$$(z^2\partial_z + w^2\partial_w + 2h(z+w))\langle \phi'(z)\phi(w) \rangle + 2z\langle \phi(z)\phi'(w) \rangle = 0.$$

With a change of coordinates $x = z - w$, $y = z + w$ this equation can be reduced to:

$$(x\partial_x + 2h)\langle \phi(z)\phi'(w) \rangle + C\frac{x+y}{yx^{2h}}.$$

This clearly can be satisfied only if $C = 0$. Thus, the logarithmic partner has a vanishing propagator and:

$$\langle\phi(z)\phi'(w)\rangle = \frac{a}{(z-w)^{2h}}.$$

Translational invariance of $\langle\phi'(z)\phi'(w)\rangle$ again implies its dependence of $(z-w)$ and variation generated by L_1 leads to the differential equation:

$$(x\partial_x + 2h)\langle\phi'(x)\phi'(w)\rangle + \frac{2c}{x^{2h}} = 0.$$

The solution is:

$$\langle\phi'(z)\phi'(w)\rangle = (z-w)^{-2h}(b - 2c\log(z-w)).$$

Consider a field γ which is not part of the Jordan cell. Following [39], [40], we call such fields non-cellular, as opposed to cellular primary fields. The only case when γ couples to ϕ is when their conformal dimensions are identical, so from now on we are going to assume that the conformal dimension of γ is h and:

$$\langle\gamma(z)\gamma(w)\rangle = d(z-w)^{-2h}$$

$$\langle\gamma(z)\phi(w)\rangle = E(z-w)^{-2h}.$$

Varying $\langle\gamma\phi'\rangle$ with respect to L_1 sets E to 0 and yields:

$$\langle\gamma(z)\phi'(w)\rangle = e(z-w)^{-2h}.$$

These findings can be generalized as follows [37]. Consider a n+1-dimensional Jordan cell with corresponding fields $\phi_0 := \phi$, $\phi_1 := \phi'$,..., ϕ_n. Then L_0 acts on the corresponding highest-weight states in the following way:

$$L_0|\phi_0\rangle = h|\phi_0\rangle$$

$$L_0|\phi_i\rangle = h|\phi_i\rangle| + |\phi_{i-1}\rangle, \quad 1 \le i \le n.$$

The action of the Virasoro generators on the fields is:

$$[L_n, \phi_0(z)] = (z^{n+1}\partial_z + h(n+1)z^n)\phi_0(z)$$

$$[L_n, \phi_i(z)] = (z^{n+1}\partial_z + h(n+1)z^n)\phi_i(z) + (n+1)z^n\phi_{i-1}(z), \quad 1 \le i \le n-1.$$

Again, one can apply the derivation trick, setting:

$$\phi_i = \frac{1}{i!}\frac{\partial^i\phi_0}{\partial h^i}. \tag{3.13}$$

We can simplify the notation by setting $\phi_{-1} = 0$. Consider the two-point function $\langle\phi_i\phi_j\rangle$. We obtain:

$$(z^2\partial_z + w^2\partial_w + 2h)\langle\phi_i(z)\phi_j(w)\rangle + z\langle\phi_{i-1}(z)\phi_j(w)\rangle + w\langle\phi_i(z)\phi_{j-1}(w)\rangle.$$

This requires:

$$\langle\phi_i(z)\phi_{j-1}(w)\rangle = \langle\phi_{i-1}(z)\phi_j(w)\rangle.$$

Setting ϕ_{-1} inducts the relation:

$$\langle\phi_i(z)\phi_j(w)\rangle = 0, \quad i+j < n-1.$$

Therefore, the last term on the right side of the Ward identity disappears for $i=0$, $j=n-1$ and any i, $j=n-1-i$ and the solution reads:

$$\langle\phi_i(z)\phi_{n-1-i}(w)\rangle = a(z-w)^{-2h}.$$

Now, we can insert this solution back into the Ward identity for $\langle\phi_1\phi_n\rangle$ or any i, $j=n+1-i$:

$$\langle\phi_i(z)\phi_{n+1-i}(w)\rangle = (z-w)^{-2h}(b-2a\,\log(z-w)).$$

Reinserting obtained solutions we arrive at:

$$\langle\phi_i(z)\phi_j(w)\rangle = (z-w)^{-2h}\sum_{j=0}^{i} a_{ij}(\log(z-w))^j$$

$$(j+1)a_{i,j+1} + 2a_{i-1,j} = 0.$$

This recursion relation can be written as:

$$a_{i,j+1} = \frac{-2}{j+1}a_{i-1,j} = \frac{(-2)^{j+1}}{(j+1)}a_{i-j-1,0} \equiv \frac{(-2)^{j+1}}{(j+1)}a_{i-j-1}.$$

Now we consider an even more general case, with eventually more Jordan cells of different sizes, which we label by I, J. The two-point function generalizes so we can write it as:

$$\langle\phi_i^I(z)\phi_j^J(w)\rangle = (z-w)^{-2h}\sum_{k=0}^{i+j-n}\frac{(-2)^k}{k!}a_{n-k}^{IJ}(\log(z-w))^k. \tag{3.14}$$

for $i+j \geq n$ and 0 otherwise.

The tree-point function can be calculated following a similar way. We remember that the fields in a logarithmic block are degenerate in h. If the correlator does not contain the logarithmic field, we obtain the ordinary three-point function:

$$\langle\phi_0(z_1)\phi_0(z_2)\phi_0(z_3\rangle = a(z_{12}z_{23}z_{13})^{-h}. \tag{3.15}$$

Acting on $\langle\phi_1\phi_0\phi_0\rangle$ with variation corresponding to one of the $SL(2, \mathsf{C}$ again leads to a set of differential equations, solution of which is:

$$\langle\phi_1(z_1)\phi_0(z_2)\phi_0(z_3\rangle = (b+a\log)(\frac{z_{12}}{z_{23}z_{31}})(z_{12}z_{23}z_{31})^{-h}. \tag{3.16}$$

Introducing some short-hand notation $\xi_1 := z_{12}$, $\xi_2 := z_{23}$, $\xi_3 := z_{13}$ the correlator $\langle\phi_1\phi_1\phi_0\rangle$ reads:

$$\langle\phi_1(z_1)\phi_1(z_2)\phi_0(z_3)\rangle = (c - \textstyle\sum_i c_i\log(\xi_i)$$
$$+\textstyle\sum_{ij} c_{ij}\log(\xi_i)\log(\xi_j)))(\xi_1\xi_2\xi_3)^{-h}.$$

The constants are fixed by the Ward identities, so in the final form we have:

$$\begin{gathered}\langle\phi_1(z_1)\phi_1(z_2)\phi_0(z_3)\rangle = \\ (c - 2b\log(\xi_3) + a((\tfrac{\log(\xi_1)}{\log(\xi_2)})^2 + (\log(\xi_3)^2))(\xi_1\xi_2\xi_3)^{-h}.\end{gathered} \tag{3.17}$$

To write down $\langle\phi_1(z_1)\phi_1(z_2)\phi_1(z_3)\rangle$ it is useful to introduce functions of ξ_i:

$$D_1 \equiv \log(\xi_1\xi_2\xi_3)$$

$$D_2 \equiv \log(\xi_1)\log(\xi_2) + \log(\xi_2)\log(\xi_3) + \log(\xi_1)\log(\xi_3)$$

$$D_3 \equiv \log(\xi_1)\log(\xi_2)\log(\xi_3).$$

Then the correlator of three logarithmic fields yields:

$$\begin{gathered}\langle\phi_1(z_1)\phi_1(z_2)\phi_1(z_3)\rangle = \\ (d + d_1D_1 + d_2D_2 + d_2'D_1^2 + d_3D_3 + d_3'D_1D_2 + d_3'D_1^3)(\xi_1\xi_2\xi_3)^{-h}.\end{gathered}$$

Finally, the Ward identities fix the constants up to:

$$\begin{gathered}\langle\phi_1(z_1)\phi_1(z_2)\phi_1(z_3)\rangle = \\ (d - cD_1 + 4bD_2 - bD_1^2 + 8aD_3 - 4aD_1D_2 + aD_1^3)(\xi_1\xi_2\xi_3)^{-h}.\end{gathered} \tag{3.18}$$

It can be seen that the relations (3.11), (3.13) can be used to obtain correlators of logarithmic fields without solving the conformal Ward identities by treating the fields and normalization constants as formally dependent on the fields' conformal weighs. Consider a three-point function $\langle\phi_i\phi_0\phi_0\rangle$:

$$\langle\phi_i(z_1)\phi_0(z_2)\phi_0(z_3)\rangle = \frac{1}{i}\frac{\partial^i}{\partial h_i^i}\langle\phi_i(z_1)\phi_0(z_2)\phi_0(z_3)\rangle.$$

For an arbitrary correlator, one has to treat the conformal weights as independent variables, although they take the same value. The correlator of ordinary primary fields then reads:

$$\langle\phi_0(z_1)\phi_0(z_2)\phi_0(z_3)\rangle = \xi_1^{-h_2-h_3+h_1}\xi_2^{-h_3-h_1+h_2}\xi_3^{-h_2-h_1+h_3}.$$

Differentiating with respect to h_1 and setting $h_1 = h_2 = h_3$ and $\frac{\partial a}{\partial h} = b$ returns exactly (3.15-3.18). This works for all correlation functions except for $n = 2$, which is the only correlation function in which the dependence on h is not continuous. A more explicit study of logarithmic four-point functions can be found in [47].

Chapter 4

Super Virasoro Theories and Jordan cell structure in SCFT

Supersymmetric extensions of the Virasoro algebra, motivated by physical considerations were suggested in [48], [49]. Interestingly, they also were independently discovered in [50], motivated by infinite dimensional Lie algebras. Although there is no evidence for particle supersymmetry, the N=1 tricritical Ising model can be realized experimentally by absorbing ^{4}He atoms on krypton-plated graphite [51].

Beginning with the notion of a supermanifold, super Viraroro algebras with N fermionic generators are discussed in this chapter. Supermanifolds allow to study supersymmetric theories in a unified way by introducing superfields which contain bosonic and fermionic components. A superconformal theory decomposes in exactly the same way as ordinary CFT in holomorphic and antiholomorphic parts, and only the holomorphic part will be studied here.

A supermanifold is obtained by extending an ordinary manifold by a fibre bundle of N anticommutative rings. Then the holomorphic part of the field theory is formulated on a map of the manifold with coordinates $\{z, \theta_1, ...\theta_N\}$. The ordinary CFT can be seen a field theory on the trivial $N = 0$ superconformal extension of the complex plane.

As we will see, one can easily construct Jordan cells for primary fields in superconformal theories and determine their transformational properties. From ordinary correlation functions, logarithmic correlation functions can be derived.

4.1 Superconformal Transformations

To obtain a supersymmetrically extended, well-defined CFT, two approaches can be taken. The algebraic one imposes a grading of the superconformal algebra, denoted $|x|$, which is either zero or one for even and odd generators, respectively. The Lie bracket is graded, which means it is an anticommutator for odd elements of the algebra. The super Jacobi identity reads:

$$(-1)^{|z||x|}[x,[y,z]] + (-1)^{|x||y|}[y,[z,x]] + (-1)^{|y||z|}[z,[x,y]] = 0. \tag{4.1}$$

The anticommutator is often denoted as $\{.,.\}$. An example of the analytic approach is [71], where superprimary fields are treated as sections of a sheaf over the graded Riemann sphere and OPEs and commutation relations of the operators in the case N=3 are obtained.

The rather analytic approach discussed in [72], [73] starts from the notion of the supermanifold. Superconformal transformations of coordinates are obtained from the superconformal condition. Uncharged primary fields are then defined on the supermanifold and their various quantities are calculated.

To obtain a supermanifold, one defines a fiber bundle of N anticommutative rings over a manifold. The coordinates of a two-dimensional supermanifold are given by $\mathsf{Z} = z, \theta_1, ...\theta_N$, where θ_i are anticommuting, and therefore nilpotent Grassmannian variables satisfying:

$$\theta_1\theta_2 = -\theta_2\theta_1.$$

Derivatives acing on θ_i satisfy:

$$\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j} = -\frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i}, \qquad \frac{\partial}{\partial\theta_i}\theta_j = \delta_{i,j} - \theta_j\frac{\partial}{\partial\theta_i}.$$

A general infinitesimal transformation acts on coordinates of superspace via:

$$z' = z + \delta z, \qquad \theta'_j = \theta_j + \delta\theta_j. \tag{4.2}$$

We define a one-form as:

$$\omega = dz - \sum_i d\theta_i\theta_i.$$

The superconformal transformations which are the homomorphisms between charts are the invertible transformations which preserve the one-form up to a function, which is called the superconformal condition:

$$\omega' = \kappa(z,\theta_i)\omega. \tag{4.3}$$

In fact, this is the defining condition of supeconformal transformations. There is another choice of the one-form:

$$\omega = dz - \sum_i d\theta.$$

This is a convenient choice for following calculations. The basis of the dual space (the space of derivations) is given by $(\partial_z,\ D_i)$ with D_i the covariant derivative:

$$D_i = \frac{\partial}{\partial\theta_i} + \theta_i\frac{\partial}{\partial\theta_i},$$

satisfying:

$$D_i^2 = \partial_z.$$

The set of superpoints together with the supercovariant derivative defines the notion of the superconformal manifold. In N=2 theory and higher, there is a continuous symmetry group transforming fermionic components in each other as we will see. Under (4.2), the covariant derivative transforms as:

$$D_i = (D_i\theta'_j)D'_j + (D_i z' - \theta'_j D_i\theta'_j)\partial_z. \tag{4.4}$$

The first term is the homogenious part. The second, inhomogenious part arises due to the general nature of the transformations. We want to obtain those transformations under which the covariant derivative transforms as a vector, which is equivalent to (4.3). This is similar to ordinary CFT, where $\frac{\partial}{\partial z} = \frac{\partial z'}{\partial z}\frac{\partial}{\partial z'}$. The equivalence of (4.3) and (4.4) can be seen as follows.

The superconformal condition implies that ω transforms as (using the Einstein summation convention):

$$\omega' = (\frac{\partial z'}{\partial z} + \theta'_i\frac{\partial\theta'_i}{\partial z})dz + (-\frac{\partial z'}{\partial\theta_j} + \theta'_i\frac{\partial\theta'_i}{\partial\theta_j})d\theta_j$$

$$\Rightarrow (\frac{\partial z'}{\partial z} + \theta'_i\frac{\partial\theta'_i}{\partial z})\theta_j = -\frac{\partial z'}{\partial\theta_j} + \theta'_i\frac{\partial\theta'_i}{\partial\theta_j}.$$

This is equivalent to:

$$D_j z' = \theta'_i D_j\theta'_i.$$

Thus, the superconformal condition is equivalent to the disappearance of the inhomogenious term in (4.4).

Superdifferentials $d\mathsf{Z}_i$ are defined as elements from the space dual to the space of superderivatives:

$$D_i d\mathsf{Z}_j = \delta_{i,j}$$

where the indices i and j run from 1 to N. Naturally, superdifferentials transform as a vector:

$$d\mathsf{Z}'_i = D_i\theta'_j d\mathsf{Z}_j$$

where $D_i\theta'_j$ are elements of the super Jacobi matrix which is defined as:

$$D\theta' = \begin{pmatrix} D_1\theta'_1 & \cdots & D_N\theta'_1 \\ \vdots & & \vdots \\ D_1\theta'_N & \cdots & D_N\theta'_N \end{pmatrix}. \tag{4.5}$$

Now we are ready to do some field theory on supermanifolds. First we consider the Neveu-Schwarz theories where the functions are subring of $\mathbb{C}[z^{-1}, z]$ under addition and pointwise multiplication.

A result obtained in [74] states that the generators of superconformal transformations in the Neveu-Schwarz theories can be written as:

$$X_a(i_1,...,i_I) = (1-\frac{I}{2})z^{n-\frac{I}{2}+1}\theta_{i_1}...\theta_{i_I}\partial_z + \frac{1}{2}\sum_{p=1}^{I}(-1)^{p+I}z^{n-\frac{I}{2}+1}\theta_{i_1}...\check{\theta}_{i_p}...\theta_{i_I}\partial_{\theta_{i_p}}$$

$$+\frac{1}{2}(a-\frac{I}{2}+1)\sum_{k\in\bar{S}} z^{a-\frac{I}{2}}\theta_{i_1}...\theta_{i_I}\theta_{i_k}\partial_{\theta_k}.$$

This combinatorial result requires some further explanation. The generators are labeled by an index a, which is from $\mathbb{Z}$ if N is even and from $\mathbb{Z}_{\frac{1}{2}}$ otherwise. $\check{\theta}_{i_p}$

means that this variable is omitted in the expression. The sequence $S = (i_1, ...i_I)$ contains elements $i_1...i_I \in \{1, ..., N\}$. In the third term, the complement of S is $\bar{S}$ and is defined as the set $\bar{S} = \{1, ..., N\} \backslash S$. Superconformal generators satisfy classical (as opposed to quantum) commutation relations:

$$\begin{aligned} &[X_a(i_1, ..., i_I), X_b(j_1, ..., j_J)] \\ = &\textstyle\sum_{p=1}^{I} \sum_{q=1}^{J} \frac{(-1)^{I+p}\delta_{i_p,j_q}}{2} X_{a+b}(i_1, ...\check{i}_p, ..., i_I, j_1, ..., \check{j}_q, ..., j_J) \\ &+ \left((1 - \tfrac{I}{2})b - (1 - \tfrac{J}{2})a\right) X_{a+b}(i_1, ..., i_I, j_1, ..., j_J). \end{aligned} \tag{4.6}$$

For N=0, one easily recovers the Witt algebra (2.11).
A consequence of (4.6) is that the scaling factor of the superconformal condition under a superconformal transformation can be determined as:

$$\omega' = X(i_1, ..., i_I)\omega = (a - \frac{I}{2} + 1)z^{a-\frac{I}{2}}\theta_{i_1}...\theta_{i_I}\omega.$$

Quantum versions of commutators can be obtained by a central extension of the symmetry algebra. Remarkably, for infinite-dimensional algebras there is at most one possible central extension which will be parametrized by c, which is allowed to be non-zero for N≤4 [72].

The only conceptual notion remaining to be generalized is superintegration. If one defines:

$$D_i F(\mathsf{Z}) = f(\mathsf{Z}),$$

then:

$$F(\mathsf{Z}) = \int d\mathsf{Z}_i f(\mathsf{Z}).$$

An integral over a finite interval is consistent and obeys the fundamental theorem of calculus:

$$\int_{\mathsf{Z}_1}^{\mathsf{Z}_2} d\mathsf{Z}_i f(\mathsf{Z}) = F(\mathsf{Z}_2) - F(\mathsf{Z}_1),$$

if Z_1 and Z_2 coincide up to the coordinates z and θ_i. Of special importance for calculations in superspace are the *superdifferences*. To define them, we need to introduce additional indices since we are dealing with different superpoints $\mathsf{Z}_i = \{z_i, \theta_{i,1} \cdots \theta_{i,N}\}$ on the supermanifold. The first index $i \in \{1, 2\}$ labels the different points and the second one the Grassmannian coordinates. Then the superdifferences are defined as:

$$\begin{aligned} \mathsf{Z}_{12} &\equiv z_1 - z_2 - \theta_{1,j}\theta_{2,j} \\ \theta_{12} &\equiv \theta_{1,j} - \theta_{2,j}. \end{aligned} \tag{4.7}$$

It can be easily seen that the following equations are satisfied (without the use of Einstein summation convention in the second expression):

$$D_{2,i}\mathsf{Z}_{12}^n = n\theta_{12,i}\mathsf{Z}_{12}^{n-1}$$

$$D_{2,i}\theta_{12,i}\mathsf{Z}_{12}^n = -\mathsf{Z}_{12}^n.$$

Successive integrals of 1 can be represented using combinations of superdifferences (again, not using the Einstein summation convention):

$$\int_{\mathsf{Z}_1}^{\mathsf{Z}_2} d\mathsf{Z}_{3,i}\mathsf{Z}_{13}^n = -\theta_{12,i}\mathsf{Z}_{12}^n$$

$$\int_{\mathsf{Z}_1}^{\mathsf{Z}_2} d\mathsf{Z}_{3,i}\theta_{13,i}\mathsf{Z}_{13}^n = \frac{1}{n+1}\mathsf{Z}_{12}^{n+1}.$$

The superdifferences, viewed as new coordinates also will drastically simplify conformal Ward identities, as we will see.

Super contour integration is a linear operation which is translationally invariant. For one Grasmanian coordinate and a closed path C_0 around the origin it is defined as:

$$\oint_{C_0} d\theta_i\theta_j = \delta_{i,j}$$

$$\oint_{C_0} d\theta_i = 0.$$

Thus, contour integration over Grassmannian numbers is equivalent to differentiation with respect to Grassmannian numbers. For N anticommutative coordinates, the full contour integral returns zero whenever the integrand is missing at least one of the θ's. Expanding a general superfunction in Grassmannian coordinates:

$$f(\mathsf{Z}) = f^0 + \theta_i f_i^1(\mathsf{z}) + \cdots + \theta_1\cdots\theta_N f^N(z),$$

and defining:

$$\epsilon^N = \oint_{C_0} d\theta_1\cdots d\theta_N = (-1)^{\frac{N(N-1)}{2}},$$

it is obvious that only the last term contributes to the integral:

$$\oint_C d\mathsf{Z}f(\mathsf{Z}) = \oint_C dz d\theta_1...\theta_N f(\mathsf{Z}) = \epsilon^N \oint_{C'} dz f^N(z).$$

Here, C' is a projection of C on the complex plane.
Volume integrals, which are necessary for formulation of action in superspace, are defined as:

$$\int_V d\mathsf{Z}f(\mathsf{Z}) = \int_V d\mathsf{Z}_1\cdots d\mathsf{Z}_N f(\mathsf{Z}).$$

Under a conformal transformation, the volume integral is required to be invariant. This leads to:

$$\int_V d\mathsf{Z}f(\mathsf{Z}) = \int_{V'} d\mathsf{Z}'f(\mathsf{Z}')\det D\theta'.$$

4.2 Ordinary N=1 SCFT

The simplest case of a conformal supersymmetric theory is the N=1 SCFT. In addition to Virasoro modes L_n there are additional fermionic generators G_r. The complete set of generators satisfies following (anti-)commutation relations:

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{n,-m}$$

$$[L_n, G_r] = (\frac{1}{2}n - r)G_{n+r} \tag{4.8}$$
$$\{G_r, G_s\} = 2L_{r+s} + \frac{1}{3}c(r^2 - \frac{1}{4})\delta_{r,-s}.$$

Here, r is a half-integer moded for the Neveu-Schwarz sector and integer moded for the Ramond sector. We expect the NS-vacuum to be globally invariant under a maximum number of modes G_r. The maximal set of such G_r is given by $G_{-\frac{1}{2}}, G_{\frac{1}{2}}$. This is the group $OSp(2|1)$.

Making use of superspace formalism, the super stress-energy tensor is written as:

$$\mathsf{T}(z, \theta) = G(z) + \theta T(z),$$

and primary superfields, expanded in the nilpotent variable, possess one bosonic and one fermionic field:

$$\Phi(z, \theta) = \phi(z) + \theta\psi(z).$$

We can write the infinitesimal transformations of superfields as:

$$[L_n, \Phi(z, \theta)] = \left(z^{n+1}\partial_z + (n+1)(h + \frac{1}{2}\theta\partial_\theta)z^n\right)\Phi(z, \theta)$$
$$[G_r, \Phi(z, \theta)] = \left(z^{r+\frac{1}{2}}\partial_\theta - \theta\left(z^{r+\frac{1}{2}}\partial_z + h(2r+1)z^{r-\frac{1}{2}}\right)\right)\Phi(z, \theta). \tag{4.9}$$

This implies operator product expansions:

$$\mathsf{T}(\mathsf{Z}_1)\mathsf{T}(\mathsf{Z}_2) = \frac{c}{6}\frac{1}{(\mathsf{Z}_{12})^3} + \frac{2\theta_{12}}{(\mathsf{Z}_{12})^2}\mathsf{T}(\mathsf{Z}_2) + \frac{1}{2\mathsf{Z}_{12}}D_{\mathsf{Z}_2}\mathsf{T}(\mathsf{Z}_2) + \frac{\theta_{12}}{\mathsf{Z}_{12}}\partial_{\mathsf{Z}_2}\mathsf{T}(\mathsf{Z}_2)$$
$$\mathsf{T}(\mathsf{Z}_1)\Phi(\mathsf{Z}_2) = \frac{h\theta_{12}}{(\mathsf{Z}_{12})^2}\Phi(\mathsf{Z}_2) + \frac{1}{2\mathsf{Z}_{12}}D_{\mathsf{Z}_2}\Phi(\mathsf{Z}_2) + \frac{\theta_{12}}{\mathsf{Z}_{12}}\partial_{z_2}\Phi(\mathsf{Z}_2)$$

with Z_{ij}, θ_{ij} defined as in (4.7) for N=1:

$$\mathsf{Z}_{ij} \equiv z_i - z_j - \theta_i\theta_j$$
$$\theta_{ij} \equiv \theta_i - \theta_j.$$

The minimal models are found for central charges:

$$c = \frac{3}{2}(1 - \frac{8}{m(m+2)}), \quad m \in \{\mathbb{N}|m > 2\}.$$

4.3 Logarithmic N=1 SCFT

In [41], it was shown that a logarithmic N=1 field Φ' and corresponding transformations under supersymmetry generators can be defined by assuming indecomposability in the bosonic component $\phi'(z)$ of a superfield and demanding consistency with the Jacobi identity. Then the infinitesimal transformation of the bosonic part under L_n can be written as:

$$[L_n, \phi'(z, \theta)] = \left(z^{n+1}\partial + h(n+1)\right)\phi' z + (n+1)z^n\phi(z). \tag{4.10}$$

From action of (4.9) on he bosonic component the ansatz for a logarithmic superpartner ψ' is taken as:

$$[G_r, \phi'_z] \equiv z^{r+\frac{1}{2}} \psi'_r(z).$$

To get transformation properties of ψ' under L_m, one acts on both sides with $[L_m,]$ and make use of the super Jacobi identity(4.1):

$$[L_m, \psi'_r(z)] = z^{-r-\frac{1}{2}}[L_m, [G_r, \phi'(z)]]$$

$$= z^{-r-\frac{1}{2}}[[L_m, G_r].\phi(z)'] + z^{-r-\frac{1}{2}}[G_r, [L_m, \phi(z)']].$$

Now, using (4.8), (4.10) and (4.9).

$$[L_m, \psi'_r(z)] = (\frac{m}{2} - r) z^m (\psi'_{m+r} - \psi'_r)$$

$$+ \left(z^{m+1}\partial_z + (h + \frac{1}{2})(m+1)z^m \right) \psi'_r + (m+1) z^m \psi.$$

The only choice consistent with L_{-1} acting as generator of translations, i.e. derivative with respect to z is $\psi'_r = \psi' \; \forall \psi'_r$. Since the N=1 algebra consists only of L_n, G_r, one obtains the complete set of superconformal transformations of a NS superconformal logarithmic field.
The OPEs of the fields with the stress-energy tensor are given by:

$$\mathsf{T}(\mathsf{z}_1)\Phi(\mathsf{z}_2) = \frac{h\theta_{12}}{(\mathsf{z}_{12})^2}\Phi(\mathsf{z}_2) + \frac{1}{2\mathsf{z}_{12}} D_{\mathsf{z}_2}\Phi(\mathsf{z}_2) + \frac{\theta_{12}}{\mathsf{z}_{12}}\partial_{\mathsf{z}_2}\Phi(\mathsf{z}_2)$$

$$\mathsf{T}(\mathsf{z}_1)\Phi'(\mathsf{z}_2) = \frac{h\theta_{12}}{(\mathsf{z}_{12})^2}\Phi'(\mathsf{z}_2) + \frac{\theta_{12}}{(\mathsf{z}_{12})^2}\Phi(\mathsf{z}_2) + \frac{1}{2\mathsf{z}_{12}} D_{\mathsf{z}_2}\Phi'(\mathsf{z}_2) + \frac{\theta_{12}}{\mathsf{z}_{12}}\partial_{\mathsf{z}_2}\Phi'(\mathsf{z}_2).$$

The second term on the right side of the last equation is the consequence of the fact that Φ' is not a primary superfield but is the logarithmic partner of Φ.

In state formalism the indecomposable state manifests itself in Jordan block structure of the dilation operator acting on a two-dimensional column vector with fields as components. From here, the OPE of the logarithmic field Φ' with $T(\mathsf{z})$ can be deduced and it can be seen that it can be obtained from the OPE of the ordinary field Φ with $T(\mathsf{z})$ using the derivation trick.

4.4 Correlation functions in N=1 SCFT and LSCFT

Therefore, the conformal Ward identities for primary fields and generators L_{-1}, L_0, L_1, $G_{-\frac{1}{2}}$, and $G_{\frac{1}{2}}$, respectively are given by:

$$\langle \Phi_{h_1}(\mathsf{Z}_1)\Phi_{h_2}(\mathsf{Z}_2)...\Phi_{h_n}(\mathsf{Z}_\mathsf{n})\rangle = F_n(\mathsf{Z}_1, \mathsf{Z}_2, ...\mathsf{Z}_n)$$

$$\sum_i \partial_{z_1} F_n = 0$$

$$\sum_i \left(z_i\partial_{z_i} + h_i + \frac{1}{2}\theta_i\partial_{\theta_i} \right) F_n = 0$$

$$\sum_i \left(z_i^2 \partial_{z_i} + 2z_i(h_i + \frac{1}{2}\theta_i z_i \partial_{\theta_i}) \right) F_n = 0$$

$$\sum_i \left(\partial_{\theta_i} - \theta_i \partial_{z_i} \right) \ F_n = 0$$

$$\sum_i \left(z_i \partial_{\theta_i} - \theta_i z_i \partial_{z_i} - 2h_i \theta_i \right) F_n = 0.$$

As in N=0, the first identity implies $F_2 = F_2(z_{12}, \theta_1, \theta_2) = F_2((z_1 - z_2), \theta_1, \theta_2)$. The fourth one can be transformed via a change of coordinates $\theta_{12} \equiv \theta_1 - \theta_2$ to:

$$\left(\partial_{\theta_1} + \partial_{\theta_2} - (\theta_1 - \theta_2) \partial_{z_{12}} \right) F_2 = 0.$$

Any function of $\mathsf{Z}_{12} \equiv z_1 - z_2 - \theta_1 \theta_2$ satisfies this equation. Thus, $G_{\frac{1}{2}}$ implies that the solution of the two point function depends on the superdifference Z_{12}. Proceeding with the usual reasoning, the first three Ward identities imply that for the correlator to be non-zero, the conformal dimensions of the superfields have to be the same number and:

$$\langle \Phi(\mathsf{Z}_1) \Phi(\mathsf{Z}_2) \rangle = a \mathsf{Z}_{12}^{-2h}.$$

Expanded around $\frac{\theta_1 \theta_2}{z_1 - z_2}$, this relation reads:

$$\langle \Phi(\mathsf{Z}_1) \Phi(\mathsf{Z}_2) \rangle = \frac{a}{(z_1 - z_2)^{-2h}} \left(1 + \frac{2h \theta_1 \theta_2}{z_1 - z_2} \right).$$

The expansion is exact after the first term since $\theta_1 \theta_2$ is nilpotent. It can be easily checked that this relation satisfies the remaining Ward identity for $G_{\frac{1}{2}}$. This is expected since $G_{\frac{1}{2}} = [L_1, G_{-\frac{1}{2}}]$.
Consider a two-point function in LSCFT. In the following, we use techniques analogous to those we developed in 3.2. The two-point function can be written as a function of $\mathsf{Z}_{12} = z_1 - z_2 - \theta_1 \theta_2$. Then the whole reasoning behind (3.14) applies directly to the logarithmic case. One finds the term $\log(\mathsf{Z}_{12})$ which can be expanded to $\log(z_1 - z_2) - \frac{\theta_1 \theta_2}{z_1 - z_2}$.

We can write:

$$\langle \Phi_i(\mathsf{Z}_1) \Phi_j(\mathsf{Z}_2) \rangle = (\mathsf{Z}_{12})^{-2h} \sum_{k=0}^{i+j-n} \frac{(-2)^k}{k!} a_{n-k} \log^k(\mathsf{Z}_{12})$$

with:

$$\log^k(\mathsf{Z}_{12}) = \log^k(z_{12}) - \frac{k \log(z_{12}) \theta_1 \theta_2}{z_{12}}. \tag{4.11}$$

The $N = 1$ ordinary three-point function was obtained in [42] and reads:

$$\begin{aligned} &\langle \Phi(\mathsf{Z}_1) \Phi(\mathsf{Z}_2) \Phi(\mathsf{Z}_3) \rangle = \textstyle\prod_{i<j} \mathsf{Z}_{ij}^{h - 2h_i - 2h_j} (a + bW) \\ &= \mathsf{Z}_{12}^{h_3 - h_1 - h_2} \mathsf{Z}_{13}^{h_2 - h_1 - h_3} \mathsf{Z}_{23}^{h_1 - h_2 - h_3} (a + bW) \end{aligned} \tag{4.12}$$

with b an undetermined Grassmannian constant and:

$$h \equiv \sum_i h_i$$

$$W \equiv \frac{\theta_1 \mathsf{Z}_{23} - \theta_2 \mathsf{Z}_{13} + \theta_3 \mathsf{Z}_{23} + \theta_1\theta_2\theta_3}{(\mathsf{Z}_{12}\mathsf{Z}_{13}\mathsf{Z}_{23})^{\frac{1}{2}}}.$$

The first term of the super three-point function is (2.16) with z_{ij} replaced by Z_{ij}. Expanding (4.12) in the Grassmann variables yields the expression:

$$\langle \Phi(\mathsf{Z}_1)\Phi(\mathsf{Z}_2)\Phi(\mathsf{Z}_3)\rangle = (z_{12})^{-h_1-h_2+h_3}(z_{13})^{-h_1+h_2-h_3}(z_{23})^{h_1-h_2-h_3}$$
$$\cdot \left(a + \frac{b(\theta_1 z_{23} - \theta_2 z_{13} + \theta_3 z_{12})}{(z_{12}z_{13}z_{23})^{\frac{1}{2}}} + \frac{a}{2}\left(\frac{\theta_1\theta_2}{z_{12}} + \frac{\theta_1\theta_3}{z_{13}} + \frac{\theta_2\theta_3}{z_{23}}\right) + \frac{b\theta_1\theta_2\theta_3(h_1+h_2+h_3-\frac{1}{2})}{(z_{12}z_{13}z_{23})^{\frac{1}{2}}}\right).$$

It is satisfactory to see the different contributions from component fields. The first term is just the ordinary CFT three-point function (2.16), the other terms arise from the different contractions of the fermionic and bosonic components. We will not write down the logarithmic counterparts to the three-point functions, since their expressions are quite long and cumbersome. They can be easily obtained by differentiating the three-point function of ordinary N=1 SCFT, treating undetermined constants as formal functions of h.
Considering the four-point function of superprimary fields, one needs to find all invariants of the action of the algebra $OSp(2|1)$. The expression:

$$Y \equiv \prod_{i<j} (\mathsf{Z}_{\mathsf{ij}})^{h-h_i-h_j}$$

satisfies the action of the subalgebra generated by $\{L_\pm, L_0, G_{\pm\frac{1}{2}}\}$ on the super four-point function. The first invariant is an obvious modification of the anharmonic ratio:

$$X \equiv \frac{\mathsf{Z}_{12}\mathsf{Z}_{34}}{\mathsf{Z}_{13}\mathsf{Z}_{24}}.$$

There are three more invariants containing θ_i and z_i. The four-point function can be found to be:

$$\langle \Phi(\mathsf{Z}_1)\Phi(\mathsf{Z}_2)\Phi(\mathsf{Z}_3)\Phi(\mathsf{Z}_4)\rangle = Y(a + b_1 \mathsf{W}_{234} + b_4 \mathsf{W}_{123} + cV)$$

with a, c undetermined constants and b_1, b_4 undetermined Grassmannian constants:

$$\mathsf{W}_{ijk} \equiv \frac{\theta_i \mathsf{Z}_{jk} - \theta_j \mathsf{Z}_{ik} + \theta_k \mathsf{Z}_{ij} + \theta_i\theta_j\theta_k}{(\mathsf{Z}_{ij}\mathsf{Z}_{ik}\mathsf{Z}_{jk})^{\frac{1}{2}}}$$

and:

$$V \equiv \frac{\theta_1\theta_2\mathsf{Z}_{34}}{\mathsf{Z}_{13}\mathsf{Z}_{24}} + \frac{\theta_3\theta_4\mathsf{Z}_{12}}{\mathsf{Z}_{13}\mathsf{Z}_{24}} + \frac{\theta_1\theta_4\mathsf{Z}_{23}}{\mathsf{Z}_{13}\mathsf{Z}_{24}} + \frac{\theta_2\theta_3\mathsf{Z}_{14}}{\mathsf{Z}_{13}\mathsf{Z}_{24}} - \frac{\theta_1\theta_3}{\mathsf{Z}_{13}} - \frac{\theta_2\theta_4}{\mathsf{Z}_{24}} + \frac{3\theta_1\theta_2\theta_3\theta_4}{\mathsf{Z}_{13}\mathsf{Z}_{24}}.$$

We see that the nilpotency of θ determines the four-point function completely up to constants and functions of harmonic ratios. Again, the logarithmic counterparts are determined by formally differentiating the fields and the constants with respect to h (derivation trick).

4.5 Ordinary and Logarithmic N=2 SCFT

A N=2 SCFT contains two fermionic generators G^1_r and G^2_r of dimension $\frac{3}{2}$. The second fermionic generator G^2 is the superpartner to an additional $\widehat{U(1)}$-current J_r. The operator product expansions between the $T(z)$ and $G^{1,2}(z)$ of the theory read [77](using $z_{ij} \equiv z_i - z_j$) :

$$T(z_1)T(z_2) = \frac{c}{z_{12}^4} + \frac{2}{z_{12}^2}T(z_2) + \frac{1}{z_{12}}\partial T(z_2) \tag{4.13}$$

$$T(z_1)G^{1,2}(z_2) = \frac{3}{2z_{12}^2}G^{1,2}(z_2) + \frac{1}{z_{12}}\partial G^{1,2}(z_2)$$

$$T(z_1)J(z_2) = \frac{1}{z_{12}^2}J(z_2) + \frac{1}{z_{12}}\partial J(z_2)$$

$$J(z_1)J(z_2) = \frac{c}{12z_{12}^2}$$

$$J(z_1)G(z_2)^{1,2} = \mp\frac{1}{2z_{12}}G^{2,1}(z_2)$$

$$G^{1,2}(z_1)G^{1,2}(z_2) = \frac{2c}{3z_{12}^3} + \frac{2}{z_{12}}T(z_2)$$

$$G^1(z_1)G^2(z_2) = \frac{4}{z_{12}^2}J(z_2) + \frac{2}{z_{12}}\partial J(z_2).$$

The super-stress-energy tensor $\mathsf{T}(\mathsf{z}) = J(z) + \theta^+G^-(z) + \theta^-G^+(z) + \theta^+\theta^-T(z)$ we have the following OPE with itself:

$$\begin{aligned}\mathsf{T}(\mathsf{Z}_1)\mathsf{T}(\mathsf{Z}_2) = -\tfrac{c}{\mathsf{Z}_{12}^2} - \tfrac{\theta_{12,1}\theta_{12,2}}{\mathsf{Z}_{12}^2}\mathsf{T}(\mathsf{Z}_2) + \tfrac{\theta_{12,2}D_{2,1}}{2\mathsf{Z}_{12}^2}\mathsf{T}(\mathsf{Z}_2)\\ -\tfrac{\theta_{12,1}D_{2,2}}{2\mathsf{Z}_{12}^2}\mathsf{T}(\mathsf{Z}_2) - \tfrac{\theta_{12,1}\theta_{12,2}}{\mathsf{Z}_{12}^2}\partial_{z_2}\mathsf{T}(\mathsf{Z}_2)\end{aligned} \tag{4.14}$$

$$\mathsf{Z}_{12} = z_1 - z_2 - \theta_{1,1}\theta_{2,2} - \theta_{2,1}\theta_{1,2}. \tag{4.15}$$

Superfields $\Phi_{h,q}(\mathsf{Z})$ satisfy the OPE:

$$\begin{aligned}\mathsf{T}(\mathsf{Z}_1)\Phi_{h,q}(\mathsf{Z}_2) = -\tfrac{h\theta_{12,2}\theta_{12,2}}{\mathsf{Z}_{12}^2}\Phi_{h,q}(\mathsf{Z}_2) + \tfrac{\theta_{12,2}D_{2,1}}{2\mathsf{Z}_{12}^2}\Phi_{h,q}(\mathsf{Z}_2) - \tfrac{\theta_{12,1}D_{2,2}}{2\mathsf{Z}_{12}^2}\Phi_{h,q}(\mathsf{Z}_2)\\ -\tfrac{\theta_{12,1}\theta_{12,2}}{\mathsf{Z}_{12}}\partial_{z_2}\Phi_{h,q}(\mathsf{Z}_2) - \tfrac{iq}{2\mathsf{Z}_{12}^2}\Phi_{h,q}(\mathsf{Z}_2).\end{aligned} \tag{4.16}$$

The centerless, classical algebra acting on the space of functions is given by:

$$l_m = -z^m(z\partial_z + \frac{1}{2}(m+1)(\theta_1\partial_{\theta_1} + \theta_2\partial_{\theta_2})) \tag{4.17}$$

$$t_m = -z^m(\theta_1\partial_{\theta_2} - \theta_2\partial_{\theta_1}) \tag{4.18}$$

$$g_r^i = z^{r-\frac{1}{2}}(z\theta_i\partial_z - z\partial_{\theta_i} + (r+\frac{1}{2})\theta_i\theta_j\partial_{\theta_j}). \tag{4.19}$$

A more convenient choice of basis for the fermionic generators is the $U(1)$-diagonal basis:

$$\theta^{\pm} = \frac{1}{\sqrt{2}}(\theta_1 \pm i\theta_2)$$

$$D^{\pm} = \frac{1}{\sqrt{2}}(D_1 \pm D_2)$$

$$G^{\pm}(z) = \frac{1}{\sqrt{2}}(G^1 \pm iG^2)$$

$$J \to -iJ.$$

It should be noted that for ordinary superfunctions, (4.18) implies that only $q \in \{-1, 0, 1\}$ are allowed. However, we are considering a larger class of representations where in principle every real value of q is allowed, but not every representation can be easily constructed.
The transformed generators of the centerless algebra (4.17)-(4.19) are:

$$l_m = -z^m(z\partial_z + \frac{1}{2}(m+1)(\theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-})) \tag{4.20}$$

$$t_m = z^m(\theta^-\partial_{\theta^-} - \theta_+\partial_{\theta^+}) \tag{4.21}$$

$$g_r^\pm = z^{r-\frac{1}{2}}(z\theta^\pm\partial_z - z\partial_{\theta^\pm} \pm (r + \frac{1}{2})\theta^+\theta^-\partial_{\theta^\mp}). \tag{4.22}$$

The operator product expansion of G^+ and G^- with themselves is non-singular and their OPE with J reads:

$$J(z_1)G^\pm(z_2) = \pm\frac{1}{2z_{12}}G^\pm(z_2)$$

$$G^+(z_1)G^-(z_2) = \frac{2c}{3z^3_{z_{12}}}J(z_2) + \frac{2}{z_{12}}(T(z_2) + \partial J(z_2))$$

$$G^+(z_1)G^+(z_2) = G^-(z_1)G^-(z_2) = 0.$$

Analogously to the $N = 1$ case, the boundary conditions can be chosen to be either periodic or anti-periodic, which on the complex plane corresponds to the Neveu-Schwarz and the Ramond sectors, respectively. Introducing a parameter λ with $\lambda = 0$ in the NS sector and $\lambda = 0$ in the Ramond sector, the boundary conditions and the mode expansions of the fermionic currents can be expressed as:

$$G^\pm(e^{2\pi i}z) = e^{\lambda\pi i}G^\pm(z)$$

$$G(z) = \sum_{n=-\infty}^{+\infty}\frac{1}{2}G_{n+(1-\lambda)/2}z^{-n-2+\lambda/2}.$$

There is also a third sector in the $N = 2$ theory. The boundary conditions of the current J can be twisted, that is chosen to have anti-periodic boundary conditions. Then the current, expanded in terms of half-integer modes, reads:

$$J(e^{2\pi i}z) = -J(z)$$

$$J(z) = \sum_{n=-\infty}^{+\infty} z^{-n-1/2}J_{n+1/2}.$$

The full (anti-)commutation relations of the NS-sector of the $N = 2$-extended Virasoro algebra read:

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{n,-m},$$

$$[L_n, G_r^\pm] = (\frac{1}{2}n - r)G^\pm_{n+r}$$

$$\{G_r^-, G_s^+\} = 2L_{r+s} - (r-s)J_{r+s} + \frac{1}{3}c(r^2 - \frac{1}{4})\delta_{r,-s}$$
$$[L_n, J_m] = -mJ_{n+m}$$
$$[J_n, J_m] = \frac{1}{3}cn\delta_{m,-n}$$
$$[J_n, G_r^{\pm}] = \pm G_{n+r}^{\pm}.$$

In contrast to the $N = 0$ and $N = 1$ cases, the Cartan subalgebra of the $N = 2$ theory is two-dimensional and is spanned by the generators L_0 and J_0, which means that these two operators are simultaneously diagonalizable and the highest-weight representations are labeled by their eigenvalues h and q:

$$L_0|h,q\rangle = h|h,q\rangle$$
$$J_0|h,q\rangle = q|h,q\rangle$$
$$L_n|h,q\rangle = G_\alpha^{\pm}|h,q\rangle = J_m|h,q\rangle = 0, \quad \alpha,\ n,\ m > 0.$$

The global symmetry subgroup of the NS sector is therefore the orthogonal symplectic group $OSp(2|2)$, generated by $\{L_0, L_{\pm 1}, G_{\pm 1/2}, J_0, J_{\pm 1}\}$. The vacuum is as usual the unique state which annihilated by all these generators in addition to all the positively-moded ones:

$$L_n|0\rangle = J_n|0\rangle = G_\alpha|0\rangle = 0, \qquad n \geq -1,\ \alpha \geq -1/2.$$

From the OPE (4.16) one can extract infinitesimal variations of the superfield generated by the modes of the super-energy tensor:

$$\begin{aligned}[L_m, \Phi(\mathsf{Z})] = (h(m+1)z^m + \tfrac{1}{2}(m+1)z^m(\theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-}) \\ + z^{m+1}\partial_z + \tfrac{q}{2}\theta^+\theta^- z^{m-1}m(m+1))\Phi'(\mathsf{Z})\end{aligned} \tag{4.23}$$

$$[J_m, \Phi(\mathsf{Z})] = (2h\theta^+\theta^- m z^{m-1} + z^m(\theta^-\partial_{\theta^-} - \theta^+\partial_{\theta^+}) + qz^m)\Phi(\mathsf{Z}) \tag{4.24}$$

$$\begin{aligned}[G_r^{\pm}, \Phi(\mathsf{Z})] = (z^{r-\frac{1}{2}}(r+\tfrac{1}{2})(2h \pm q)\theta^{\pm} - z^{r+\frac{1}{2}}\partial_{\theta^{\mp}} + z^{r+\frac{1}{2}}\theta^{\pm}\partial_z \\ \pm z^{r-\frac{1}{2}}(r+\tfrac{1}{2})\theta^+\theta^-\partial_{\theta^{\mp}}))\Phi(\mathsf{Z}).\end{aligned} \tag{4.25}$$

In [56] it was shown that irreducible unitary highest-weight representations with respect to N=2 supersonformal algebra exist only for the minimal series:

$$c = \frac{3k}{k+2} \qquad k \in \mathbb{N}_0.$$

One of the remarkable properties of the N=2 superconformal case is that the Ramond and the Neveu-Schwarz sectors are equivalent . They are connected by the spectral flow α_η, an automorphism of the superconformal algebra parametrized by $\eta \in \mathbb{R}$ [76] [78]:

$$\alpha_\eta(G_r^{\pm}) = G_{r\mp\eta}^{\pm}$$

$$\alpha_\eta(L_n) = L_n - \eta J_n + \hat{c}\frac{\eta^2}{2}\delta_{n,0}$$
$$\alpha_\eta(J_n) = J_n - \hat{c}\eta\delta_{n,0}.$$

For $\eta = \frac{1}{2}$ the half-integer moded fermionic generators are mapped on the integer moded ones and vice versa. This fact shows the obvious difference to the $N = 1$ case, where the Ramond and the Neveu-Schwarz sectors are very different. However, the spectral flow deforms the Verma modules, so that the representation theory of the two sectors of the algebra is different. In [80] it was shown that all rational representations of the N=2 Virasoro algebra are necessarily unitary. For logarithmic superprimary fields with a degeneracy in h, the generalization is straightforward. Consider a Jordan cell of rank 2. The infinitesimal transformations are:

$$\begin{aligned}[L_n, \Phi'(\mathsf{Z})] = (h(m+1)z^m + \tfrac{1}{2}(m+1)z^m(\theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-}) + z^{m+1}\partial_z \\ +\tfrac{q}{2}\theta^+\theta^- z^{m-1}m(m+1))\Phi'(\mathsf{Z}) + z^m(m+1)\Phi(\mathsf{Z})\end{aligned} \tag{4.26}$$

$$\begin{aligned}[G_r^\pm, \Phi'(\mathsf{Z})] = (-z^{r+\frac{1}{2}}(\partial_{\theta^\mp} - \theta^\mp\partial_z) + z^{r-\frac{1}{2}}(r+\tfrac{1}{2})((2h \pm q)\theta^\pm \\ \pm\theta^+\theta^-\partial_{\theta^\mp}))\Phi'(\mathsf{Z}) + z^{r-\frac{1}{2}}(2r+1)\theta^\pm\Phi(\mathsf{Z})\end{aligned} \tag{4.27}$$

$$\begin{aligned}[J_n, \Phi'(\mathsf{Z})] = (2h\theta^+\theta^- m z^{m-1} + z^m(\theta^-\partial_{\theta^-} - \theta^+\partial_{\theta^+}) + qz^m)\Phi'(\mathsf{Z}) \\ +2h\theta^+\theta^- m z^{m-1}\Phi(\mathsf{Z}).\end{aligned} \tag{4.28}$$

Now one can consider Jordan cells with respect to J_0. Define a two-dimensional Jordan cell:

$$J_0|\Phi'\rangle = q|\Phi'\rangle + |\Phi\rangle$$
$$J_0|\Phi\rangle = q|\Phi\rangle.$$

Then the logarithmic superfield transforms, according to (4.23), (4.25), (4.24) as:

$$\begin{aligned}[L_n, \Phi'(\mathsf{Z})] = (h(m+1)z^m + \tfrac{1}{2}(m+1)z^m(\theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-}) + z^{m+1}\partial_z \\ +\tfrac{q}{2}\theta^+\theta^- z^{m-1}m(m+1))\Phi'(\mathsf{Z}) + \tfrac{1}{2}\theta^+\theta^- z^{m-1}m(m+1)\Phi(\mathsf{Z})\end{aligned}$$

$$\begin{aligned}[G_r^\pm, \Phi'(\mathsf{Z})] = (-z^{r+\frac{1}{2}}(\partial_{\theta^\mp} - \theta^\mp\partial_z) + z^{r-\frac{1}{2}}(r+\tfrac{1}{2})((2h \pm q)\theta^\pm \\ \pm\theta^+\theta^-\partial_{\theta^\mp}))\Phi'(\mathsf{Z}) \pm z^{r-\frac{1}{2}}(r+\tfrac{1}{2})\theta^\pm\Phi(\mathsf{Z})\end{aligned}$$

$$[J_n, \Phi'(\mathsf{Z})] = (2h\theta^+\theta^- m z^{m-1} + z^m(\theta^-\partial_{\theta^-} - \theta^+\partial_{\theta^+}) + qz^m)\Phi'(\mathsf{Z}) + z^m\Phi(\mathsf{Z}).$$

Although this kind of fields can be considered, we will see that J_0 is always diagonal. Even if one assumes an operator J_0 with Jordan block, the corresponding "logarithmic partners" decouple from the theory, in other words all correlation functions containing "logarithmic partners" are trivial. We will show this in the next sections.

4.6 Correlation Functions in N=2 SCFT

The ordinary N=2 vacuum is invariant under the orthosymplectic group OSp(2|2). The two point function is:

$$F_2 \equiv \langle \Phi(\mathsf{Z}_1)\Phi(\mathsf{Z}_2)\rangle.$$

It satisfies the following eight superdifferential equations corresponding to L_{-1}, L_0, L_1, J_0, $G^+_{-\frac{1}{2}}$, $G^-_{-\frac{1}{2}}$ $G^+_{\frac{1}{2}}$ and $G^-_{\frac{1}{2}}$, respectively:

$$(L^1_{-1}+L^2_{-1})F_2 = (\partial_{z_1}+\partial_{z_2})F_2 = 0 \tag{4.29}$$

$$\begin{aligned}(L^1_0+L^2_0)F_2 = (h_1+h_2+\tfrac{1}{2}\left(\theta^+_1\partial_{\theta^+_1}+\theta^-_1\partial_{\theta^-_1}+\theta^+_2\partial_{\theta^+_2}+\theta^-_2\partial_{\theta^-_2}\right)\\ +z_1\partial_{z_1}+z_2\partial_{z_2})F_2 = 0\end{aligned} \tag{4.30}$$

$$\begin{aligned}(L^1_1+L^2_1) = (2h_1z_1+2h_2z_2+z_1(\theta^+_1\partial_{\theta^+_1}+\theta^-_1\partial_{\theta^-_1})+z_2(\theta^+_2\partial_{\theta^+_2}+\theta^-_2\partial_{\theta^-_2})\\ +z_1^2\partial_{z_1}+z_2^2\partial_{z_2}+q_1\theta^+_1\theta^-_1+q_2\theta^+_2\theta^-_2)F_2 = 0\end{aligned} \tag{4.31}$$

$$(J^1_0+J^2_0)F_2 = (\theta^-_1\partial_{\theta^-_1}-\theta^+_1\partial_{\theta^+_1}+\theta^-_2\partial_{\theta^-_2}-\theta^+_2\partial_{\theta^+_2}+q_1+q_2)F_2 = 0 \tag{4.32}$$

$$(G^+_{-\frac{1}{2}}+G^+_{-\frac{1}{2}})F_2 = (-\partial_{\theta^-_1}-\partial_{\theta^-_2}+\theta^+_1\partial_{z_1}+\theta^+_2\partial_{z_2})F_2 = 0 \tag{4.33}$$

$$(G^-_{-\frac{1}{2}}+G^-_{-\frac{1}{2}})F_2 = (-\partial_{\theta^+_1}-\partial_{\theta^+_2}+\theta^-_1\partial_{z_1}+\theta^-_2\partial_{z_2})F_2 = 0 \tag{4.34}$$

$$\begin{aligned}(G^+_{\frac{1}{2}}+G^+_{\frac{1}{2}})F_2 = ((2h_1+q_1)\theta^+_1+(2h_2+q_2)\theta^+_2-z_1\partial_{\theta^-_1}-z_2\partial_{\theta^-_2}\\ +z_1\theta^+_1\partial_{z_1}+z_2\theta^+_2\partial_{z_2}+\theta^+_1\theta^-_1\partial_{\theta^-_1}+\theta^+_2\theta^-_2\partial_{\theta^-_2})F_2 = 0\end{aligned} \tag{4.35}$$

$$\begin{aligned}(G^-_{\frac{1}{2}}+G^-_{\frac{1}{2}})F_2 = ((2h_1-q_1)\theta^-_1+(2h_2-q_2)\theta^-_2-z_1\partial_{\theta^+_1}-z_2\partial_{\theta^+_2}\\ +z_1\theta^-_1\partial_{z_1}+z_2\theta^-_2\partial_{z_2}-\theta^+_1\theta^-_1\partial_{\theta^+_1}-\theta^+_2\theta^-_2\partial_{\theta^+_2})F_2 = 0.\end{aligned} \tag{4.36}$$

Since the two-point function depends on six coordinates it is fixed up to a constant. It can be expanded around the four Grassmannian variables θ^+_1, θ^-_1, θ^+_2 and θ^-_2, yielding sixteen components. One component is purely bosonic and not conjugate to any θ. Four components appear as functions of z_1, z_2 times one of the Grassmannian variables. Another six are conjugate to $\theta^+_1\theta^-_1$. $\theta^+_1\theta^+_2$, $\theta^+_1\theta^-_2$, $\theta^+_2\theta^-_2$, $\theta^-_1\theta^+_2$ and $\theta^-_1\theta^-_2$, there are four terms coming with one of the combinations of three θ's ($\theta^+_1\theta^-_1\theta^+_2$, $\theta^+_1\theta^-_1\theta^-_2$, $\theta^+_1\theta^+_2\theta^-_2$ and $\theta^-_1\theta^+_2\theta^-_2$). Additionally, there is one term proportial to the combination $\theta^+_1\theta^-_1\theta^+_2\theta^-_2$. The components have to be of same gradation, so the constants associated with combinations of one or three Grassmannian coordinates are Grassmannian-valued as well.
Using (4.29) and (4.30) constraints the ansatz to:

$$\begin{aligned}F_2 = z_{12}^{-h_1-h_2}(a+(b_1\theta^+_1+b_2\theta^-_1+b_3\theta^+_2+b_4\theta^-_2)z_{12}^{\frac{1}{2}}\\ +(c_1\theta^+_1\theta^-_1+c_2\theta^+_1\theta^+_2+c_3\theta^+_1\theta^-_2+c_4\theta^+_2\theta^-_2+c_5\theta^-_1\theta^+_2+c_6\theta^-_1\theta^-_2)z_{12}^{-1}\\ (d_1\theta^+_1\theta^-_1\theta^+_2+d_2\theta^+_1\theta^-_1\theta^-_2+d_3\theta^+_1\theta^+_2\theta^-_2+d_4\theta^-_1\theta^+_2\theta^-_2)z_{12}^{-\frac{3}{2}}\\ +e\theta^+_1\theta^-_1\theta^+_2\theta^-_2z_{12}^{-2})\end{aligned} \tag{4.37}$$

where b_i and d_i are Grassmannian constants. Using (4.31) yields $h_1 = h_2$. Constants b_i and d_i are fixed to be zero. This is because for $\Phi_{h_1,q_1}(\mathsf{Z}_1) = \phi_1(z_1)+\theta_1^+\psi_1^-(z_1)+\theta_1^-\psi_1^+(z_1)+\theta_1^+\theta_1^-g_1(z_1)$, the correlators of some component fields are zero:

$$\begin{aligned}\langle\phi_1(z_1)\theta_2^+\psi_2^-(z_2)\rangle = \langle\phi_1(z_1)\theta_2^-\psi_2^+(z_2)\rangle \\ = \langle\theta_1^+\theta_1^-g_1(z_1)\theta_2^+\psi_2^-(z_2)\rangle = \langle\theta_1^+\theta_1^-g(z_2)\theta_2^-\psi_2^+(z_2)\rangle = 0\end{aligned} \tag{4.38}$$

and corresponding equations with exchanged lower indices. Remaining constants satisfy $c_1 = -aq_1$, $c_4 = aq_2$, $q_1 = -q_2$. Equations (4.35) and (4.36) return $c_6 = c_2 = 0$, $c_1 = a(-h_1 + h_2 - q_1)$, $c_3 = a(2h_1 + q_1)$, $c_4 = aq_2)$ and $c_5 = a(2h_2+q_2)$ for the components proportional to one anticommuting variable, which in particular means:

$$\langle\theta^+\psi_1^-(z_1)\theta_2^+\psi_2^-(z_2)\rangle = \langle\theta_1^-\psi_1^+(z_1)\theta_2^-\psi_2^+(z_2)\rangle = 0.$$

The only remaining constant is e and it can be fixed as $e = h(h+1)$ by solving equation (4.35) for level three. Putting all constants together (4.37) reads:

$$\begin{aligned}\langle\Phi(\mathsf{Z}_1)\Phi(\mathsf{Z}_2)\rangle = \tfrac{1}{z_{12}^{2h}}(1 + \tfrac{q_2(\theta_1^+\theta_1^-+\theta_2^+\theta_2^-)}{z_{12}} \\ +\tfrac{(2h-q_2)\theta_1^+\theta_2^-}{z_{12}} + \tfrac{(2h+q_2)\theta_1^-\theta_2^+}{z_{12}} + \tfrac{2h(2h+1)\theta_1^+\theta_2^-\theta_1^-\theta_2^+}{z_{12}^2}).\end{aligned} \tag{4.39}$$

Now the usefulness of the set of coordinates we call the superdifferences Z_{12}, θ_{12}^+, θ_{12}^- for solving conformal Ward identities can be demonstrated. We introduce a complimentary set of coordinates W_{12}, ξ_{12}^+, ξ_{12}^- (which one might call the "supersums"):

$$\mathsf{Z}_{12} = z_1 - z_2 - \theta_1^+\theta_2^- - \theta_1^-\theta_2^+ \tag{4.40}$$

$$\mathsf{W}_{12} = z_1 + z_2 - \theta_1^+\theta_2^- - \theta_1^-\theta_2^+ \tag{4.41}$$

$$\theta_{12}^+ = \theta_1^+ - \theta_2^+, \qquad \theta_{12}^- = \theta_1^- - \theta_2^- \tag{4.42}$$

$$\xi_{12}^+ = \theta_1^+ + \theta_2^+, \qquad \xi_{12}^- = \theta_1^- + \theta_2^-. \tag{4.43}$$

Rewriting the conformal Ward identities (4.29)-(4.36) in the new coordinates and simplyfing leads a set of equations which are more convenient to solve. In particular, for the Ward identity corresponding to L_{-1} one obtains:

$$\partial_{\mathsf{W}_{12}}F_2 = 0 \tag{4.44}$$

$G_{\frac{1}{2}}^+$ and $G_{\frac{1}{2}}^-$, annihilating the vacuum, produce the following set of simple equations that immediately imply $F_2 = F_2(\mathsf{Z}_{12}, \theta_{12}^+, \theta_{12}^-)$:

$$\partial_{\xi_{12}^+}F_2 = 0 \tag{4.45}$$

$$\partial_{\xi_{12}^-}F_2 = 0. \tag{4.46}$$

This can be used to simplify the equation corresponding to L_0, producing:

$$(h_1 + h_2 + \mathsf{Z}_{12}\partial_{\mathsf{Z}_{12}} + \tfrac{1}{2}(\theta_{12}^+\partial_{\theta_{12}^+} + \theta_{12}^-\partial_{\theta_{12}^-}))F_2 = 0. \tag{4.47}$$

We take the ansatz:

$$F_2(\mathsf{Z}_{12},\theta_{12}^+,\theta_{12}^-)=f(\mathsf{Z}_{12})+\theta_{12}^+f_+(\mathsf{Z}_{12})+\theta_{12}^-f_-(\mathsf{Z}_{12})+\theta_{12}^+\theta_{12}^-f_{+-}(\mathsf{Z}_{12})$$

to take the form:

$$F_2=a\mathsf{Z}_{12}^{-h_1-h_2}+b_+\theta_{12}^+\mathsf{Z}_{12}^{-h_1-h_2}+b_-\theta_{12}^-\mathsf{Z}_{12}^{-h_1-h_2-\frac{1}{2}}+c\theta_{12}^+\theta_{12}^-\mathsf{Z}_{12}^{-h_1-h_2-1}.$$

Introducing the new version of J_0-action (4.32) first, using (4.45) and (4.46) yields:

$$(q_1+q_2-\theta_{12}^+\partial_{\theta_{12}^+}+\theta_{12}^-\partial_{\theta_{12}^-})F_2=0. \tag{4.48}$$

Using the ansatz, we obtain three possible solutions:

$$q_1+q_2=b_+=b_-=0 \tag{4.49}$$

$$q_1+q_2+1=a=b_-=c=0 \tag{4.50}$$

$$q_1+q_2-1=a=b_+=c=0. \tag{4.51}$$

The action of L_1 on the vacuum results in the following equation:

$$\begin{gathered}(2(h_1-h_2)(2\mathsf{Z}_{12}+\theta_{12}^+\xi_{12}^-+\theta_{12}^-\xi_{12}^+)\\ +2\mathsf{Z}_{12}(\xi_{12}^+\partial_{\theta_{12}^+}+\xi_{12}^-\partial_{\theta_{12}^-})-(\theta_{12}^+\xi_{12}^-+\theta_{12}^-\xi_{12}^+)(\theta_{12}^+\partial_{\theta_{12}^+}+\theta_{12}^-\partial_{\theta_{12}^-})\\ +2q_2(\theta_{12}^+\xi_{12}^-+\xi_{12}^+\theta_{12}^-))F_2=0.\end{gathered} \tag{4.52}$$

This equation rules out (4.50) and (4.51) and fixes $h_1=h_2$ and $c=aq_2$. Thus, the two-point function is:

$$F_2=a(\mathsf{Z}_{12}^{-h_1-h_2}+q_2\theta_{12}^+\theta_{12}^-\mathsf{Z}_{12}^{-h_1-h_2-1}). \tag{4.53}$$

This result appeared in [73] as:

$$\langle\Phi(\mathsf{Z}_1)\Phi(\mathsf{Z}_2)\rangle=\mathsf{Z}_{12}^{-2h_1}\mathrm{e}^{q_2\frac{\theta_{12}^+\theta_{12}^-}{\mathsf{Z}_{12}}}\delta_{h_1,h_2}\delta_{q_1,-q_2}. \tag{4.54}$$

$G_{\frac{1}{2}}^{\pm}$ leads to the following set of Ward identities:

$$\begin{gathered}(((h_1-h_2)\pm\tfrac{1}{2}(q_1-q_2))\theta_{12}^{\pm}+((h_1+h_2)\pm\tfrac{1}{2}(q_1+q_2))\xi_{12}^{\pm}\\ +(\xi_{12}^{\pm}\theta_{12}^{\mp}-\mathsf{Z}_{12})\partial_{\theta_{12}^{\mp}}+\mathsf{Z}_{12}\xi_{12}^{\pm}\partial_{\mathsf{Z}_{12}})F_2=0.\end{gathered}$$

which are not necessary due to $[L_1,G_{-\frac{1}{2}}^{\pm}]=G_{\frac{1}{2}}^{\pm}$. In the same way, (4.47) is not necessary due to $[L_1,L_{-1}]=2L_0$.
The three-point function F_3 is dependent on nine variables:

$$F_3=F_3(z_1,z_2,z_3,\theta_1^+,\theta_1^-,\theta_2^+,\theta_2^-,\theta_3^+,\theta_3^-,\xi_1^+,\xi_1^-,\xi_2^+,\xi_2^-,\xi_3^+,\xi_3^-). \tag{4.55}$$

The superdifferences and supersums read:

$$\mathsf{Z}_{ij}=z_i-z_j-\theta_i^+\theta_j^--\theta_i^-\theta_j^+$$

$$\mathsf{W}_{ij}=z_i+z_j-\theta_i^+\theta_j^--\theta_i^-\theta_j^+$$

$$\theta^{\pm}_{ij} = \theta^{\pm}_i - \theta^{\pm}_j$$
$$\xi^{\pm}_{ij} = \theta^{\pm}_i + \theta^{\pm}_j.$$

The three-point function was found in [73]. For the purpose of this thesis, it is necessary to re-derive this result using superconformal Ward identities to analyse logarithmic behaviour of superfields. The three-point function depends on nine variables, but spans a six-dimensional space. Since there are eight Ward identities, there must be a quantity $R(\mathsf{Z}_{12}, \mathsf{Z}_{31}, \mathsf{Z}_{23}, \theta^+_{12}, \theta^+_{31}, \theta^+_{23}, \theta^-_{12}, \theta^-_{31}, \theta^-_{23})$ which is invariant under the remaining differential operators. The general solution is a special solution multiplied by an arbitrary function of the invariant. This invariant is known [73] and reads:

$$R = \frac{\theta^+_{12}\theta^-_{12}}{\mathsf{Z}_{12}} + \frac{\theta^+_{31}\theta^-_{31}}{\mathsf{Z}_{31}} + \frac{\theta^+_{23}\theta^-_{23}}{\mathsf{Z}_{23}}. \tag{4.56}$$

Although not obvious at the first glance, this invariant is nilpotent: $R^2 = 0$. As usual, we first use translational invariance:

$$(L^1_{-1} + L^2_{-1} + L^3_{-1})F_3 = (\partial_{z_1} + \partial_{z_2} + \partial_{z_3})F_3 = 0. \tag{4.57}$$

Using differences $z_{ij} = z_i - z_j$ and sums $w_{ij} = w_i + w_j$ we first realize that the three differences are not linearly independent. Thus, to span the whole three-dimensional space we must use one of the w_{ij} to build a basis. For the choice $\{z_{12}, z_{31}, w_{23}\}$ equation (4.57) returns $\partial_{w_{23}} F_3 = 0$. This and other choices are equivalent to the following differential equations:

$$\partial_{\mathsf{W}_{12}} F_3 = \partial_{\mathsf{W}_{31}} F_3 = \partial_{\mathsf{W}_{23}} F_3 = 0.$$

Thus, translational invariance imposes the dependence on Z_{12}, Z_{31}, Z_{23} without missing any solutions.

Because the differences span a two-dimensional space, dependence of the three-point function on only two of the z_{ij} is required, though we keep all three of them to give the solution a more symmetric form. We use the identities imposed by the invariance under $G^{\pm}_{\frac{1}{2}}$:

$$\begin{gathered}(G^{+,1}_{-\frac{1}{2}} + G^{+,2}_{-\frac{1}{2}} + G^{+,3}_{-\frac{1}{2}})F_3 = \\ (-\partial_{\theta^-_1} - \partial_{\theta^-_2} - \partial_{\theta^-_3} + \theta^+_1\partial_{z_1} + \theta^+_2\partial_{z_2} + \theta^+_3\partial_{z_3})F_3 = 0\end{gathered} \tag{4.58}$$

$$\begin{gathered}(G^{-,1}_{-\frac{1}{2}} + G^{-,2}_{-\frac{1}{2}} + G^{-,3}_{-\frac{1}{2}})F_3 = \\ (-\partial_{\theta^+_1} - \partial_{\theta^+_2} - \partial_{\theta^+_3} + \theta^-_1\partial_{z_1} + \theta^-_2\partial_{z_2} + \theta^-_3\partial_{z_3})F_3 = 0.\end{gathered} \tag{4.59}$$

In terms of supersums and superdifferences these equations read:

$$(\partial_{\xi^-_{12}} + \partial_{\xi^-_{31}} + \partial_{\xi^-_{23}})F_3 = 0 \tag{4.60}$$

$$(\partial_{\xi^+_{12}} + \partial_{\xi^+_{31}} + \partial_{\xi^+_{23}})F_3 = 0. \tag{4.61}$$

Again, we can eliminate the dependence of two of the $\xi^{\pm}_{ij}$ to obtain:

$$\partial_{\xi^+_{12}} F_3 = \partial_{\xi^+_{31}} F_3 = \partial_{\xi^+_{23}} F_3 = \partial_{\xi^-_{12}} F_3 = \partial_{\xi^-_{31}} F_3 = \partial_{\xi^-_{23}} F_3 = 0.$$

Global dilation invariance implies:

$$\begin{gathered}(L_0^1+L_0^2+L_0^3)F_3=\\(h_1+h_2+h_3+\tfrac{1}{2}\left(\theta_1^+\partial_{\theta_1^+}+\theta_1^-\partial_{\theta_1^-}+\theta_2^+\partial_{\theta_2^+}+\theta_2^-\partial_{\theta_2^-}+\theta_3^+\partial_{\theta_3^+}+\theta_3^-\partial_{\theta_3^-}\right)\\+z_1\partial_{z_1}+z_2\partial_{z_2}+z_3\partial_{z_3})F_3=0.\end{gathered}\tag{4.62}$$

This translates to:

$$\begin{gathered}(L_0^1+L_0^2+L_0^3)F_3=\\(h_1+h_2+h_3+\mathsf{Z}_{12}\partial_{\mathsf{Z}_{12}}+\mathsf{Z}_{31}\partial_{\mathsf{Z}_{31}}+\mathsf{Z}_{23}\partial_{\mathsf{Z}_{23}}\\+\tfrac{1}{2}(\theta_{12}^+\partial_{\theta_{12}^+}+\theta_{12}^-\partial_{\theta_{12}^-}+\theta_{31}^+\partial_{\theta_{31}^+}+\theta_{31}^-\partial_{\theta_{31}^-}+\theta_{23}^+\partial_{\theta_{23}^+}+\theta_{23}^-\partial_{\theta_{23}^-}))F_3=0.\end{gathered}\tag{4.63}$$

Expanding the ansatz, one obtains sixty-four term with corresponding functions of Z_{ij}. Taking a closer look at (4.63), it is obvious that functions conjugate $n,m,k\in\{1,2\}$ Grassmannian variables of the type $\theta_{12}^{\pm},\theta_{31}^{\pm},\theta_{23}^{\pm}$ are of the form $\mathsf{Z}_{12}^{-h_1-h_2-\frac{n}{2}+h_3}\mathsf{Z}_{31}^{-h_1-h_3-\frac{n}{2}+h_2}\mathsf{Z}_{23}^{-h_2-h_3-\frac{n}{2}+h_1}$. The differential equation generated by J_0-invariance reads:

$$\begin{gathered}(J_0^1+J_0^2+J_0^3)F_3=\\(\theta_1^-\partial_{\theta_1^-}-\theta_1^+\partial_{\theta_1^+}+\theta_2^-\partial_{\theta_2^-}-\theta_2^+\partial_{\theta_2^+}+\theta_3^-\partial_{\theta_3^-}-\theta_3^+\partial_{\theta_3^+}+q_1+q_2+q_3)F_3=0.\end{gathered}\tag{4.64}$$

It translates to the new variables :

$$\begin{gathered}(J_0^1+J_0^2+J_0^3)F_3=\\(\theta_{12}^-\partial_{\theta_{12}^-}-\theta_{12}^+\partial_{\theta_{12}^+}+\theta_{23}^-\partial_{\theta_{23}^-}-\theta_{23}^+\partial_{\theta_{23}^+}+\theta_{31}^-\partial_{\theta_{31}^-}-\theta_{31}^+\partial_{\theta_{31}^+}+q_1+q_2+q_3)F_3=0.\end{gathered}\tag{4.65}$$

This equation is satisfied for terms where a variable $\theta_{ij}\pm$ always appears together with its counterpart variable $\theta_{ij}^{\mp}$. In this case, we have $q_1+q_2+q_3=0$. The number of terms is reduced to eight.

$$\begin{gathered}(L_1^1+L_1^2+L_1^3)F_3=(2h_1z_1+2h_2z_2+2h_3z_3\\+z_1(\theta_1^+\partial_{\theta_1^+}+\theta_1^-\partial_{\theta_1^-})+z_2(\theta_2^+\partial_{\theta_2^+}+\theta_2^-\partial_{\theta_2^-})+z_3(\theta_3^+\partial_{\theta_3^+}+\theta_3^-\partial_{\theta_3^-})\\+z_1^2\partial_{z_1}+z_2^2\partial_{z_2}+z_3^2\partial_{z_3}+q_1\theta_1^+\theta_1^-+q_2\theta_2^+\theta_2^-+q_3\theta_3^+\theta_3^-)F_3=0.\end{gathered}\tag{4.66}$$

Substituting and simplifying, the Ward identity can be brought in the following form:

$$\begin{aligned}
&(L_1^1 + L_1^2 + L_1^3)F_3 = \\
&(h_1(\mathsf{W}_{31} - \mathsf{Z}_{31}) + h_2(\mathsf{W}_{12} - \mathsf{Z}_{12}) + h_3(\mathsf{W}_{23} - \mathsf{Z}_{23}) \\
&+\tfrac{1}{2}(\mathsf{Z}_{12}\xi_{12}^+ + \mathsf{W}_{12}\theta_{12}^+ + \tfrac{1}{2}(\xi_{12}^- + \theta_{12}^-)\xi_{12}^+\theta_{12}^+)\partial_{\theta_{12}^+} \\
&+\tfrac{1}{2}(\mathsf{Z}_{12}\xi_{12}^- + \mathsf{W}_{12}\theta_{12}^- + \tfrac{1}{2}(\xi_{12}^+ + \theta_{12}^+)\xi_{12}^-\theta_{12}^-)\partial_{\theta_{12}^-} \\
&+\tfrac{1}{2}(\mathsf{Z}_{23}\xi_{23}^+ + \mathsf{W}_{23}\theta_{23}^+ + \tfrac{1}{2}(\xi_{23}^- + \theta_{23}^-)\xi_{23}^+\theta_{23}^+)\partial_{\theta_{23}^+} \\
&+\tfrac{1}{2}(\mathsf{Z}_{23}\xi_{23}^- + \mathsf{W}_{23}\theta_{23}^- + \tfrac{1}{2}(\xi_{23}^+ + \theta_{23}^+)\xi_{23}^-\theta_{23}^-)\partial_{\theta_{23}^-} \\
&+\tfrac{1}{2}(\mathsf{Z}_{31}\xi_{31}^+ + \mathsf{W}_{31}\theta_{31}^+ + \tfrac{1}{2}(\xi_{31}^- + \theta_{31}^-)\xi_{31}^+\theta_{31}^+)\partial_{\theta_{31}^+} \\
&+\tfrac{1}{2}(\mathsf{Z}_{31}\xi_{31}^- + \mathsf{W}_{31}\theta_{31}^- + \tfrac{1}{2}(\xi_{31}^+ + \theta_{31}^+)\xi_{31}^-\theta_{31}^-)\partial_{\theta_{31}^-} \\
&+(\mathsf{W}_{12} - \tfrac{1}{2}(\xi_{12}^+\theta_{12}^- + \xi_{12}^-\theta_{12}^+))\mathsf{Z}_{12}\partial_{\mathsf{Z}_{12}} \\
&+(\mathsf{W}_{23} - \tfrac{1}{2}(\xi_{23}^+\theta_{23}^- + \xi_{23}^-\theta_{23}^+))\mathsf{Z}_{23}\partial_{\mathsf{Z}_{23}} \\
&+(\mathsf{W}_{31} - \tfrac{1}{2}(\xi_{31}^+\theta_{31}^- + \xi_{31}^-\theta_{31}^+))\mathsf{Z}_{31}\partial_{\mathsf{Z}_{31}} \\
&+\tfrac{q_1}{4}(\xi_{12}^+ + \theta_{12}^+)(\xi_{12}^- + \theta_{12}^-) + \tfrac{q_2}{4}(\xi_{23}^+ + \theta_{23}^+)(\xi_{23}^- + \theta_{23}^-) \\
&+\tfrac{q_3}{4}(\xi_{31}^+ + \theta_{31}^+)(\xi_{31}^- + \theta_{31}^-))F_3 = 0.
\end{aligned} \tag{4.67}$$

Using the ansatz:

$$\Delta_{ij} = h_j + h_j - h_k$$

$$\eta(\mathsf{Z}) = \mathsf{Z}_{12}^{-\Delta_{12}}\mathsf{Z}_{31}^{-\Delta_{31}}\mathsf{Z}_{23}^{-\Delta_{23}}$$

$$F_3 = \eta(\mathsf{Z})\sigma(\mathsf{Z}, \theta)$$

where $\sigma(\mathsf{Z}, \theta)$ is a function with dimension one, and writing:

$$(L_1^1 + L_1^2 + L_1^3)F_3 = (L_q + L_h + L_{\partial_\theta} + L_{\partial_\mathsf{Z}})F_3 = 0$$

where L_q, L_h, L_{∂_θ} and L_{∂_Z} are terms containing q, h, derivatives with respect to one of the θ's and Z's, respectively, we can obtain a differential equation for σ. First we note that:

$$(L_{\partial_\mathsf{Z}} + L_h)\eta = 0.$$

This leads to the equation:

$$L_q\sigma + L_{\partial_\theta}\sigma + L_{\partial_\mathsf{Z}}\sigma = 0. \tag{4.68}$$

From the structure of this equation one might guess the suitable ansatz:

$$\begin{aligned}
F_3 &= \mathsf{Z}_{12}^{-\Delta_{12}}\mathsf{Z}_{31}^{-\Delta_{31}}\mathsf{Z}_{23}^{-\Delta_{23}} \\
&\cdot \exp(A_{12}\tfrac{\theta_{12}^+\theta_{12}^-}{\mathsf{Z}_{12}} + A_{31}\tfrac{\theta_{31}^+\theta_{31}^-}{\mathsf{Z}_{31}} + A_{23}\tfrac{\theta_{23}^+\theta_{23}^-}{\mathsf{Z}_{23}}).
\end{aligned}$$

Indeed, this equation satisfies the Ward identity for L_1. Expanding σ we have:

$$\begin{aligned}
\sigma = 1 &+ A_{12}\tfrac{\theta_{12}^+\theta_{12}^-}{\mathsf{Z}_{12}} + A_{31}\tfrac{\theta_{31}^+\theta_{31}^-}{\mathsf{Z}_{31}} + A_{23}\tfrac{\theta_{23}^+\theta_{23}^-}{\mathsf{Z}_{23}} \\
+A_{12}A_{23}\tfrac{\theta_{12}^+\theta_{12}^-\theta_{23}^+\theta_{23}^-}{\mathsf{Z}_{12}\mathsf{Z}_{23}} &+ A_{12}A_{31}\tfrac{\theta_{12}^+\theta_{12}^-\theta_{31}^+\theta_{31}^-}{\mathsf{Z}_{12}\mathsf{Z}_{31}} + A_{23}A_{31}\tfrac{\theta_{23}^+\theta_{23}^-\theta_{31}^+\theta_{31}^-}{\mathsf{Z}_{23}\mathsf{Z}_{31}}.
\end{aligned} \tag{4.69}$$

The next term disappears due to $\theta^{\pm}_{12}\theta^{\pm}_{23}\theta^{\pm}_{31} = 0$. All of the terms chancel in (4.68) except for L_q and terms containing $\mathsf{Z}_{ij}\partial_{\theta^{\pm}_{ij}}$:

$$\begin{aligned} L_q\sigma + \tfrac{1}{2}(\mathsf{Z}_{12}\xi^{+}_{12}\partial_{\theta^{+}_{12}} + \mathsf{Z}_{12}\xi^{+-}_{12}\partial_{\theta^{-}_{12}} + \mathsf{Z}_{23}\xi^{+}_{23}\partial_{\theta^{+}_{23}} + \mathsf{Z}_{23}\xi^{-}_{23}\partial_{\theta^{-}_{23}} \\ +\mathsf{Z}_{31}\xi^{+}_{31}\partial_{\theta^{+}_{31}} + \mathsf{Z}_{31}\xi^{-}_{31}\partial_{\theta^{-}_{31}})\sigma \equiv \\ (L_q + L_{\mathsf{Z}\xi\partial_\theta})\sigma = 0. \end{aligned} \tag{4.70}$$

Now it can be verified easily that (the index i on σ indicates the therm containing i nilpotent variables):

$$\begin{aligned} 0 =& \quad L_{\mathsf{Z}\xi\partial_\theta}\sigma_0 \\ L_q\sigma_0 =& \quad -L_{\mathsf{Z}\xi\partial_\theta}\sigma_2 \\ L_q\sigma_2 =& \quad -L_{\mathsf{Z}\xi\partial_\theta}\sigma_4 \\ L_q\sigma_4 =& \quad 0, \end{aligned} \tag{4.71}$$

with following A_{ij}:

$$\begin{aligned} q_1 &= A_{31} - A_{12} \\ q_2 &= A_{12} - A_{23} \\ q_3 &= A_{23} - A_{31} \\ \Rightarrow q_1 &+ q_2 + q_3 = 0. \end{aligned}$$

Thus, the complete ordinary N=2 three-point function reads:

$$\begin{aligned} F_3 = C_{123}\mathsf{Z}_{12}^{-\Delta_{12}}\mathsf{Z}_{31}^{-\Delta_{31}}\mathsf{Z}_{23}^{-\Delta_{23}} \\ \cdot \exp(A_{12}\frac{\theta^{+}_{12}\theta^{-}_{12}}{\mathsf{Z}_{12}} + A_{31}\frac{\theta^{+}_{31}\theta^{-}_{31}}{\mathsf{Z}_{31}} + A_{23}\frac{\theta^{+}_{23}\theta^{-}_{23}}{\mathsf{Z}_{23}})(1+\alpha R)\delta_{q_1+q_2+q_3,0} = 0. \end{aligned}$$

It is easy to see that multiplying by the invariant R is equivalent to addition of α to the coefficients A_{ij}. For completeness, we present the Ward identities corresponding to $G^{\pm}_{-\frac{1}{2}}$, which in new coordinates read:

$$\begin{aligned} \big(-(\mathsf{Z}_{12} - \xi^{\pm}_{12}\theta^{\mp}_{12})\partial_{\theta^{\mp}_{12}} - (\mathsf{Z}_{23} - \xi^{\pm}_{23}\theta^{\mp}_{23})\partial_{\theta^{\mp}_{23}} - (\mathsf{Z}_{31} - \xi^{\pm}_{31}\theta^{\mp}_{31})\partial_{\theta^{\mp}_{31}} \\ +\xi^{\pm}_{12}\mathsf{Z}_{12}\partial_{\mathsf{Z}_{12}} + \xi^{\pm}_{23}\mathsf{Z}_{23}\partial_{\mathsf{Z}_{23}} + \xi^{\pm}_{31}\mathsf{Z}_{31}\partial_{\mathsf{Z}_{31}} \\ +(\xi^{\pm}_{12} + \theta^{\pm}_{12})(h_1 \pm \tfrac{q_1}{2}) + (\xi^{\pm}_{23} + \theta^{\pm}_{23})(h_2 \pm \tfrac{q_2}{2}) + (\xi^{\pm}_{31} + \theta^{\pm}_{31})(h_3 \pm \tfrac{q_3}{2})\big)F_3 = 0. \end{aligned}$$

The general n-point function in N=2 superconformal theory reads:

$$F_n = \left(\prod_{i<j}^{n} \mathsf{Z}_{ij}^{-\Delta_{ij}}\right) \exp\left(\sum_{i<j}^{n} A_{ij}\frac{\theta^{+}_{ij}\theta^{-}_{ij}}{\mathsf{Z}_{ij}}\right) f(x_1, ...x_{3n-8})\delta_{\sum_{i=1}^{n} q_i = 0}. \tag{4.72}$$

Here, $f(x_1, ...x_{3n-8})$ is a function of the $3n-8$ invariants and the constants satisfy:

$$A_{ij} = -A_{ji}, \qquad \Delta_{ij} = \Delta_{ji}, \qquad \sum_{j=1, i\neq j}^{n} A_{ij} = -q_i, \qquad \sum_{i=1, i\neq j}^{n} \Delta_{ij} = 2\Delta_i.$$

4.7 Correlation Functions in N=2 LSCFT

There have been different approaches considering correlation functions in logarithmic field theory. We will mostly follow [45], this approach is very general and whose notation is the most compact. For a one-form dz^h on the Riemann sphere, we can consider replacing $h \to h\mathbb{I}_n$. The differential form acts on the n-component vector, containing conformal fields of dimension h:

$$\Phi = \begin{pmatrix} \phi_0 \\ \vdots \\ \phi_{n-1} \end{pmatrix}.$$

One can go further and try to generalize the one form by adding a non-diagonal term $h\mathbb{I} \to h\mathbb{I} + H$, which represents the Jordan cell structure of the dilation generator L_0 (the unit matrix is going to be omitted from now on). This matrix is obviously nilpotent of degree n: $H_1^n = 0$. Using $a^b = \exp(b \log a)$, we arrive at the following expression:

$$dz^{h+H_1} = dz^h \sum_{n=0}^{n-1} \frac{H^i (\log dz)^i}{i!}.$$

Although there appears not to be a rigorous treatment of the somewhat unusual quantity $\log dz$ in the literature, the differential form seems to be well-defined and leads to the transformation law:

$$\Phi'(z) = \left(\frac{\partial z'}{\partial z}\right)^{h+H} \Phi(z').$$

In the next step, we change the description of Φ from a vector bundle to the associated G-bundle, which amounts to the replacement of vector Φ by a matrix. To write down the matrix explicitly, one has to fix the convention since we have the choice between H of the form:

$$H = \begin{pmatrix} 0 & 1 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & 1 \\ 0 & 0 & \ddots & 0 \end{pmatrix},$$

which is called a *upper shift matrix*, or of the form:

$$H = \begin{pmatrix} 0 & \ddots & 0 & 0 \\ 1 & \ddots & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & 1 & 0 \end{pmatrix},$$

which is a *lower shift matrix.* By acting on a matrix with a upper (lower) shift matrix literally shifts the components one row up (down) and replaces the last (first) row with zeroes. Here, without loss of generality, we introduce non-diagonalizable structure using upper shift matrices. Then the (quasi-)primary fields are expressed as:

$$\Phi = \begin{pmatrix} \phi_0 \\ \vdots \\ \phi_{n-1} \end{pmatrix} \quad \rightarrow \quad \mathbf{\Phi} = \sum_{i=0}^{n-1} H^i \phi_{n-1-i} = \begin{pmatrix} \phi_{n-1} & \phi_{n-2} & \ddots & \phi_0 \\ 0 & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \phi_{n-2} \\ 0 & \ddots & 0 & \phi_{n-1} \end{pmatrix}.$$

The n-point function can be expressed as the vacuum expectation value of the tensor product of the fields, with explicit dependence on H_i:

$$\mathbf{F}_n(z_1, ..., z_n, \theta_1^+, ..., \theta_n^+, \theta_1^-, ..., \theta_n^-, H_1, ..., H_n) =$$
$$\langle 0|\mathbf{\Phi}(z_1, \theta_1^+, \theta_1^-, H_1) \otimes ... \otimes \mathbf{\Phi}(z_n, \theta_1^+, \theta_n^-, H_n)|0\rangle.$$

This matrix-valued correlation function contains all the information about correlation functions between primary and quasiprimary fields which are or are not part of a Jordan cell. The tensor product signs are going to be dropped in the following. Let us turn our attention to the logarithmic version of the two-point function. One can consider Jordan cells in correlation functions by replacing:

$$h_1 \rightarrow \mathbb{I}h_1 + J, \qquad h_2 \rightarrow \mathbb{I}h_2 + K$$
$$q_1 \rightarrow \mathbb{I}q_1 + P, \qquad h_2 \rightarrow \mathbb{I}h_2 + Q.$$

The matrices J, K, P, Q take the off-diagonal action of L_0 into account and have rank M, N, R and S, respectively. They are nilpotent:

$$J^M = K^N = P^R = Q^S = 0$$
$$J^{M-1}, K^{N-1}, P^{R-1} =, Q^{S-1} \neq 0.$$

Equations (4.44), (4.45) and (4.46) do not involve h, q and are not modified by off-diagonal action so we can formally write:

$$F_2(z_1, z_2, \theta_1^+, \theta_1^-, \theta_2^+, \theta_2^-, J, K, P, Q) = F_2(\mathsf{Z}_{12}, \theta_{12}^+, \theta_{12}^-, J, K, P, Q).$$

Equation (4.47) now reads:

$$(h_1 + J + h_2 + K + \mathsf{Z}_{12}\partial_{\mathsf{Z}_{12}} + \tfrac{1}{2}(\theta_{12}^+\partial_{\theta_{12}^+} + \theta_{12}^-\partial_{\theta_{12}^-}))F_2 = 0.$$

This has the solution:

$$F_2 = A\mathsf{Z}_{12}^{-h_1-J-h_2-K} + B_+\theta_{12}^+\mathsf{Z}_{12}^{-h_1-J-h_2-K-\frac{1}{2}} + B_-\theta_{12}^-\mathsf{Z}_{12}^{-h_1-J-h_2-K-\frac{1}{2}}$$
$$+C\theta_{12}^+\theta_{12}^-\mathsf{Z}_{12}^{-h_1-J-h_2-K-1}.$$

Here, the coefficients are constant matrices which depend on the choice of the Jordan cells involved. Equation (4.48) in the logarithmic case reads:

$$(q_1 + P + q_2 + Q - \theta^+_{12}\partial_{\theta^+_{12}} + \theta^-_{12}\partial_{\theta^-_{12}})F_2 = 0.$$

This returns following conditions on the prefactors:

$$(q_1 + q_2 + P + Q)C = 0.$$

$$(q_1 + q_2 + P + Q + 1)B_+ = 0$$
$$(q_1 + q_2 + P + Q - 1)B_- = 0$$
$$(q_1 + q_2 + P + Q)A = 0.$$

The matrix $P + Q$ is per definition invertible because it is non-singular, so is $\mathbb{I}(q_1 + q_2)$. Multiplying with the respective inverse matrices yields three non-trivial (where not all the coefficient matrices are zero) solutions:

$$q_1 + q_2 = 0 \Rightarrow\rightarrow B_+ = B_- = 0, \quad (P + Q)A = (P + Q)C = 0$$

$$q_1 + q_2 = 1 \Rightarrow A = B_+ = C = 0, \quad (P + Q)B_- = 0$$
$$q_1 + q_2 = -1 \Rightarrow A = B_- = C = 0, \quad (P + Q)B_+ = 0.$$

Finally, we use the logarithmic version of (4.52):

$$\begin{gathered}(2(h_1 + J - h_2 - K)(2\mathsf{Z}_{12} + \theta^+_{12}\xi^-_{12} + \theta^-_{12}\xi^+_{12}) \\ +2\mathsf{Z}_{12}(\xi^+_{12}\partial_{\theta^+_{12}} + \xi^-_{12}\partial_{\theta^-_{12}}) - (\theta^+_{12}\xi^-_{12} + \theta^-_{12}\xi^+_{12})(\theta^+_{12}\partial_{\theta^+_{12}} + \theta^-_{12}\partial_{\theta^-_{12}}) \\ +2(q_2 + Q)(\theta^+_{12}\xi^-_{12} + \xi^+_{12}\theta^-_{12}))F_2 = 0.\end{gathered}$$

The only possible non-trivial solution remaining is $q_1 + q_2 = 0$. Additionally, one obtains $C = A(q_2 + Q)$, $h_1 = h_2$ and $(J - K)A = 0$. Thus, the most general logarithmic two-point function is:

$$F_2 = A(\mathsf{Z}_{12}^{-2h_1-J-K} + (q_2 + Q)\theta^+_{12}\theta^-_{12}\mathsf{Z}^{-2h_1-J-K-1})\delta_{h_1,h_2}\delta_{q_1,-q_2} \tag{4.73}$$
$$(J - K)A = (P + Q)A = 0.$$

It can easily be seen that logarithmic behaviour is reproduced for indecomposable L_0. For a two-dimensional Jordan cell of fields containing Φ' and Φ with $P = Q = 0$, one can retrieve correlation functions:

$$\langle\Phi_{h,q}\Phi_{h,-q}\rangle = 0$$

$$\langle\Phi'_{h,q}\Phi_{h,-q}\rangle = a(\mathsf{Z}_{12}^{-2h} + q_2\theta^+_{12}\theta^-_{12}\mathsf{Z}_{12}^{-2h_1-1})$$
$$\langle\Phi'_{h,q}\Phi'_{h,-q}\rangle = \mathsf{Z}_{12}^{-2h}(1 + q_2\theta^+_{12}\theta^-_{12}\mathsf{Z}^{-1})(a + b\log(\mathsf{Z}_{12}).$$

In theories with an extended Cartan algebra, one has to account for the eventually appearing indecomposable representations with respect to the additional elements of the algebra. By introducing Jordan blocks by hand, one obtains a large variety of possible transformation properties on the space of fields.

Previously in the literature [45], it was assumed that the zero mode of the affine $U(1)$-current J which appears in the N=2 theory is in general non-diagonalizable and can account for a more complex logarithmic structure compared with N=0 and N=1 cases. Although in general operators of an extended algebra can posses non-diagonalizable structure, in case of supersymmetrically extended algebras this turns out to be incorrect.

First of all, from the N=2 algebra we know that since J_0 and L_0 commute, they are both simultaneously triangularizable. In correlation functions of degenerate fields, logarithms appear when singular vectors in a Virasoro module lead to Fuchsian differential equations, which solutions can not be expanded in series but also posses logarithmic terms. A consequence of this is that the OPE of two certain degenerate fields has logarithmic terms as well. Although explicit, simple formulae for singular vectors are not known for $N = 2$ (some progress was made in [72]), Fuchsian differential equations of second order do not exist for nilpotent variables in general.

If one introduces indecomposable representations with respect to J_0, the resulting logarithmic partners decouple completely from the theory. Let us assume that J_0 is non-diagonalizable, acting on a two-component field containing Φ' and Φ. Then from (4.73) we can extract the correlation functions of the components:

$$\langle \Phi_{h_1,q_1} \Phi_{h_2,q_2} \rangle = 0$$

$$\langle \Phi'_{h_1,q_1} \Phi_{h_2,q_2} \rangle = 0$$

$$\langle \Phi'_{h_1,q_1} \Phi'_{h_2,q_2} \rangle = \mathsf{Z}_{12}^{-2h}(1 + q_2 \theta_{12}^{+} \theta_{12}^{-} \mathsf{Z}_{12}^{-1}) \delta_{h_1,h_2} \delta_{q_1,-q_2}.$$

This result means that the two-point function of two logarithmic fields behaves exactly like a two-point function of two regular primary superfields. This already leads to the conclusion that defining a Jordan cell for J_0 has no effect on the correlators of the theory if we set $\Phi' \to \Psi$ and $\Phi \to 0$, where Ψ is an ordinary primary field. The correlation functions with respect to L_0 behave exactly as in $N = 0, 1$-theory. Replacing h_i by $h_i + H_i$ the ansatz $\prod_{i<j} \mathsf{Z}_{ij}^{\Delta_{ij}}$ is solved for:

$$2(h_i + H_i) = \sum_{j=2, i<j}^{n} \Delta_{ij}.$$

Thus, for $n > 2$, the correlation functions can be obtained from ordinary ones using the derivation trick.

4.8 N=3 SCFT

Generators of superconformal transformations closing to the N=3 super Virasoro algebra act on the space spanned by one complex and three Grassmannian variables. We will denote the nilpotent variables as $\theta^1, \theta^2, \theta^3$ and use lower indices to designate the position. Schwimmer and Seiberg ([76]) found that there are only two unitary representations of the N=3 algebra, which are labeled by $c = \frac{3}{2}, h = 0, q = 0$ and $c = \frac{3}{2}, h = \frac{1}{4}, q = \frac{1}{2}$. Obviously, the algebra turns out to be very restrictive. We will discuss the N=3 in its full generality, making use of this important result later.

The most general infinitesimal transformation consistent with the conformal condition reads (using the Einstein summation convention from now on):

$$z \mapsto z + a(z) + \alpha_i(z)\theta^i + \frac{1}{2}\alpha_{ij}(z)\theta^i\theta^j + \alpha_{123}(z)\theta^1\theta^2\theta^3.$$

The eight classical generators of the superconformal transformations read:

$$l_m = -z^m(z\partial_z + \frac{1}{2}(m+1)\theta^i\partial_{\theta^i})$$

$$\begin{aligned} g^i_r &= z^{r-\frac{1}{2}}(z\theta^i\partial_z - z\partial_{\theta^i} + (r+\frac{1}{2})\theta^i\theta^j\partial_{\theta^j}) \\ t^i_m &= z^{m-1}(z\epsilon_{ijk}\theta^j\partial_{\theta^k} - m\theta^1\theta^2\theta^3\partial_{\theta^i}) \\ \psi_r &= -z^{r-\frac{1}{2}}(\theta^1\theta^2\theta^3\partial_z + \frac{1}{2}\epsilon_{ijk}\theta^i\theta^j\partial_{\theta^k}). \end{aligned}$$

Additionally to g^i_r, we have another odd generator ψ_r. These transformations give rise to following classical graded algebra:

$$\begin{aligned} [l_m, l_n] &= (m-n)l_{m+n} \\ \{g^i_r, g^j_s\} &= 2\delta_{ij}l_{r+s} + \epsilon_{ijk}(r-s)t^k_{r+s} \\ [l_m, g^i_r] &= (\frac{m}{2} - r)g^i_{r+m} \\ [t^i_m, t^j_n] &= -\epsilon_{ijk}t^k_{m+n} \\ [l_m, t^i_n] &= -nt^i_{m+n} \\ [t^i_m, g^j_r] &= \delta_{ij}m\psi_{r+m} - \epsilon_{ijk}g^k_{r+m} \\ \{\psi_m, \psi_n\} &= 0 \\ [l_m, \psi_s] &= -(\frac{m}{2} + s)\psi_{m+s} \\ \{g^i_r, \psi_s\} &= t^i_{r+s} \\ [t^i_m, \psi_s] &= 0. \end{aligned}$$

The set of operators annihilating the vacuum reads:

$$X|0\rangle = 0 \Leftrightarrow X \in \{l_n, g^i_r, t^i_m, \psi_s : n \geq -1, r \geq -\frac{1}{2}, m \geq 0, s \geq \frac{1}{2}\}.$$

The generators of the group Osp(2|3) are the closed subset of this set of operators:

$$\{l_{-1}, l_0, l_1, g^i_{\frac{1}{2}}, g^i_{-\frac{1}{2}}, t^i_0\},$$

containing twelve generators. This leads to twelve potentially useful Ward identities.

The gauge transformations that leave the z-coordinate invariant are t^i_m, which satisfy a su(2) algebra. Thus, a representation of the N=3 superconformal algebra carries an implicit su(2) index transforming under an operator J_i:

$$[J_i, J_j] = -\frac{1}{2}\epsilon_{ijk}J_k \tag{4.74}$$

so that the OPE of the stress-energy tensor:

$$\mathsf{T}(\mathsf{Z}) = \theta^1\theta^2\theta^3 L(z) + \frac{1}{2}\epsilon_{ijk}\theta^i\theta^j G^k(z) + \theta^i T^i(z) + \psi(z) \tag{4.75}$$

with a superfield reads:

$$\begin{aligned}\mathsf{T}(\mathsf{Z}_1)\Phi(\mathsf{Z}_2) = &\tfrac{h\theta^1_{12}\theta^2_{12}\theta^3_{12}}{\mathsf{Z}^2_{12}}\Phi(\mathsf{Z}_2) + \tfrac{\theta^1_{12}\theta^2_{12}\theta^3_{12}}{\mathsf{Z}_{12}}\partial_{z_1}\Phi(\mathsf{Z}_2)\\ &+\tfrac{\epsilon_{ijk}\theta^i_{12}\theta^j_{12}D^k_2}{4\mathsf{Z}_{12}}\Phi(\mathsf{Z}_2) + \tfrac{\theta^i_{12}J_i}{\mathsf{Z}_{12}}\Phi(\mathsf{Z}_2).\end{aligned}$$

The covariant derivative is defined as:

$$D^i = \partial_{\theta^i} + \theta^i\partial_z,$$

and the superdifferences read:

$$\mathsf{Z}_{12} = (z_1 - z_2 - \theta^i_1\theta^i_2)$$
$$\theta^i_{12} = (\theta^i_1 - \theta^i_2).$$

The infinitesimal transformations on primary superfields are:

$$[L_m, \Phi(\mathsf{Z})] = z^m(h(m+1) + z\partial_z + \frac{1}{2}(m+1)\theta^i\partial_{\theta^i} + \frac{m(m+1)}{2z}\epsilon_{ijk}\theta^i\theta^j J_k)\Phi(\mathsf{Z})$$

$$\begin{aligned}[G^i_s, \Phi(\mathsf{Z})] = &-z^{s-\frac{1}{2}}(h(s+\tfrac{1}{2})\theta^i + \tfrac{1}{2}\theta^i z\partial_z - \tfrac{1}{2}z\partial_{\theta^i} + \tfrac{1}{2}(s+\tfrac{1}{2})\theta^i\theta^j\partial_{\theta^j}\\ &+(s+\tfrac{1}{2})(\epsilon_{ijk}\theta^j J_k - \tfrac{1}{z}(s^2-\tfrac{1}{4})\theta^1\theta^2\theta^3 J_i)\Phi(\mathsf{Z})\end{aligned}$$

$$\begin{aligned}[T^i_m, \Phi(\mathsf{Z})] = &z^{m-1}(\tfrac{mh}{2}\epsilon_{ijk}\theta^j\theta^k - \tfrac{z}{2}\epsilon_{ijk}\theta^j\partial_{\theta^k} + \tfrac{m}{2}\theta^1\theta^2\theta^3\partial_{\theta^i}\\ &+zJ_i - m\theta^i\theta^k J_k)\Phi(\mathsf{Z})\end{aligned}$$

$$\begin{aligned}[\psi_s, \Phi(\mathsf{Z})] = &z^{s-\frac{1}{2}}(-\tfrac{h}{z}(s-\tfrac{1}{2})\theta^1\theta^2\theta^3 + \tfrac{1}{2}\theta^1\theta^2\theta^3\partial_z\\ &+\tfrac{1}{4}\epsilon_{ijk}(\theta^i\theta^j\partial_{\theta^k} - \theta^i J_i)\Phi(\mathsf{Z}).\end{aligned}$$

The OPE of the super stress-energy tensor with itself is:

$$\mathsf{T}(\mathsf{Z}_1)\mathsf{T}(\mathsf{Z}_2) = \frac{c}{\mathsf{Z}_{12}} + \frac{\theta^1_{12}\theta^2_{12}\theta^3_{12}}{2\mathsf{Z}^2_{12}}\mathsf{T}(\mathsf{Z}_2) + \frac{\theta^1_{12}\theta^2_{12}\theta^3_{12}}{\mathsf{Z}_{12}}\partial_{z_2}\mathsf{T}(\mathsf{Z}_2) + \frac{\epsilon_{ijk}\theta^i_{12}\theta^j_{12}D^k_2}{4\mathsf{Z}_{12}}\mathsf{T}(\mathsf{Z}_2).$$

This OPE gives rise to the quantum N=3 algebra (note the extra factors of $\frac{1}{2}$ appearing compared to the classical algebra, besides the usual central extension):

$$[L_m, L_n] = (m-n)L_{m+n} - cm(m^2+1)\delta_{m+n,0}$$

$$\{G^i_r, G^j_s\} = \frac{\delta_{ij}}{2}L_{r+s} + \frac{\epsilon_{ijk}}{2}(r-s)T^k_{r+s} - c(r^2 - \frac{1}{4})\delta_{r+s,0}\delta_{ij}$$

$$[L_m, G^i_r] = (\frac{m}{2} - r)G^i_{r+m}$$

$$[T^i_m, T^j_n] = -\frac{\epsilon_{ijk}}{2} T^k_{m+n} + mc\delta_{ij}\delta_{m+n,0}$$
$$[L_m, T^i_n] = -nT^i_{m+n}$$
$$[T^i_m, G^j_r] = \frac{1}{2}(\delta_{ij} m\psi_{r+m} - \epsilon_{ijk} G^k_{r+m})$$
$$\{\psi_r, \psi_s\} = c\delta_{r+s,0}$$
$$[L_m, \psi_s] = -(\frac{m}{2} + s)\psi_{m+s}$$
$$\{G^i_r, \psi_s\} = \frac{1}{2} T^i_{r+s}$$
$$[T^i_m, \psi_s] = 0.$$

For representation theory, it is useful to make a coordinate transformation map the generators to a diagonal su(2) basis. We define:

$$\theta^+ = 2(i\theta^1 - \theta^2)$$
$$\theta^- = 2(i\theta^1 + \theta^2)$$
$$\theta^H = i\theta^3$$

The central charge is fixed:

$$k = -4c.$$

Although this basis turns out to be very convenient to work with, the algebra, written down explicitly, becomes rather lengthy:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{k}{4} m(m^2 - 1)\delta_{m+n,0}$$
$$\{G^H_r, G^H_s\} = -32L_{r+s} - 16k(r^2 - \frac{1}{4})\delta_{r+s,0}$$
$$\{G^+_r, G^-_s\} = 16L_{r+s} + 8k(r^2 - \frac{1}{4})\delta_{r+s,0} + 8(r-s)T^H_{r+s}$$
$$\{G^\pm_r, G^H_s\} = 8(r-s)T^\pm_{r+s}$$
$$[L_m, G^{\pm,H}_r] = (\frac{m}{2} - r)G^{\pm,H}_{r+m}$$
$$[T^H_n, T^H_m] = km\delta_{m+n,0}$$
$$[T^H_m, T^\pm_n] = \pm T^\pm_{m+n}$$
$$[T^+_m, T^-_n] = 2T^H_{m+n} + 2km\delta_{m+n,0}$$
$$[L_m, T^{\pm,H}_n] = -nT^{\pm,H}_{m+n}$$
$$[T^H_m, G^\pm_r] = \pm G^\pm_{r+m}$$
$$[T^H_m, G^H_r] = -2T^H_{r+s}$$
$$[T^\pm_m, G^H_r] = -2G^\pm_{m+r}$$
$$[T^\mp_m, G^\pm_r] = -G^H_{r+m} \pm 8m\psi_{r+m}$$

$$
\begin{aligned}
[\psi_r, \psi_s] &= -\frac{k}{4}\delta_{r+s,0} \\
[L_m, \psi_s] &= -(\frac{m}{2}+s)\psi_{m+s} \\
\{\psi_s, G^H_r\} &= -2T^H_{r+s} \\
\{\psi_s, G^\pm_r\} &= \mp T^\pm_{r+s} \\
[T^\pm_m, T^\pm_n] = [T^\pm_m, G^\pm_r] &= \{G^\pm_r, G^\pm_s\} = [T^{\pm,H}_m, \psi_s] = 0.
\end{aligned}
$$

The twelve generators of superconformal transformations relevant for solving super Ward identities, written in the new basis, read explicitly:

$$
[L_{-1}, \Phi(\mathsf{Z})] = \partial_z \Phi(\mathsf{Z}) \tag{4.76}
$$

$$
[L_0, \Phi(\mathsf{Z})] = (h + z\partial_z + \tfrac{1}{2}(\theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-} + \theta^H\partial_{\theta^H}))\Phi(\mathsf{Z}) \tag{4.77}
$$

$$
\begin{aligned}
[L_1, \Phi(\mathsf{Z})] = (2hz + z(z\partial_z + \theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-} + \theta^H\partial_{\theta^H}) \\
+\tfrac{1}{8}\theta^+\theta^- J^H + \tfrac{1}{4}\theta^+\theta^H J^- - \tfrac{1}{4}\theta^-\theta^H J^+)\Phi(\mathsf{Z})
\end{aligned} \tag{4.78}
$$

$$
[G^H_{-\frac{1}{2}}, \Phi(\mathsf{Z})] = -4(\theta^H\partial_z + \partial_{\theta^H})\Phi(\mathsf{Z}) \tag{4.79}
$$

$$
\begin{aligned}
[G^H_{\frac{1}{2}}, \Phi(\mathsf{Z})] = (-8h\theta^H - 4\theta^H z\partial_z - 4z\partial_{\theta^H} - 4\theta^H\theta^-\partial_{\theta^-} \\
-4\theta^H\theta^+\partial_{\theta^+} + \theta^- J^+ - \theta^+ J^-)\Phi(\mathsf{Z})
\end{aligned} \tag{4.80}
$$

$$
[G^\pm_{-\frac{1}{2}}, \Phi(\mathsf{Z})] = \pm(\theta^\pm\partial_z + 8\partial_{\theta^\mp})\Phi(\mathsf{Z}) \tag{4.81}
$$

$$
\begin{aligned}
[G^\pm_{\frac{1}{2}}, \Phi(\mathsf{Z})] = (\pm 2h\theta^\pm \pm \theta^\pm z\partial_z \pm 8z\partial_{\theta^\mp} + \theta^+\theta^-\partial_{\theta^\mp} \\
+2\theta^H J^\pm + \theta^\pm J^H)\Phi(\mathsf{Z})
\end{aligned} \tag{4.82}
$$

$$
[T^H_0, \Phi(\mathsf{Z})] = (\theta^-\partial_{\theta^-} - \theta^+\partial_{\theta^+} + J^H)\Phi(\mathsf{Z}) \tag{4.83}
$$

$$
[T^\pm_0, \Phi(\mathsf{Z})] = (\mp\tfrac{1}{2}\theta^\pm\partial_{\theta^H} \pm 4\theta^H\partial_{\theta^\mp} + J^\pm)\Phi(\mathsf{Z}). \tag{4.84}
$$

We note that by constructing superfields from superfunctions, we can obtain $q = 0$ and $q = 1$ representations explicitly. Making use of three Grassmannian variables, there are eight possible terms in the expansion of a superfunction. One can regard the set:

$$
\{1, \theta^H, \theta^+, \theta^-, \theta^H\theta^+, \theta^+\theta^-, \theta^H\theta^-, \theta^H\theta^+\theta^-\}
$$

as a basis of a superfield. The coefficients are functions of Z_{12} which are fixed by L_0. Then the bases of the eigenstates of $\overrightarrow{J}^2$ and J^H are given by:

$$
\begin{aligned}
j = 0,\, q = 0 &: \{1, \theta^+\theta^-\theta^H\} \\
j = 1,\, q = 1 &: \{\theta^+, \theta^+\theta^H\} \\
j = 1,\, q = 0 &: \{\theta^H, \theta^+\theta^-\} \\
j = 1,\, q = -1 &: \{\theta^-, \theta^-\theta^H\}.
\end{aligned} \tag{4.85}
$$

Thus, it is easy to construct $j = 0$ and $j = 1$ representations in superspace using ordinary functions containing nilpotent, anticommuting variables. Other su(2)-representations require more complicated constructions.

4.9 Indecomposable Representations with Respect to su(2)-Operators

From the onset, it is not quite obvious how to introduce irreducible representations with respect to a given su(2) algebra and how they transform under ladder operators. This was noticed in [45]. We will answer this question here.

There are two commuting (simultaneously triangulazable) operators $\overrightarrow{J}^2$, J^H. Naively, both of them can possess Jordan cell structure independently from each other. The operators satisfy following commutation relations:

$$[J_i, J_k] = -\frac{1}{2}\epsilon_{ijk}J_k$$

$$J^+ = 2(iJ_1 - J_2)$$
$$J^- = 2(iJ_1 + J_2)$$
$$J^H = -2iJ_3.$$

The highest-weight state with respect to J^H is mapped to:

$$|q\rangle \to \frac{i}{2}|q\rangle.$$

The computation of the correlation functions is much more complicated in N=3 theory due to the presence of su(2) generators. We will elaborate on that in the next section. The representation theory of the su(2) subspace is completely analogous to the quantum mechanical angular momentum. The generators in the diagonal basis satisfy:

$$[J^\pm, J^H] = \mp J^\pm \tag{4.86}$$
$$[J^+, J^-] = 2J^H. \tag{4.87}$$

The diagonal operator $\overrightarrow{J}^2$ can be expressed as:

$$\overrightarrow{J}^2 = J^+J^- + (J^H)^2 - J^H. \tag{4.88}$$

Thus, primary superfields can be labeled by quantum numbers j, q which are eigenvalues of $\overrightarrow{J}^2$ and J^H, respectively. From the action of (4.88) on highest-weight states we know that action on corresponding highest-weight states is:

$$\overrightarrow{J}^2|h, j, q\rangle = j(j+1)|h, j, q\rangle$$

and that the highest-weight states have $q = j$. Using $J^+|q\rangle = a^+(q)|q+1\rangle$ and $J^-|q\rangle = a^-(q)|q-1\rangle$:

$$\langle q|J^+J^-|q\rangle = (\langle q|(J^-)^\dagger)J^-|q\rangle = a^-(q)^2 = j(j+1) - q(q-1)$$
$$\Rightarrow a^-(q) = \sqrt{j(j+1) - q(q-1)}$$

$$\langle q|J^-J^+|q\rangle = (\langle q|(J^+)^\dagger)J^+|q\rangle = a^+(q)^2 = j(j+1) - q(q+1)$$
$$\Rightarrow a^+(q) = \sqrt{j(j+1) - q(q+1)}.$$

Thus, we obtain the usual action of the ladder operators on highest-weight states, with $J^{-}|q=-j\rangle = 0$.

Preservation of commutation relations on a given indecomposable representation with respect to J^H or $\overrightarrow{J}^2$ determines the embedding structure under ladder operators.

One could try to construct indecomposable $\overrightarrow{J}^2$. In this case J^H must be diagonal and the ladder operators have to mix between different representations j to satisfy (4.88). However, it can be easily shown that in this case it is not possible to satisfy (4.86). In fact, $\overrightarrow{J}^2$, as a quadratic Casimir operator, exists only on a given representation and is not part of the algebra, since an algebra is equipped with a bilinear form only and the notion of multiplication is not defined for elements of an algebra.

For a logarithmic pair with respect to J^H, that is for a rank 2 Jordan cell living in a representation j one has:

$$\begin{aligned} J^H|j,q'\rangle &= q'|j,q'\rangle + |j,q\rangle \\ J^H|j,q\rangle &= q|j,q\rangle \\ \overrightarrow{J}^2|j,q'\rangle &= j(j+1)|q'\rangle \\ \overrightarrow{J}^2|j,q\rangle &= j(j+1)|q\rangle. \end{aligned}$$

The state $|j,q\rangle$ is assumed to be a regular su(2)-representation. Equation (4.86) implies that the ladder operators act as lowering and raising operators on the representation $|q'\rangle$, obeying:

$$J^H(J^{\pm}|q'\rangle) = J^{\pm}((q\pm 1)|q'\rangle + |q\rangle). \tag{4.89}$$

This equation suggests that the ansatz for the action of ladder operators on components of the logarithmic pair is given by:

$$\begin{aligned} J^{\pm}|q'\rangle &= a^{\pm}(q)|(q\pm 1)'\rangle + b(q)^{\pm}|q\pm 1\rangle. \\ J^{\pm}|q\rangle &= c^{\pm}(q)|q\rangle. \end{aligned}$$

Using this ansatz on both sides of equation (4.89) returns $a^{\pm}(q) = c^{\pm}(q)$. Equation (4.87), acting on $|q'\rangle$ returns:

$$a^{-}(q)a^{+}(q-1) = a^{+}(q)a^{-}(q+1) + 2q \tag{4.90}$$

$$a^{-}(q)b^{+}(q-1) + b^{-}(q)a^{+}(q-1) = a^{+}(q)b^{-}(q+1) + b^{+}(q)a^{-}(q+1) + 1. \tag{4.91}$$

Similar conditions can be obtained by acting on different states of the theory, e.g. the logarithmic partner and highest- and lowest states of the theory. Since $|q\rangle$ is assumed to be an ordinary su(2) representation, the action of $J^{\pm}$ on that state is:

$$J^{\pm}|q\rangle = a^{\pm}|q\rangle = \sqrt{j(j+1) - q(q\pm 1)}|q\rangle. \tag{4.92}$$

A remarkable observation is that the derivation trick also works for logarithmic su(2) representations. In particular:

$$b^{\pm}(q) = \partial_q a^{\pm}(q).$$

Thus, the action of $J^{\pm}$ on $|q'\rangle$ is given by:

$$J^{\pm}|q'\rangle = \sqrt{j(j+1) - q(q \pm 1)}|(q \pm 1)'\rangle + \frac{-q \mp \frac{1}{2}}{\sqrt{j(j+1) - q(q \pm 1)}}|q \pm 1\rangle.$$

We obtain a logarithmic representation of su(2) with respect to J^H. The unusual feature is that the ladder operators acting on logarithmic fields do not produce a descendent of the logarithmic field alone but rather a linear combination involving the logarithmic partner. For a rank two Jordan cell, one obtains the following embedding diagram (the arrows indicate the action of J^-):

$$\begin{array}{lcccl}
|q=j\rangle & \bullet & & \bullet & |q'=j\rangle \\
 & \downarrow & \swarrow & \downarrow & \\
|q=j-1\rangle & \bullet & & \bullet & |q'=j-1\rangle \\
 & \downarrow & \swarrow & \downarrow & \\
 & \bullet & & \bullet & \\
 & \vdots & \vdots & \vdots & \\
 & \bullet & & \bullet & \\
 & \downarrow & \swarrow & \downarrow & \\
|q=-j+1\rangle & \bullet & & \bullet & |q'=-j+1\rangle \\
 & \downarrow & \swarrow & \downarrow & \\
|q=-j\rangle & \bullet & & \bullet & |q'=-j\rangle
\end{array} \tag{4.93}$$

The derivation trick also holds for higher-dimensional Jordan cells, in which case we obtain the embedding diagram (denoting the position of a field in the Jordan cell by an upper index):

$$\begin{array}{ccccccccc}
|q=j\rangle & & |(q=j)^{(1)}\rangle & & \cdots & & |(q=j)^{(n-1)}\rangle & & |(q=j)^{(n)}\rangle \\
\bullet & & \bullet & & & & \bullet & & \bullet \\
\downarrow & \swarrow & \downarrow & \swarrow & \cdots & \swarrow & \downarrow & \swarrow & \downarrow \\
\bullet & & \bullet & & & & \bullet & & \bullet \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
\bullet & & \bullet & & & & \bullet & & \bullet \\
\downarrow & \swarrow & \downarrow & \swarrow & \cdots & \swarrow & \downarrow & \swarrow & \downarrow \\
\bullet & & \bullet & & & & \bullet & & \bullet \\
|q=-j\rangle & & |(q=-j)^{(1)}\rangle & & \cdots & & |(q=-j)^{(n-1)}\rangle & & |(q=-j)^{(n)}\rangle
\end{array}$$

The main result of this section is the fact that it is indeed possible to define an arbitrary, non-diagonal J^H acting on a column of fields, which in turn can be transformed to a Jordan cell form. However, the N=3 super Virasoro theory does not contain such operators due to the mentioned fact that Fuchsian differential equations of second order in nilpotent variables would be required.

4.10 The Two-Point Functions in N=3 SCFT

As in N=2 theory, we perform yet another coordinate transformation to solve the Ward identities. Again, the supersums and superdifferences turn out to be very convenient for this purpose:

$$\theta_{ij}^{\pm,H} = \theta_i^{\pm,H} - \theta_j^{\pm,H}$$

$$\xi_{ij}^{\pm,H} = \theta_i^{\pm,H} + \theta_j^{\pm,H}$$

$$\mathsf{Z}_{ij} = z_i - z_j + \frac{1}{8}(\theta_i^-\theta_j^+ + \theta_i^+\theta_j^-) + \theta_i^H\theta_j^H$$

$$\mathsf{W}_{ij} = z_i + z_j + \frac{1}{8}(\theta_i^-\theta_j^+ + \theta_i^+\theta_j^-) + \theta_i^H\theta_j^H.$$

We note that the subgroup SL(2, $\mathbb{C}$) acts on the space of functions in the completely analogous way as in N=0 theory. We also know that the supertranslations in direction of nilpotent dimensions $G_{\frac{1}{2}}^{\pm,H}$ eliminate the dependence on $\xi_{ij}^{\pm,H}$. Taking these facts into account, the derivatives transform as (ignoring derivatives with respect to W_{ij} and $\xi_{ij}^{\pm,H}$):

$$\partial_{z_1} = \partial_{\mathsf{Z}_{12}}$$

$$\partial_{z_2} = -\partial_{\mathsf{Z}_{12}}$$

$$\partial_{\theta_1^\pm} = \partial_{\theta_{12}^\pm} + \frac{1}{8}\theta_2^\mp\partial_{\mathsf{Z}_{12}}$$

$$\partial_{\theta_2^\pm} = -\partial_{\theta_{12}^\pm} - \frac{1}{8}\theta_1^\mp\partial_{\mathsf{Z}_{12}}$$

$$\partial_{\theta_1^H} = \partial_{\theta_{12}^H} + \theta_2^H\partial_{\mathsf{Z}_{12}}$$

$$\partial_{\theta_2^H} = -\partial_{\theta_{12}^H} - \theta_1^H\partial_{\mathsf{Z}_{12}}.$$

Since the transformation is straightforward and we discussed the details explicitly in N=2 case already. We skip the technicalities and present the superconformal Ward identities for the two-point function $F_2(\mathsf{Z}_{12}, \theta_{12}^+, \theta_{12}^-, \theta_{12}^H)$ in terms of supersums and superdifferences:

$$(L_{-1}^{(1)} + L_{-1}^{(2)})F_2 = 0 \tag{4.94}$$

$$(L_0^{(1)} + L_0^{(2)})F_2 = (h_1 + h_2 + \mathsf{Z}_{12}\partial_{\mathsf{Z}_{12}} + \frac{1}{2}(\theta_{12}^+\partial_{\theta_{12}^+} + \theta_{12}^-\partial_{\theta_{12}^-} + \theta_{12}^H\partial_{\theta_{12}^H}))F_2 = 0$$

$$\begin{aligned}
&(L_1^{(1)} + L_1^{(2)})F_2 = \\
&(h_1(\mathsf{W}_{12} + \mathsf{Z}_{12} - \tfrac{1}{8}(\theta_{12}^-\xi_{12}^+ + \theta_{12}^+\xi_{12}^-) - \theta_{12}^H\xi_{12}^H) + h_2(\mathsf{W}_{12} - \mathsf{Z}_{12}) \\
&+\tfrac{1}{16}(\xi_{12}^+ + \theta_{12}^+)(\xi_{12}^H + \theta_{12}^H)J_1^- + \tfrac{1}{16}(\xi_{12}^+ - \theta_{12}^+)(\xi_{12}^H - \theta_{12}^H)J_2^- \\
&-\tfrac{1}{16}(\xi_{12}^- + \theta_{12}^-)(\xi_{12}^H + \theta_{12}^H)J_1^+ - \tfrac{1}{16}(\xi_{12}^- - \theta_{12}^-)(\xi_{12}^H - \theta_{12}^H)J_2^+ \\
&+\tfrac{1}{32}(\xi_{12}^+ + \theta_{12}^+)(\xi_{12}^- + \theta_{12}^-)J_1^H + \tfrac{1}{32}(\xi_{12}^+ - \theta_{12}^+)(\xi_{12}^- - \theta_{12}^-)J_2^H \\
&+(\mathsf{W}_{12}\mathsf{Z}_{12} - \tfrac{1}{16}\mathsf{Z}_{12}(\theta_{12}^-\xi_{12}^+ + \theta_{12}^+\xi_{12}^-) - \tfrac{1}{2}\mathsf{Z}_{12}\theta_{12}^H\xi_{12}^H)\partial_{\mathsf{Z}_{12}} \\
&+\tfrac{1}{2}(\mathsf{Z}_{12}\xi_{12}^+ + \mathsf{W}_{12}\theta_{12}^+ + \tfrac{1}{16}\theta_{12}^+\xi_{12}^+(\xi_{12}^- + \theta_{12}^-) - \tfrac{1}{2}\theta_{12}^H\xi_{12}^H(\xi_{12}^+ + \theta_{12}^+))\partial_{\theta_{12}^+} \\
&+\tfrac{1}{2}(\mathsf{Z}_{12}\xi_{12}^- + \mathsf{W}_{12}\theta_{12}^- + \tfrac{1}{16}\theta_{12}^-\xi_{12}^-(\xi_{12}^+ + \theta_{12}^+) - \tfrac{1}{2}\theta_{12}^H\xi_{12}^H(\xi_{12}^- + \theta_{12}^-))\partial_{\theta_{12}^-} \\
&+\tfrac{1}{2}(\mathsf{Z}_{12}\xi_{12}^H + \mathsf{W}_{12}\theta_{12}^H - \tfrac{1}{16}(\theta_{12}^-\xi_{12}^+ + \theta_{12}^+\xi_{12}^-)(\xi_{12}^H + \theta_{12}^H))\partial_{\theta_{12}^H})F_2 = 0
\end{aligned} \tag{4.95}$$

$$(G^{H(1)}_{-\frac{1}{2}} + G^{H(2)}_{-\frac{1}{2}})F_2 = 0 \tag{4.96}$$

$$\begin{gathered}(G^{H(1)}_{\frac{1}{2}} + G^{H(2)}_{\frac{1}{2}})F_2 = (-4((h_1+h_2)\xi^H_{12} + (h_1-h_2)\theta^H_{12}) \\ -\tfrac{1}{2}(\xi^+_{12}+\theta^+_{12})J^-_1 - \tfrac{1}{2}(\xi^+_{12}-\theta^+_{12})J^-_2 + \tfrac{1}{2}(\xi^-_{12}+\theta^-_{12})J^+_1 + \tfrac{1}{2}(\xi^-_{12}-\theta^-_{12})J^+_2 \\ -4\mathsf{Z}_{12}\xi^H_{12}\partial_{\mathsf{Z}_{12}} - 2(\xi^H_{12}\theta^-_{12} + \theta^H_{12}\xi^-_{12})\partial_{\theta^-_{12}} - 2(\xi^H_{12}\theta^+_{12} + \theta^H_{12}\xi^+_{12})\partial_{\theta^+_{12}} \\ -4(\mathsf{Z}_{12} - \tfrac{1}{16}(\theta^-_{12}\xi^+_{12} + \theta^+_{12}\xi^-_{12}) - \tfrac{1}{2}\theta^H_{12}\xi^H_{12})\partial_{\theta^H_{12}})F_2 = 0\end{gathered} \tag{4.97}$$

$$(G^{\pm(1)}_{-\frac{1}{2}} + G^{\pm(2)}_{-\frac{1}{2}})F_2 = 0 \tag{4.98}$$

$$\begin{gathered}(G^{\pm(1)}_{\frac{1}{2}} + G^{\pm(2)}_{\frac{1}{2}})F_2 = (\pm h_1(\xi^\pm_{12}+\theta^\pm_{12}) \pm h_2(\xi^\pm_{12}-\theta^\pm_{12}) \\ +(\xi^H_{12}+\theta^H_{12})J^\pm_1 + (\xi^H_{12}-\theta^H_{12})J^\pm_2 \\ +\tfrac{1}{2}(\xi^\pm_{12}+\theta^\pm_{12})J^H_1 + \tfrac{1}{2}(\xi^\pm_{12}-\theta^\pm_{12})J^H_2 \pm \xi^\pm_{12}\mathsf{Z}_{12}\partial_{\mathsf{Z}_{12}} \\ +(\pm 8\mathsf{Z}_{12} \mp \theta^\mp_{12}\xi^\pm_{12} - 4\theta^H_{12}\xi^H_{12})\partial_{\theta^\mp_{12}} \pm \tfrac{1}{2}(\theta^\pm_{12}\xi^H_{12} + \xi^\pm_{12}\theta^H_{12})\partial_{\theta^H_{12}})F_2 = 0\end{gathered} \tag{4.99}$$

$$(T^{H(1)}_0 + T^{H(2)}_0)F_2 = (\theta^-_{12}\partial_{\theta^-_{12}} - \theta^+_{12}\partial_{\theta^+_{12}} + J^H_1 + J^H_2)F_2 = 0 \tag{4.100}$$

$$(T^{\pm(1)}_0 + T^{\pm(2)}_0)F_2 = (\mp\tfrac{1}{2}\theta^\pm_{12}\partial_{\theta^H_{12}} \pm 4\theta^H_{12}\partial_{\theta^\mp_{12}} + J^\pm_1 + J^\pm_2)F_2 = 0. \tag{4.101}$$

Equation (4.88) can be easily verified by consecutive application of generators on the right-hand side on one of four eigenstates of the su(2) algebra:

$$\begin{gathered}((\tfrac{1}{2}\theta^+\partial_{\theta^H} - 4\theta^H\partial_{\theta^-})(-\tfrac{1}{2}\theta^-\partial_{\theta^H} + 4\theta^H\partial_{\theta^+}) \\ +(\theta^+\partial_{\theta^+} - \theta^-\partial_{\theta^-})((\theta^+\partial_{\theta^+} - \theta^-\partial_{\theta^-}) - 1))\Phi_{j,q} = j(j+1)\Phi_{j,q}\end{gathered}$$

$$\begin{gathered}\Rightarrow 2(\theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-} + \theta^H\partial_{\theta^H} \\ -\theta^+\theta^-\partial_{\theta^-}\partial_{\theta^+} - \theta^+\theta^H\partial_{\theta^H}\partial_{\theta^+} - \theta^-\theta^H\partial_{\theta^H}\partial_{\theta^-})\Phi_{j,q} = j(j+1)\Phi_{j,q}.\end{gathered} \tag{4.102}$$

As a matter of fact, one can use this equation to generate another Ward identity. We first assume that the superdifferences appear only with either integer powers or powers which are products of h_i with an integer. The Ward identity for L_0 returns the right functions associated with each element of the basis. Thus, the ansatz for the two-point function is of the form:

$$\begin{gathered}F_2 = \frac{a}{\mathsf{Z}_{12}^{h_1+h_2}} + \frac{b^+\theta^+_{12}}{\mathsf{Z}_{12}^{h_1+h_2+\frac{1}{2}}} + \frac{b^-\theta^-_{12}}{\mathsf{Z}_{12}^{h_1+h_2+\frac{1}{2}}} + \frac{b^H\theta^H_{12}}{\mathsf{Z}_{12}^{h_1+h_2+\frac{1}{2}}} \\ +\frac{c^{+H}\theta^+_{12}\theta^H_{12}}{\mathsf{Z}_{12}^{h_1+h_2+1}} + \frac{c^{-H}\theta^-_{12}\theta^H_{12}}{\mathsf{Z}_{12}^{h_1+h_2+1}} + \frac{c^{+-}\theta^+_{12}\theta^-_{12}}{\mathsf{Z}_{12}^{h_1+h_2+1}} + \frac{d\theta^+_{12}\theta^-_{12}\theta^H_{12}}{\mathsf{Z}_{12}^{h_1+h_2+\frac{3}{2}}}.\end{gathered} \tag{4.103}$$

In [71], two solutions of the two-point function were presented. The author solved the superconformal Ward identities by substituting terms involving J^-

into each other. The two solutions read (ignoring the constants and setting $h_1 = h_2 = h$):

$$\langle \Phi_{q_1} \Phi_{q_2} \rangle = \frac{1}{\mathsf{Z}_{12}^{2h}},\ q_1 = q_2 = 0 \tag{4.104}$$

$$\langle \Phi_{q_1} \Phi_{q_2} \rangle = \frac{\theta_{12}^{+} \theta_{12}^{H}}{\mathsf{Z}_{12}^{2h+1}},\ q_1 + q_2 = 1,\ q_1, q_2 \neq 0. \tag{4.105}$$

We find that although correct, this picture is far from complete, since the author of [71] failed to recognize the necessity of interpreting the eigenvalue of J^H as quantized superconformal isospin. Since we know that q must be an integer or a half-integer and both q involved belong to highest-weight states, the only possibility for the second solution is $q_1 = q_2 = \frac{1}{2}$. beginning from that, we can derive all the other two-point functions of the theory.

It is an important observation that the quadratic Casimir operator J^2 exists in N=3 theory, generating an additional, although not independent Ward identity. To define a Casimir operator, one needs to consecutively apply operators at the same superspace "points". Since the points involve Grassmannian variables only, there is no problem with singularities and we don't need to worry about normal ordering. Consider a correlator of primary fields with $q_1 = j_1$ and $q_2 = j_2$. A Casimir operator can be defined as:

$$T^2 = T^- T^+ + (T^H)^2 + T^H. \tag{4.106}$$

The first term annihilates the two-point function of primary fields and the third term generates an already known Ward identity. The second term translates to:

$$((T^{H(1)})^2 + 2T^{H(1)} T^{H(2)} + (T^{H(2)})^2) F_2 = 0. \tag{4.107}$$

In terms of superdifferences, this relation reads (abbreviating $q_1 + q_2 = x$):

$$(-2\theta_{12}^{+}\theta_{12}^{-}\partial_{\theta_{12}^{-}}\partial_{\theta_{12}^{+}} + (1-2x)\theta_{12}^{+}\partial_{\theta_{12}^{+}} + (1+2x)\theta_{12}^{-}\partial_{\theta_{12}^{-}} + x^2)F_2 = 0. \tag{4.108}$$

This Ward Identity holds for correlators of both lowest- and highest-weight fields. The only nontrivial solutions are given by terms containing $\theta_{12}^{+}, \theta_{12}^{+}\theta_{12}^{H}$ for $x = 1$ and $\theta_{12}^{-}, \theta_{12}^{-}\theta_{12}^{H}$ for $x = -1$. Since the terms containing one nilpotent variable are already ruled out, we find one other candidate for a two-point function, which indeed satisfies all the remaining Ward identities:

$$\langle \Phi_{-\frac{1}{2}} \Phi_{-\frac{1}{2}} \rangle = \frac{\theta_{12}^{-} \theta_{12}^{H}}{\mathsf{Z}_{12}^{2h+1}}. \tag{4.109}$$

Using (4.97) on this function, we obtain conditions of the form:

$$\langle (J^- \Phi_{\frac{1}{2}}) \Phi_{\frac{1}{2}} \rangle + \langle \Phi_{\frac{1}{2}} (J^- \Phi_{\frac{1}{2}}) \rangle = \langle \Phi_{-\frac{1}{2}} \Phi_{\frac{1}{2}} \rangle + \langle \Phi_{\frac{1}{2}} \Phi_{-\frac{1}{2}} \rangle = \frac{\theta_{12}^{-} \theta_{12}^{+}}{2\mathsf{Z}_{12}^{2h+1}} \tag{4.110}$$

$$\langle (J^- \Phi_{\frac{1}{2}}) \Phi_{\frac{1}{2}} \rangle - \langle \Phi_{\frac{1}{2}} (J^- \Phi_{\frac{1}{2}}) \rangle = \langle \Phi_{-\frac{1}{2}} \Phi_{\frac{1}{2}} \rangle - \langle \Phi_{\frac{1}{2}} \Phi_{-\frac{1}{2}} \rangle = \frac{8}{2\mathsf{Z}_{12}^{2h}} \tag{4.111}$$

from which we obtain mixed correlators:

$$\langle \Phi_{-\frac{1}{2}} \Phi_{\frac{1}{2}} \rangle = \frac{\theta_{12}^{-} \theta_{12}^{+}}{4\mathsf{Z}_{12}^{2h+1}} + \frac{4}{\mathsf{Z}_{12}^{2h}} \tag{4.112}$$

$$\langle \Phi_{\frac{1}{2}} \Phi_{-\frac{1}{2}} \rangle = \frac{\theta^{-}_{12}\theta^{+}_{12}}{4\mathsf{Z}^{2h+1}_{12}} - \frac{4}{\mathsf{Z}^{2h}_{12}}. \tag{4.113}$$

Thus, all five non-zero two-point functions of the N=3 theory have been found. There are two fields in the theory which give non-trivial correlation functions: the identity $\mathbb{I}$ and a su(2)-doublet $\Phi_{\frac{1}{2}}$. These are the two fields identified in [76] in the $c = \frac{3}{2}$ theory, where two representations in the NS sector, labeled by $h(q)$: $0(0)$ and $\frac{1}{4}(\frac{1}{2})$ have been found. It is a very intriguing fact that only $q = 0$, $q = \frac{1}{2}$- states return non-zero two-point functions; presumably, unitary N=3 representations for all $c = \frac{3}{2}k$ contain only fields with these two values of q.

4.11 The N=3 n-Point Functions

For n-point functions, we have $\displaystyle\sum_{a=1}^{n}(n-a) = \frac{1}{2}n(n-1)$ coordinates labeled by $\{(i,j)|1 \leq i < j \leq n\}$. We take as the general ansatz:

$$F_n = \xi_n(\mathsf{Z})\sigma_{q_1,\ldots,q_k}(\mathsf{Z},\theta)$$
$$\xi_n(\mathsf{Z}) = \prod_{i<j} \mathsf{Z}_{ij}^{-\Delta_{ij}}.$$

The number of fields in the correlator with $q_i \neq 0$ is k.

Acting on this ansatz with superdifferential Ward operators produces a set of equations for $\sigma_{q_1,\ldots,q_k}$. We define $\sigma_{i,j}$, the σ-part if the two-point function at points i and j, as:

$$\sigma_{i,j} = \begin{cases} 1 & q_i = q_j = 0 & \\ \frac{\theta^{-}_{ij}\theta^{H}_{ij}}{\mathsf{Z}_{ij}} & q_i = -\frac{1}{2} & q_j = -\frac{1}{2} \\ \frac{\theta^{+}_{ij}\theta^{H}_{ij}}{\mathsf{Z}_{ij}} & q_i = +\frac{1}{2} & q_j = +\frac{1}{2} \\ 4 + \frac{\theta^{-}_{ij}\theta^{+}_{ij}}{4\mathsf{Z}_{ij}} & q_i = -\frac{1}{2} & q_j = +\frac{1}{2} \\ -4 + \frac{\theta^{-}_{ij}\theta^{+}_{ij}}{4\mathsf{Z}_{ij}} & q_i = +\frac{1}{2} & q_j = -\frac{1}{2} \\ 0 & \text{otherwise.} & \end{cases}$$

The Ward identity for L_0, for example (from (4.77)) produces a linear superdifferential equation for $\sigma_{q_1,\ldots,q_k}$:

$$\mathcal{L}_0\sigma_{q_1,\ldots,q_k} = \sum_{i<j}(\mathsf{Z}_{ij}\partial_{\mathsf{Z}_{ij}} + \frac{1}{2}(\theta^{+}_{ij}\partial_{\theta^{+}_{ij}} + \theta^{-}_{ij}\partial_{\theta^{-}_{ij}} + \theta^{H}_{ij}\partial_{\theta^{H}_{ij}}))\sigma_{q_1,\ldots,q_k} = 0. \tag{4.114}$$

From T^{H}_{0} we have the condition:

$$\mathcal{T}^{H}_{0}\sigma_{q_1,\ldots,q_k} = (\sum_{i<j}(\theta^{-}_{ij}\partial_{\theta^{-}_{ij}} - \theta^{+}_{ij}\partial_{\theta^{+}_{ij}}) + \sum_{i} J^{H}_{i})\sigma_{q_1,\ldots,q_k} = 0. \tag{4.115}$$

The three-point correlation function $\langle \Phi_{\frac{1}{2}}\Phi_{\frac{1}{2}}\Phi_{\frac{1}{2}} \rangle$ is trivial, as is every correlator containing an odd number of su(2)-doublets. Invariance under $T^{\pm}_{0}$ produces another equation:

$$\mathcal{T}_0^{\pm}\sigma_{q_1,\ldots,q_k} = (\sum_{i<j}(\mp\frac{1}{2}\theta_{ij}^{\pm}\partial_{\theta_{ij}^H} \pm 4\theta_{ij}^H\partial_{\theta_{ij}^{\mp}}) + \sum_i J_i^{\pm})\sigma_{q_1,\ldots,q_k} = 0, \qquad (4.116)$$

and invariance under $G^H_{\frac{1}{2}}$ leads to:

$$\begin{gathered}\mathcal{G}^H_{-\frac{1}{2}}\sigma_{q_1,\ldots,q_k} = (\sum_{i<j}(-4\mathsf{Z}_{ij}\xi_{ij}^H\partial_{\mathsf{Z}_{ij}} - 2(\xi_{ij}^H\theta_{ij}^- + \theta_{ij}^H\xi_{ij}^-)\partial_{\theta_{ij}^-} \\ -2(\xi_{ij}^H\theta_{ij}^+ + \theta_{ij}^H\xi_{ij}^+)\partial_{\theta_{ij}^+} - 4(\mathsf{Z}_{ij} - \tfrac{1}{16}(\theta_{12}^-\xi_{ij}^+ + \theta_{ij}^+\xi_{12}^-) - \tfrac{1}{2}\theta_{ij}^H\xi_{ij}^H)\partial_{\theta_{ij}^H}) \\ + \sum_i(-\theta_i^+J_i^- + \theta_i^-J_i^+))\sigma_{q_1,\ldots,q_k} = 0.\end{gathered} \qquad (4.117)$$

Because of $[G^H_{\frac{1}{2}}, G^H_{\frac{1}{2}}] = -32L_1$ and $[T_0^{\pm}, G^H_{\frac{1}{2}}] = -2G^{\pm}_{\frac{1}{2}}$, all the other Ward identities are satisfied without returning new constraints.

Every $\sigma_{i,j}$ satisfies (4.114)-(4.117), in that case the sum simplifies to one term which is the part of super Ward identities acting on $\sigma_{i,j}$. With k=4 we have the set of six coordinates labeled by $\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$. There are three pairwise contractions $\{((1,2),(3,4)),((1,3),(2,4)),((1,4),(2,3))\}$. In general, there are $(k-1)!!$ ways to perform pairwise contractions of k elements. This cooresponds to the number of choices of $\frac{k}{2}$ coordinates with uniquely distributed indices $1,..,k$. If l labels elements of the ordered set of possible pairwise contractions of k fields, we write for a particular realization l of pairwise contractions:

$$(\sigma_{q_1,\ldots,q_k})_l = (\sigma_{i_1,j_1}\sigma_{i_2,j_2}\ldots\sigma_{i_{\frac{k}{2}},j_{\frac{k}{2}}})_l = \prod_{m=1}^{\frac{k}{2}}(\sigma_{i_m,j_m})_l$$

$$i_m < j_m, i_1 \neq \ldots \neq i_{\frac{k}{2}} \neq j_1 \neq \ldots \neq j_{\frac{k}{2}}.$$

Every particular contraction is a solution of (4.114)-(4.117).

The general solution can be obtained from a contraction of pairs of su(2)-doublets and a summation over all possible contractions:

$$\sigma_{q_1,\ldots,q_k} = \sum_{l=1}^{(k-1)!!}\prod_m^{\frac{k}{2}}(\sigma_{i_m,j_m})_l.$$

This is how it works in case n=4, k=4:

$$\sigma_{q_1,q_2,q_3,q_4} = \sigma_{q_1,q_2}\sigma_{q_3,q_4} + \sigma_{q_1,q_3}\sigma_{q_2,q_4} + \sigma_{q_1,q_4}\sigma_{q_2,q_3}.$$

We can give the q_i particular values, for example to obtain the sum of all possible contractions:

$$\begin{gathered}\sigma_{(\frac{1}{2})_1,(-\frac{1}{2})_2,(\frac{1}{2})_3,(-\frac{1}{2})_4} = \\ \sigma_{(\frac{1}{2})_1,-(\frac{1}{2})_2}\sigma_{(\frac{1}{2})_3,(\frac{1}{2})_4} + \sigma_{(\frac{1}{2})_1,(\frac{1}{2})_3}\sigma_{-(\frac{1}{2})_2,(-\frac{1}{2})_4} + \sigma_{(\frac{1}{2})_1,-(\frac{1}{2})_4}\sigma_{-(\frac{1}{2})_2,(\frac{1}{2})_3} = \\ -\frac{\theta_{12}^+\theta_{12}^-}{\mathsf{Z}_{12}} + \frac{\theta_{34}^+\theta_{34}^-}{\mathsf{Z}_{34}} + \frac{\theta_{14}^+\theta_{14}^-}{\mathsf{Z}_{14}} - \frac{\theta_{23}^+\theta_{23}^-}{\mathsf{Z}_{23}} \\ + \frac{\theta_{12}^+\theta_{12}^-\theta_{34}^+\theta_{34}^-}{16\mathsf{Z}_{12}\mathsf{Z}_{34}} + \frac{\theta_{14}^+\theta_{14}^-\theta_{23}^+\theta_{23}^-}{16\mathsf{Z}_{14}\mathsf{Z}_{23}} + \frac{\theta_{13}^+\theta_{13}^-\theta_{24}^+\theta_{24}^-}{16\mathsf{Z}_{13}\mathsf{Z}_{24}}.\end{gathered} \qquad (4.118)$$

Equation (4.117) is then satisfied:

$$
\begin{aligned}
&\mathcal{G}_0^H \sigma_{(\frac{1}{2})_1,(-\frac{1}{2})_2,(\frac{1}{2})_3,(-\frac{1}{2})_4} = \\
&\left((-\theta_2^- J_2^+ + \theta_1^+ J_1^-)\sigma_{(\frac{1}{2})_1,(-\frac{1}{2})_2}\right)\sigma_{(\frac{1}{2})_3,(-\frac{1}{2})_4} \\
&+\sigma_{(\frac{1}{2})_1,(-\frac{1}{2})_2}\left((-\theta_3^- J_4^+ + \theta_4^+ J_3^-)\sigma_{(\frac{1}{2})_3,(-\frac{1}{2})_4}\right) \\
&\left((\theta_1^+ J_1^- + \theta_3^+ J_3^-)\sigma_{(\frac{1}{2})_1,(\frac{1}{2})_3}\right)\sigma_{(-\frac{1}{2})_2,(-\frac{1}{2})_4} \\
&+\sigma_{(\frac{1}{2})_1,(\frac{1}{2})_3}\left((-\theta_2^- J_2^+ - \theta_4^- J_4^+)\sigma_{(\frac{1}{2})_3,(-\frac{1}{2})_4}\right) \\
&\left((-\theta_4^- J_4^+ + \theta_1^+ J_1^-)\sigma_{(\frac{1}{2})_1,(-\frac{1}{2})_4}\right)\sigma_{(-\frac{1}{2})_2,(\frac{1}{2})_3} \\
&+\sigma_{(\frac{1}{2})_1,(-\frac{1}{2})_4}\left((-\theta_2^- J_2^+ + \theta_3^+ J_3^-)\sigma_{(-\frac{1}{2})_2,(\frac{1}{2})_3}\right) \\
&+(\theta_2^- J_2^+ + \theta_4^- J_4^+ - \theta_1^+ J_1^- - \theta_3^+ J_3^-)\sigma_{(\frac{1}{2})_1,(-\frac{1}{2})_2,(\frac{1}{2})_3,(-\frac{1}{2})_4} = 0.
\end{aligned}
\tag{4.119}
$$

The equation is satisfied because acting with $J_i^\pm$ on the right and comparing different $\theta_i^\pm$ we obtain contractions of different σ_{q_1,q_2,q_3,q_4}, for example for θ_1^+:

$$
\begin{aligned}
\sigma_{(-\frac{1}{2})_1,(-\frac{1}{2})_2,(\frac{1}{2})_3,(-\frac{1}{2})_4} &= \sigma_{(-\frac{1}{2})_1,(-\frac{1}{2})_2}\sigma_{(\frac{1}{2})_3,(-\frac{1}{2})_4} + \sigma_{(-\frac{1}{2})_1,(\frac{1}{2})_3}\sigma_{(-\frac{1}{2})_2,(-\frac{1}{2})_4} \\
&+\sigma_{(-\frac{1}{2})_1,(-\frac{1}{2})_4}\sigma_{(-\frac{1}{2})_2,(\frac{1}{2})_3}.
\end{aligned}
$$

After all this preparation, we can write the general n-point function (up to functions in $\frac{1}{2}n(n-3)$ independent cross-ratios), as:

$$
\langle q_1, ..., q_n\rangle \sim \begin{cases}
\prod_{i<j}^{n} \mathsf{Z}_{ij}^{-\Delta_{ij}} & k=0 \\
\prod_{i<j}^{n} \mathsf{Z}_{ij}^{\Delta_{ij}} \sum_{l=1}^{(k-1)!!} \prod_{m}^{\frac{k}{2}} (\sigma_{i_m,j_m})_l & k \text{ even} \\
0 & k \text{ odd}
\end{cases}
\tag{4.120}
$$

$$
\sum_{i=1,i<j}^{n} \Delta_{ij} = 2h_i. \tag{4.121}
$$

Due to the low representation content in q, we were able to give all n-point functions in N=3 theory as far as they are constrained by superconformal Ward identities. Null vectors could be used to provide further constraints on functions of invariants. In the context of the logarithmic theory, correlation functions with fields indecomposable with respect to L_0 can be obtained by using the derivation trick.

Conclusions and Outlook

An interesting aspect of the N=3 theory we found is that all representations with $q \neq 0, \frac{1}{2}$ seem to decouple from the theory. This is a strong indicator that the representation content in the Neveau-Schwarz sector of a given N=3 theory is limited to these values of q. We conjecture that a similar result can be obtained in the Ramond sector. In [76], it was found that in the $c = \frac{3}{2}$ theory, there are two representations given by $h(q) = \frac{1}{16}(0)$ (corresponding to spin fields Σ) and $\frac{5}{16}(\frac{1}{2})$. Obtaining Ramond correlation functions is more difficult since spin fields have to be inserted to get the right boundary conditions. A vertex algebra representation would simplify this calculation. However, to the knowledge of the author, no such representation has been found yet.

We used the method of assuming irreducible representations to obtain correlation functions of supersymmetric (logarithmic) fields. It was shown that for super Virasoro theories, logarithmic representations exist only with respect to L_0. Therefore, logarithmic theories might be encountered in supersymmetric extensions of known $N = 0$ models. The most important question remaining is which regular logarithmic representations "survive" supersymmetrisation (and if they do at all). The correlation functions of logarithmic fields are given by a product of some combination of superdifferences times the $N = 0$ logarithmic correlation functions with differences z_{ij} replaced by superdifferences Z_{12}.

During the course of writing this thesis, it became obvious that the supersymmetric logarithmic field theory is much simpler then previously conjectured. Not only are there no indecomposable representations with respect to operators of additional gauge symmetries, we did not find any evidence for suggested similarity between "nilpotent-variable"-description of logarithmic fields and superfields, since their transformation properties are already very different. The apparent similarity becomes even less visible for $N = 2$ and $N = 3$ fields, in which case superfields contain two bosonic, two fermionic and four bosonic, four fermionic components, respectively, as opposed to one bosonic field and any number of bosonic fields in conjunction with nilpotent variables in the logarithmic case.

We hope that this work will contribute to a better understanding of the regular N=3 theory and the N=2 and N=3 logarithmic theories. However, a lot of work remains to be done in the future. Even regular, two-dimensional CFT remains a vast and productive field with many open questions. The classification of all rational theories alone is a problem, and it is still not clear if it is tractable at all. Our present understanding of ordinary logarithmic theories is even less far from complete. Even seventeen years after their discovery, they stand out somewhat strange and obscure, although highly interesting and applicable area in conformal quantum field theory.

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Eigenständigkeitserklärung

Hiermit versichere ich, dass ich die vorlegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

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