

# On a Logarithmic Deformation of the Supersymmetric $bc$ -system on Curved Manifolds

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## Abstract

E. Frenkel, A. Losev and N. Nekrasov claim that a certain class of theories on compact Kähler manifolds and in particular the “gauged” supersymmetric  $bc$ -system on  $\mathbb{CP}^1$  are logarithmic conformal field theories. We discuss that proposition on a classical level for the  $bc$ -system on  $\mathbb{CP}^1$ . The outcome of our investigation conforms to their conjecture. The property of being a *logarithmic* CFT thus can be interpreted as an effect of gravity.

# 1 Introduction

This paper is the result of our attempt to tackle the following question: can we understand the attribute of a conformal field theory on curved target space to be a *logarithmic* conformal field theory as an aspect of gravity? Below, we will explain the background of that question and give an outline.

In 2006, E. Frenkel, A. Losev and N. Nekrasov (“FLN”, if we may) published the first part of a series of papers (up to now [7, 8] and a third part is to appear), in which they propose a new perspective on the non-perturbative regime of certain quantum field theories. In these models, gravitational and topological structures are entangled in a non-trivial way, allowing for an analytic solution of the topological and even of dynamical correlation functions. Thereby the authors extend the approach of topological field theories, which did already combine the effects of instantons and of curved target spaces, by the dynamical sector.

The type of theory they consider is roughly as follows. The action is a supersymmetric field theory of type  $\mathcal{N} = 2$  in Euclidean dimensions one, two or four, twisted and deformed to a topological theory with a first order Lagrangian. The target space of the theory is some Calabi-Yau or compact Kähler manifold. There is an additional topological term  $S_{top}(\lambda)$  in the action which triggers the contribution of the anti-instanton sector to the correlation functions, depending on  $\lambda \in \mathbb{R}$  :

$$S_\lambda = S_{\mathcal{N}=2} + S_{top}(\lambda) .$$

In general correlation functions, the anti-instantons are not damped for  $\lambda = 0$  , whereas for  $0 < \lambda < \infty$  they are damped. Finally, for  $\lambda = \infty$  , the instanton sector contributes, while the anti-instantons are completely damped out. In addition, CPT is broken if  $\lambda \neq 0$  . The topological correlation functions are independent of  $\lambda$  and therefore only the dynamical sector is sensitive for the different “phases” just described. The authors are mostly interested in the theories with  $\lambda = \infty$  because they turn out to be integrable in that phase, even including dynamical correlation functions.

The domain manifold is assumed to factorize according to  $\Sigma_t \times \Sigma_s$  , where  $\Sigma_t$  is a one dimensional manifold that serves as the domain for the time coordinate. One can then reduce the theory to a one dimensional super quantum mechanics, a Morse-Bott-Novikov theory, by integrating over the space coordinates. Thereby, the authors obtain a canonical quantized description of the model in which they calculate correlation functions as expectation values of (non-)topological observables. What is moreover important in doing that reduction is that the properties of the one dimensional super quantum mechanics should be mirrored in the higher dimensional theory. Therefore, the higher dimensional field theory can be investigated by means of a more simple model.

If we can say that, in our point of view the piece of work Frenkel, Losev and Nekrasov have done is outstanding in the sense that the authors open new perspectives on some of the more fundamental questions. One example which is important for our investigations is the connection between gravity and topological aspects that can be drawn from their work.

The theory they consider is defined on curved target spaces and hence belongs to the field of quantum gravity. Usually in quantum gravity one chooses either a path integral approach or the method of canonical quantization. FLN’s ansatz, however combines both and thus serves as a playground for studying the different aspects that can be analysed best by each of them, respectively. It turns out, that instanton effects and breaking of CPT-invariance are nicely described within the path integral approach, while the question of gravity is best treated from the canonical point of view. However, gravity and instantons are interwoven. They can appear as aspects in different formulations of the same structure. Already for the simplest target space and model FLN consider, which is Morse theory on  $\mathbb{C}\mathbb{P}^1$ , it becomes transparent that

non-unitarity in the non-topological states is such a structure. In the path integral picture, it appears as an effect of the anti-instantons being absent. In the canonical picture non-unitarity emerges in the shape of extra terms in the Hamiltonian as a consequence of  $\mathbb{C}\mathbb{P}^1$  being compact. This allows for the following question: in what circumstances can gravity be traded off for instantons and what is the difference in the perspective causing that either the one or the other appear?

The question we put is in the same line of substituting gravity by something else that might be better known. In their theory, Frenkel, Losev and Nekrasov claim that the models they consider turn out to be logarithmic conformal field theories (LCFTs) in the limit of  $\lambda = \infty$  [7, 8]. The motivation for that conjecture is, that the Hamiltonian of the associated Morse-Bott-Novikov theory has Jordan blocks due to the extra terms mentioned above. This is exactly the situation in generic<sup>1</sup> two dimensional LCFTs. These Jordan blocks originate from the geometry of target space and, to the best of our knowledge, this relation between gravity and LCFTs is new, though many connections between geometry and conformal field theory have been drawn before.

In order to analyse that connection, we restrict our attention to the supersymmetric  $bc$ -system with domain and target space being  $\mathbb{C}\mathbb{P}^1$ . Non-reducibility of the Hamiltonian appears in the related one dimensional Morse-Novikov theory.<sup>2</sup> However, in LCFTs, non-reducibility is a property of the energy momentum tensor or in mathematical terms of the Virasoro algebra, which is an object in the two dimensional field theory. Therefore, in order to analyse the conjecture, we consider it necessary to find out, if the derived non-reducibility of the Hamiltonian of the Morse-Novikov theory can be obtained by a deformation of the full energy momentum tensor in the two dimensional field theory which preserves the Virasoro algebra. This is what we investigated in the following sections:

In the first two chapters we will describe the model we are going to investigate and derive the operators that deform the energy momentum tensor. We will start with one of the operators that were already obtained by Frenkel, Losev and Nekrasov in [8]. In order to derive the second one, it is sufficient to give some more detailed arguments, which we will do. Our calculations verify the choices of FLN. At the end of chapter 3, we will give a short outline of the procedure that will follow and summarize some results.

In chapter 4 we will obtain the deformation operators by means of the method of J. Fjelstad, J. Fuchs, S. Hwang, A. M. Semikhatov and I. Y. Tipunin [5]. The space of states of the thus deformed  $bc$ -system will be described and we also calculate the change in the cohomology. A deformation of the supercharge was already calculated by E. Frenkel and A. Losev in [9] but by a method which usually does not apply to the situation under consideration. The result we obtain looks however very similar to theirs.

The last chapter is devoted to a discussion of our results and gives a summary of the questions that arised during our investigations and that are still open.

## 2 The Model under Consideration

Before we start with the subject, we would like to draw the attention of the reader to the appendix. Frenkel's, Losev's and Nekrasov's work connects serveral topics. When we wrote this paper we were confronted with the question how self-contained it should be. We also had a lot of conventions to choose

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<sup>1</sup>By generic we mean that most studies on LCFTs treat theories that have a non-reducible Hamiltonian, though there exist more general situations.

<sup>2</sup>An extra symmetry on target space will be implemented in the action such that we get rid of the difficulties that come along with the generalizations made by Bott.

for the calculations. Therefore we decided to make a more detailed appendix in order to fix the conventions and definitions and to give very brief introductions to the main topics “around” the subject. A reader who is not an expert in bosonization, the chiral de Rham complex or Morse-Bott-Novikov theory might start with the appendix. A reader who is well schooled in these things can easily grasp the notations and definitions by just taking a glimpse.

In this section we shortly summarize, how Frenkel, Losev and Nekrasov [8] derive the space of states of the model we will discuss here. This is important in order to elucidate the connection of the underlying Morse-Novikov theory with an associated supersymmetric  $bc$ -system. This latter two dimensional CFT will be the subject of our investigations. The notations and definitions can be found in A.1.2 and A.3.

We start with the following action for the situation  $x : \Sigma \rightarrow X$ ,  $\Sigma = \mathbb{CP}^1 = X$  :

$$S = \int_{\Sigma} -i(p'(\partial_{\bar{z}} + A_{\bar{z}})x - \pi(\partial_{\bar{z}} + A_{\bar{z}})\psi + c.c.) . \quad (2.1)$$

With non quantized “gauge field”  $A_{\bar{z}} = \frac{\alpha}{\bar{z}}$ ,  $\alpha \in (-1, 0) \subset \mathbb{R}$  .

Due to the  $A$ -field, the equation of motion is not just the condition for holomorphicity. In local coordinates of  $\mathbb{C}_0 := X \setminus \{\infty\}$  it reads

$$\partial_{\bar{z}}x + \frac{\alpha}{\bar{z}}x = 0 , \quad x(z=0) = 0 = x(z=\infty) . \quad (2.2)$$

The boundary conditions are chosen such that the solution  $x$  is nonsingular near  $z=0$  and  $z=\infty$ , where  $z \in \Sigma$ . They imply that the solutions run into the fixed points  $\{0, \infty\}$  of some  $\mathbb{C}^\times$  action with generator  $v = x\partial_x + \bar{x}\partial_{\bar{x}}$  on  $X$ .

The theory above is transformed to a Morse-Novikov theory on the universal cover of loop space (c.f. section A.3). There, one deals with mappings  $\tilde{\gamma} : D \times \bar{D} \rightarrow \mathbb{C}_0$  that satisfy the Hamiltonian flow equation

$$\partial_{\bar{z}}\tilde{\gamma} + \frac{\alpha}{\bar{z}}\tilde{\gamma} = 0 , \quad \tilde{\gamma}(z, \bar{z})|_{(0,0)} = 0 , \quad z \in D \quad (2.3)$$

and solutions  $\tilde{g} : D \times \bar{D} \rightarrow \mathbb{C}_\infty := X \setminus \{0\}$  of

$$\partial_{\bar{z}}\tilde{g} - \frac{\alpha}{\bar{z}}\tilde{g} = 0 , \quad \tilde{g}(z, \bar{z})|_{(0,0)} = 0 , \quad z \in D . \quad (2.4)$$

$D \subset \mathbb{C}$  is the disc of radius one. The choice of boundary conditions is now such, that the solutions  $\tilde{\gamma}$  and  $\tilde{g}$  are the flow lines along the descending manifold that for  $z \rightarrow 0$  run into the fixed points  $\{0\} \in X$  and  $\{\infty\} \in X$ , respectively. Analogous results are obtained for the superpartners. The solutions of the flow equations above are given by

$$\tilde{\gamma}(z, \bar{z}) = \bar{z}^{-\alpha} \sum_{n \leq 0} x_n z^{-n} \quad \text{and} \quad \tilde{g}(z, \bar{z}) = \bar{z}^{\alpha} \sum_{n < 0} \tilde{x}_n z^{-n} , \quad (2.5)$$

respectively.

Since the local coordinates along the descending manifolds of  $\widetilde{LX}$  in the related Morse-Novikov theory are identical with the modes of the solutions of the respective e.o.m., the local coordinates along  $\widetilde{LX}_{0,0}$  are  $\{x_n\}_{n \leq 0}$  and along  $\widetilde{LX}_{\infty,0}$  they are the  $\{\tilde{x}_n\}_{n < 0}$  .

The operators  $x_n$  and  $\tilde{x}_n$  are identical with the field modes of the conformal field  $x(z)$ ,  $z \in \Sigma$  in the original theory, where the tilde denotes the coordinates in the chart  $\mathbb{C}_\infty$ . Therefore, the *naive* space of

states,<sup>3</sup> associated with the descendent manifolds of the super quantum mechanics, can be modeled by the chiral de Rham complex of the supersymmetric  $bc$ -system without  $A$ -field:

$$\mathcal{F}_0 = \mathbb{C}[x_n, p'_m]_{n \leq 0, m < 0} \otimes \wedge[\psi_n, \pi_m]_{n \leq 0, m < 0} \cdot |0\rangle_0 \quad (2.6)$$

for  $\widetilde{LX}_{0,0}$  and for  $\widetilde{LX}_{\infty,0}$  it is

$$\mathcal{F}_\infty^1 = \mathbb{C}[\tilde{x}_n, \tilde{p}_m]_{n < 0, m \leq 0} \otimes \wedge[\tilde{\psi}_n, \tilde{\pi}_m]_{n < 0, m \leq 0} \cdot |1\rangle_\infty . \quad (2.7)$$

The different highest weight vectors originate from the range of the indices. Indeed, we obtained two different representations of the Heisenberg and Clifford algebras:

$$\begin{aligned} x_n|0\rangle_0 = \psi_n|0\rangle_0 = 0, \quad n > 0, \quad p'_n|0\rangle_0 = \pi_n|0\rangle_0 = 0, \quad n \geq 0 \\ \tilde{x}_n|1\rangle_\infty = \tilde{\psi}_n|1\rangle_\infty = 0, \quad n > -1, \quad \tilde{p}'_n|1\rangle_\infty = \tilde{\pi}_n|1\rangle_\infty = 0, \quad n \geq 1 \end{aligned} \quad (2.8)$$

Because of the bosons, these representations are not equivalent, c.f. appendix A.1.2.

The naive spaces of states associated with the other sheets, such as  $\widetilde{LX}_{0,n}$ , are isomorphic. They are connected by the equivariance operator  $q$  to the states above:  $q^n : \mathcal{F}_0 \ni \Psi \mapsto q^n \Psi \in q^n \mathcal{F}_0$ .

The main observation here is, that the naive space of states of the Morse-Novikov theory is related to a two dimensional, supersymmetric  $bc$ -system. That system does only indirectly know about the  $A$ -field via the representation spaces as above. In the following we will introduce the Grothendieck-Cousin operators that appear within the particular Morse-Novikov model due to the non-trivial geometry of target space. The next step will then be to analyze, how these operators appear as (logarithmic) deformations in the energy momentum tensor of the associated  $bc$ -system. This task demands some words of explanation, that we will also shift to the next section.

### 3 Beyond Naivety

Frenkel's, Losev's and Nekrasov's work [7, 8] leads to a generalization of the chiral de Rham complex. Their crucial result is that one has to include additional operators, Grothendieck-Cousin operators (GCOs), if the space of states of the Morse-Novikov theory is defined globally on target space. These operators change the cohomology of the chiral de Rham complex, extend the naive Hilbert spaces and deform the Hamiltonian. Roughly, the construction is as follows:

Characterize the descending (or ascending) manifolds. Take their closure, for example that of  $\widetilde{L\mathbb{P}^1}_{0,n}$ . Find the descending manifolds that are contained as a subset of codimension one in any such closure. The naive Hilbert space  $\mathcal{F}_I$ , associated with a descending manifold, is then non-canonically extended by  $\mathcal{F}_{II}$ , the naive Hilbert space associated with a descending submanifold of codimension one:

$$0 \rightarrow \mathcal{F}_{II} \rightarrow \mathcal{F}_{II} \oplus \mathcal{F}_I \rightarrow \mathcal{F}_I \rightarrow 0 . \quad (3.1)$$

The GCO is mapping  $\mathcal{F}_I \rightarrow \mathcal{F}_{II}$ , while it acts trivially on  $\mathcal{F}_{II}$ . It appears as an extra term in the naive Hamiltonian, such that  $H = H_{\text{naive}} + \delta$ . Therefore, the Hamiltonian becomes non-diagonalizable on certain subspaces.

Another important feature of the GCOs in question is, that they entangle the chiral and anti-chiral parts of the model. The reason is that the naive Hilbert spaces are polynomials in holomorphic and

<sup>3</sup>What is meant by "naive" will become transparent in the following section.

anti-holomorphic coordinates and have to be generalized as distributions, if they are supposed to be defined globally on target space. The Hamiltonian is a Lie derivative in direction of the Morse-Bott-Novikov-potential  $\nabla f$  and when acting on these polynomial distributions, it relates holomorphic and anti-holomorphic coordinates due to the exterior derivative. As a toy model that can be understood by the relation  $\partial_z \bar{z}^{-n-1} \sim \partial_z^n \delta(z)$  on  $\mathbb{C}$  for  $n \in \mathbb{N}$ .

To summarize: the nontrivial geometry of the target space causes a deformation of the naive space of states. The Hamiltonian is not diagonal on all states, any more, and mixes chiral and anti-chiral coordinates. These effects find an expression in additional operators, the Grothendieck-Cousin operators, that appear within the Hamiltonian. That structure is the starting point for some proposals of FLN.

### 3.1 On the LCFT Proposal

Frenkel, Losev and Nekrasov claim [8] that non-reducibility of the Hamiltonian does also appear in the two dimensional, supersymmetric  $bc$ -systems on Calabi-Yau manifolds. They propose that it is therefore a *logarithmic* conformal field theory.

In order to analyse that, we must proof that the deformation of the Hamiltonian of the Morse-Novikov theory shows up as a deformation of the energy momentum tensor in the  $bc$ -system, transforming it to a *logarithmic* conformal field theory. Now we have to do with two  $bc$ -systems, the original one (2.1) and the associated one at which we arrived at the end of section 2. The latter differs from the original  $bc$ -system in that it does only implicitly know about the  $A$ -field - it is completely shifted into the spaces of states. This is reflected by the fact, that the states in  $\mathbb{C}_\infty$  are in the representation  $\mathcal{F}_\infty^1$  and not  $\mathcal{F}_\infty^0$ . If we chose a different value of  $\alpha$ , we would again get a different representation, for instance if  $\alpha \in (0, 1)$ , the rôle of (2.6) and (2.7) would be interchanged. To conclude, the zero mode of the energy momentum tensor of the associated  $bc$ -system does not know about the  $A$ -field, it is just like usual the Lie derivative in direction of the generator of loop reparametrizations (c.f. A.1.2).

The Hamiltonian of the Morse-Novikov theory has two parts: it is given by the Lie derivative in direction of the sum of the generator of loop reparametrizations as above - which hence can be identified with the zero mode of the energy momentum tensor of the associated  $bc$ -system - plus the additional generator that comes from the  $\mathbb{C}^\times$  action on the target space. As mentioned, that Hamiltonian gains additional GCOs because of the exterior derivative  $d_{LX}$ , acting on the spaces of states that are globally defined on  $\widetilde{LX}$ . The contraction  $\iota_{\nabla f}$  does not mix the spaces of states. Up to some prefactor it acts like  $x_n \iota_{\partial_{x_n}}$  for both symmetries. Therefore, the GCOs do also appear in the associated  $bc$ -system (up to a prefactor that we neglect).

In our analysis we consider that associated  $bc$ -system on  $\mathbb{CP}^1$  with representation spaces  $\mathcal{F}_\infty^1$  and  $\mathcal{F}_0$  on the respective coordinate charts. We will try to do some deformation such that the GCOs enter the zero mode of the energy momentum tensor and investigate, if the respective model is an LCFT.

### 3.2 On the Cohomology Proposal

Frenkel, Losev and Nekrasov further propose that the cohomology of the chiral-anti-chiral de Rham complex of the associated  $bc$ -system is deformed by the GCOs. In [9] Frenkel and Losev calculated a change of the supercharge, using a method outlined in [15], from which they say in [8] that it is exactly the change that stems from the GCOs. This method acts on the level of the two dimensional theory and

is therefore interesting for our purpose. As the authors did not discuss the question of computing the logarithmic correction of the stress tensor, we are going to do that in the following. However, we choose a different method by which we obtain manifest deformation terms in the energy momentum tensor and in the supercharge field.

For the model that we just introduced, there exist two Grothendieck-Cousin operators. In the next section we will propose a different method in order to deform the  $bc$ -system and also let the choice of the GCOs undergo a detailed analysis. Especially for one of the GCOs that were already obtained in [8] we will fill in the arguments on the level of the representation spaces. But first, let us specify the algebraic structure of the free fermionic  $bc$ -system in order to identify the Grothendieck-Cousin operators.

### 3.3 Identification of the Grothendieck-Cousin Operators

To obtain the Grothendieck-Cousin operators, we consider the naive energy momentum tensor of the associated, free CFT. Mapping it to a supersymmetric  $bc$ -system, like it is done in section A.1, the energy momentum tensor reads:

$$T(z) = T^+(z) + T^-(z) , \quad (3.2)$$

with zero total central charge. Applying chiral bosonization like in A.1.2, we end up with

$$T(z) \mapsto T_J(z) = T_{J^+}(z) + (T_{J^-}(z) + T_{\eta\xi}(z)) , \quad T_{\eta\xi} =: \partial_z \xi(z) \eta(z) : \quad (3.3)$$

and the algebra is that of  $\mathcal{A}^+ \otimes \overline{N}(p)$ . The fields  $\eta$  and  $\xi$  make the auxiliary fermionic  $bc$ -system that has to be tensored to the bosons in order to produce the correct central charge. For the following reason, that system will be in the focus of many of our calculations: any effect the Grothendieck-Cousin operator does have on the energy momentum tensor must be grounded in the bosons. The mathematical argument for that is the relation  $M^+(p) \simeq M^+(p')$ ,  $\forall p, p'$  for the fermionic sector (c.f. A.1.2).

In [8], the authors conclude that there must be two Grothendieck-Cousin operators that are mappings between these spaces: Let  $q$  denote the equivariance operator as explained in A.3. Since  $\widetilde{LX}_{\infty, n} \supset \widetilde{LX}_{0, n+1}$  as a codimension one subset there is a first GCO  $\delta_1 : \mathcal{F}_\infty^1 \otimes \overline{\mathcal{F}}_\infty^1 \hookrightarrow \mathcal{F}_\times^1 \otimes \overline{\mathcal{F}}_\times^1 \rightarrow q(\mathcal{F}_0 \otimes \overline{\mathcal{F}}_0)$ . The same applies to  $\widetilde{LX}_{0, n} \supset \widetilde{LX}_{\infty, n}$  and there exists a second GCO  $\delta_2 : \mathcal{F}_0 \otimes \overline{\mathcal{F}}_0 \hookrightarrow \mathcal{F}_\times \otimes \overline{\mathcal{F}}_\times \rightarrow \mathcal{F}_\infty^1 \otimes \overline{\mathcal{F}}_\infty^1$ . We implemented a coordinate transition from  $\mathbb{C}_0 \subset X$  to the overlap of both charges,  $\mathbb{C}_\times \subset X$ , and vice versa, such that for instance  $\mathcal{F}_0 \hookrightarrow \mathcal{F}_\times$ . The GCOs demand that, for they map states localized around one pole of  $X$  to states that are localized around the other. In the following we will specify these two operators.

#### 3.3.1 The First Grothendieck-Cousin Operator

The action (2.1) is formulated in coordinates of  $\mathbb{C}_0 \subset X$  and for the moment we take the perspective of an “observer” who is “sitting” at the fixed point  $\{0\}$ .

In our “observatory” we can bosonize the representation spaces (2.8) and obtain  $\mathcal{F}_0 \simeq \mathcal{A}^+ \otimes \overline{N}(0)$  and  $\mathcal{F}_\infty^1 \simeq \mathcal{A}^+ \otimes \overline{N}(1)$ , respectively. Since the fermionic sector is the same for both and since  $\xi_0|0\rangle_{\eta\xi} = |1\rangle_{\eta\xi}$  while  $\eta_0|1\rangle_{\eta\xi} = |0\rangle_{\eta\xi}$ , we would naively expect that, after the coordinate transition to  $\mathbb{C}_\times$ , the Grothendieck-Cousin operators are basically given by  $\tilde{\eta}_0$ , acting on  $\mathcal{F}_\times^1$ , and  $\tilde{\xi}_0$ , acting on  $\mathcal{F}_\times$ .

We take the first GCO  $\tilde{\eta}_0$  for granted, since it is the same as in [8]:

$$\delta_1 := q \oint_{(0,0)} \tilde{\eta}(z) \bar{\tilde{\eta}}(\bar{z}) . \quad (3.4)$$

Very soon, we will give an additional argument for that choice but first, let us shortly comment on an obstacle it has:  $\mathcal{F}_\infty^1$  is in the kernel of  $\eta_0$  and the question appears if that resists for  $\tilde{\eta}_0$  and  $\mathcal{F}_\times^1$ . Applying the coordinate transformation  $\mathbb{C}_\infty \mapsto \mathbb{C}_\times$  on the fields according to (A.22), we find

$$\mathcal{F}_\infty^1 \hookrightarrow \mathcal{F}_\times^1 = \mathcal{F}_{0,ferm} \otimes \mathcal{F}_{\times,bos}^1 . \quad (3.5)$$

In terms of the bosonized fields,  $\mathcal{F}_{\times,bos}^1$  is generated by  $[V^-(+,z) \otimes V_{\eta\xi}^+(+,z)]^{\pm 1}$ , by  $V^-(-,z) \otimes \partial_z V_{\eta\xi}^+(-,z)$  and their derivatives. As we explain in A.1, a transformation  $[V^-(+,z) \otimes V_{\eta\xi}^+(+,z)]^{-1}$  inverts the zero mode. Therefore, the coordinate transformation above is equivalent to an inclusion of  $\xi_0$  and thus to an extension  $\bar{N}(1) \hookrightarrow N(1)$ :

$$\mathcal{F}_\times^1 \simeq \mathcal{A}^+ \otimes N(1) . \quad (3.6)$$

For convenience, we will omit the tilde on  $\eta$  and  $\xi$  from now on and hope that the context makes clear, in what coordinates we are.

Introducing the operator  $\xi_0$  as the second GCO would imply some heavy obstacles. The most harmful one is, that it would break conformal symmetry. That can be seen if we apply some calculation of M. Krohn and M. Flohr [12]. On the mode-level, the authors consider the set of deformations of some supersymmetric  $bc$ -system which preserve the Virasoro algebra. Applying the same calculations to the situation under discussion we find, that the conditions of a Virasoro algebra do not allow for a simultaneous deformation of the Hamiltonian by  $\xi_0$  and  $\eta_0$ .

Luckily, to choose  $\xi_0$  as the second GCO would be wrong and we suggest the following interpretation why. The reason has to do with the position of our ‘‘observatory’’ and with non-unitarity of the theory. As we mentioned in the introduction, the appearance of the GCOs can not only be understood as an effect of gravity but also of the presence of (only !) instantons. In short terms, they mimic the instantons, interpolating between different vacua. As CPT invariance is broken, the anti-instantons are absent, and the instantons flow along the descending manifolds into the respective fixedpoints. In our case, sitting at the fixed point  $\{0\}$ , the instantons flow from  $\{\infty\}$  to  $\{0\}$ , when  $z \rightarrow 0$ . The corresponding operator can therefore only be  $\eta_0$ . The operator  $\xi_0$  represents the outgoing anti-instanton. We obtain the corresponding dual, ‘‘anti-GCO’’ operator from  $\delta_1$  by conjugation and multiplication with  $q^{-2}$ , which is the analogue to [7, pg. 35, first eqn].

Before we derive the second GCO, let us conclude that section with just one remark. Assumed, that the supersymmetric  $bc$ -system on target space  $\mathbb{CP}^1$  is an LCFT, the discussion above links non-unitarity and the question of conformal invariance. It seems, that non-unitarity is a necessity for the supersymmetric  $bc$ -system to be a conformal theory.

### 3.3.2 The Second Grothendieck-Cousin Operator

The second GCO can only be obtained when changing our ‘‘position’’ from  $\{0\}$  to an ‘‘observatory’’ at  $\{\infty\}$ . However, we can not do that with a coordinate change  $x \mapsto x^{-1}$ , alone. The reason is that the action (2.1) is not invariant under that mapping, one further has to change the sign of the ‘‘gauge field’’. This is the crucial observation that we owe E. Frenkel: the second GCO can be derived from  $\delta_1$  by applying a composition of  $x \mapsto x^{-1}$  and a rescaling of the  $A$ -field via  $\alpha \mapsto -\alpha$ . In the following we will



explain the details of that transition up to one open question: unfortunately, we do not know if, and if yes how, a prefactor  $q^{-1}$  is introduced by that composite transition to  $\{\infty\}$ , which would pay for the factor of  $q$  in  $\delta_1$ . Therefore, we allow ourselves to add it by hand. The result will be that the second GCO is again obtained by  $\eta_0$ , but the space of states have a different representation in terms of the bosonized theory.

Before we go into details, let us refer the reader to section A.2.1, which is a summary of [8, pg. 95] in our conventions. Here we obtain the auxiliary fields  $\eta$  and  $\xi$  in terms of the original fields. For instance, the first GCO reads up to the factor of  $q$ :

$$\delta_1 \sim - \oint_{(0,0)} \Psi_+(z, \bar{z}) \pi(z) \bar{\pi}(\bar{z}), \quad \Psi_{\pm}(z, \bar{z}) = e^{\pm i \int (J^z p'(\omega) d\omega + J^{\bar{z}} \bar{p}'(\bar{\omega}) d\bar{\omega})}. \quad (3.7)$$

In the following, we will first apply the transition to  $\{\infty\}$  and investigate its effect on that operator. Afterwards we will discuss the effects on the bosonized representation spaces.

From the cited appendix we know that the coordinate change  $x \mapsto x^{-1}$  yields

$$\delta_1 \mapsto - \oint_{(0,0)} \Psi_-(z, \bar{z}) \pi(z) \bar{\pi}(\bar{z}). \quad (3.8)$$

We further know that on  $\bar{N}(p)$  it has the effect  $\bar{N}(p) \leftrightarrow N(p)$ . Therefore, we concentrate on  $\alpha \mapsto -\alpha$ .

The latter causes a change in the charges of the matter fields. This can be understood as follows. In a generic QFT one introduces an  $A$ -field by coupling it to a matter current  $j$  such that

$$\frac{\delta S_m}{\delta A_\mu} \sim j^\mu, \quad \text{where} \quad j^\mu \sim \sum_k \frac{\partial \mathcal{L}_m}{\partial (\partial_\mu \phi_k)} q_k \phi_k. \quad (3.9)$$

Here,  $S_m$  and  $\mathcal{L}_m$  denote the action and Lagrangian of the matter fields  $\phi_k$  with charge  $q_k$ . Usually one absorbs the overall scale factor in the charges and defines that the left hand side above equals the right hand side. In that respect, a mapping of  $A \mapsto -A$  causes a rescaling of the charges  $q_k \mapsto -q_k$ .

What does that mean for the bosonized system? Let us start with the  $bc$ -system as given in A.1.2. These are the changes:

$\alpha$	$-\alpha$
$j^\epsilon(z) = - : b(z)c(z) :$	$j^{\epsilon'}(z) = : b(z)c(z) :$
$j_0 p\rangle = -\epsilon p p\rangle$	$j_0' p\rangle = \epsilon p p\rangle$
$\mathcal{Q} = \epsilon$	$\mathcal{Q}' = -\epsilon$

In particular the OPEs between  $b(z)$  and  $c(z)$  and the representation space  $M^\epsilon(p)$  stay the same. Therefore  $\alpha \mapsto -\alpha$  can only have an effect on the bosonized system. And the reason for that is that, when we associate it to the already changed  $bc$ -system, the new background charge  $\mathcal{Q}'$  enters the energy momentum tensor and changes the conformal weights. We use the same definitions of  $J^\epsilon(z)$ , of  $\phi^\epsilon(z)$  and of the vertex operators. The changes are

$\alpha$	$-\alpha$
$\alpha_0 = -\frac{\epsilon}{2}$	$\alpha'_0 = \frac{\epsilon}{2}$
$\Delta_{T_{J^\epsilon}}(V^\epsilon(r, z)) = \frac{\epsilon}{2} r(r + \epsilon)$	$\Delta'_{T_{J^\epsilon}}(V^\epsilon(r, z)) = \frac{\epsilon}{2} r(r - \epsilon)$
$\Delta_{T_{J^\epsilon}}(\nu_{-\epsilon p}) = \frac{\epsilon}{2} p(p - 1)$	$\Delta'_{T_{J^\epsilon}}(\nu_{+\epsilon p}) = \frac{\epsilon}{2} p(p - 1)$

The most important remark is that the identification of states has changed, for now we have  $\Delta'_{T^\epsilon}(|p\rangle) = \Delta'_{T_{J^\epsilon}}(\nu_{\epsilon p})$ ,  $J_0 \cdot \nu_{\epsilon p} = j_0'|p\rangle$ . Hence, the state  $|p\rangle$  gets now identified with  $\nu_{\epsilon p}$ , whereas before it was

identified with  $\nu_{-ep}$ . That will make the change in the representation in the bosonized system. Before we come to that point, let us derive the correspondence between the fields. As the OPEs stay the same, we end up with

$$\begin{array}{c|c|c} -\alpha & \epsilon = + & \epsilon = - \\ \hline b(z) & V^+(+, z) & V^-(+, z) \otimes \partial_z \xi(z) \\ c(z) & V^+(-, z) & V^-(-, z) \otimes \eta(z) \end{array}$$

The auxiliary  $\eta\xi$ -system is again the same as in A.1.2.

In a next step, we can see that the second GCO, that we obtain in the “observatory” at  $\{\infty\}$  is again basically  $\eta_0$ . Therefore, we consider the changes in the representation spaces. While now  $M^+(p) \xrightarrow{\alpha \mapsto -\alpha} M^+(p) \simeq \bigoplus_{l \in \mathbb{Z}} \mathcal{A}_{\frac{1}{2}}^+ := \mathcal{A}^{+'}$ , the bosonized bosons make a bigger difference:

$$M^-(p) \xrightarrow{\alpha \mapsto -\alpha} M^-(p) \simeq \overline{N}'(p) := \bigoplus_{l \in \mathbb{Z}} \mathcal{A}_{-\frac{1}{2}}^-(-p+l) \otimes \overline{\mathcal{A}}_{-\frac{1}{2}}^+(l). \quad (3.10)$$

The sign of  $p$  is changed on the r.h.s. because of the new identification  $|p\rangle \simeq \nu_{-p}$ . Again,  $\eta_0 : \mathcal{A}_{-\frac{1}{2}}^+(l) \rightarrow \mathcal{A}_{-\frac{1}{2}}^+(l+1)$  and this time we get  $\eta_0 : N'(p) \rightarrow N'(p+1)$ . Therefore,  $\eta_0$  as “seen” from the new “observatory” is indeed a mapping  $\mathcal{F}_\times \rightarrow \mathcal{F}_0^1$ .

If we now apply the calculation already done in the appendix A.2.1, the new GCO reads in the original coordinates

$$\delta_2 := - \oint_{(0,0)} \Psi_-(z, \bar{z}) \pi(z) \bar{\pi}(\bar{z}). \quad (3.11)$$

### 3.4 Summary and Outline of Methods

We have now obtained both Grothendieck-Cousin operators and have found, that they are essentially given by  $\eta_0$ , the zero mode of the auxiliary fermionic  $bc$ -system that is introduced when bosonizing the theory. Therefore we can concentrate and restrict our analysis to the situation of the “observer” at  $\{0\}$ .

In the following, we will make a logarithmic deformation of the auxiliary  $\eta\xi$ -system in the representation of  $\mathcal{F}_0 \otimes \bar{\mathcal{F}}_0$ . The method we choose goes back to Fjelstad e.al. [5]. This will add the GCO  $\delta_1$  to the zero mode of the energy momentum tensor, which will hence serve as a logarithmic improvement term. All other modes of the energy momentum tensor will also be affected and the supercharge acquires an additional term, as well. Furthermore, this sort of deformation will automatically provide us with an extension of the algebra  $\mathcal{F}_0 \otimes \bar{\mathcal{F}}_0 \rightarrow (\mathcal{F}_0 \oplus \mathcal{F}_0^{\star}) \otimes (\bar{\mathcal{F}}_0 \oplus \bar{\mathcal{F}}_0^{\star}) \hookrightarrow \mathcal{F}_{\eta\bar{\eta}} := (\mathcal{F}_0 \oplus \mathcal{F}_\times^1) \otimes (\bar{\mathcal{F}}_0 \oplus \bar{\mathcal{F}}_\times^1)$ , where the “ $\star$ ” denotes the kernel of  $\xi_0$ . As we already explained, the space of states  $\mathcal{F}_0^{\star}$  is enlarged by applying the coordinate transformation from  $\mathbb{C}_0$  to  $\mathbb{C}_\times$ . The GCO acts on that latter space.

The result of these examinations will be that the deformation transforms the theory to an LCFT with logarithmic partners on each level.

## 4 Logarithmic Deformation and the First Grothendieck-Cousin Operator

We will start with a short summary of the method introduced by [5]. Afterwards, we derive and apply a special version to the auxiliary  $\eta\xi$ -system in order to introduce the first GCO in such a way, that it appears as an improvement term in the Hamiltonian.

Fjelstad et al. consider a class of deformations of CFTs that lead to logarithmic extensions. Basically the idea is to enlarge the conformal algebra by introducing additional fields, such that the energy momentum tensor gains an improvement term, leading to a non-reducible representation.

Let  $\mathcal{C}$  denote some chiral algebra of conformal fields and  $\mathcal{F}$  the corresponding Fock space with conformally invariant highest weight vector  $|0\rangle_{\mathcal{F}}$ . We will assume that there exists a fermionic field  $E \in \mathcal{C}$  of conformal weight one such that  $E_0|0\rangle_{\mathcal{F}} = 0$  and  $E(z)E(\omega) = 0$ . The authors deform the fields  $f(z) \in \mathcal{C}$  by introducing a new field  $\delta_E(z)$  and vector space  $\mathcal{K}$  such that

$$\begin{aligned} \delta_E(z) : \mathcal{C} &\rightarrow \mathcal{C} \otimes \text{End}(\mathcal{K}), \quad f(z) \mapsto f_E(z) := \exp(-\beta \delta_E(0)) : f(z), \\ \delta_E(z) &= 1_{\mathcal{F}} \otimes \delta - \int^z E(\omega) d\omega \otimes 1_{\mathcal{K}}, \quad \int^z E(\omega) d\omega = E_0 \log z - \sum_{n<0} \frac{E_n}{n} z^{-n} - \sum_{n>0} \frac{E_n}{n} z^{-n}. \end{aligned} \quad (4.1)$$

$\delta, \beta \in \text{End}(\mathcal{K})$  are Grassman valued. We assume that they satisfy the condition  $[\delta, \beta] = 1_{\mathcal{K}}$  and that we have chosen a vector  $|0\rangle_{\mathcal{K}} \in \mathcal{K}$  such that  $\beta|0\rangle_{\mathcal{K}} = 0$ . The OPE of  $\delta_E$  with a field  $F(z) = f(z) \otimes \sigma$ ,  $\sigma \in \text{End}(\mathcal{K})$  is given by

$$\delta_E(z)F(\omega) = \left( -[E, f]_1 \log(\omega - z) + \sum_{n \geq 1} \frac{1}{n} \frac{[E, f]_{n+1}}{(z - \omega)^n} \right) \otimes \sigma. \quad (4.2)$$

In particular, the energy momentum tensor gets deformed to

$$T(z) \mapsto T_E(z) = T(z) + \frac{\beta}{z} E(z). \quad (4.3)$$

That we introduced  $\delta_E$  and  $\mathcal{K}$  causes the Virasoro algebra to have a non-reducible representation on certain composite fields:

$$\begin{aligned} T_E(z)\Psi_{E,f}(\omega) &= \sum_{m \geq 3} \frac{[E, f_E]_{m-1}}{(z - \omega)^m} + \frac{\Delta_T(f)\Psi_{E,f} + [E, f_E]_1}{(z - \omega)^2} + \frac{\partial_\omega \Psi_{E,f}}{z - \omega}, \\ \Psi_{E,f}(z) &:= - : \delta_E(z) f_E(z) : . \end{aligned} \quad (4.4)$$

Since  $[\delta, \beta] = 1_{\mathcal{K}}$ , the ground state of the extended Fock space is degenerate by the additional vacuum  $E_0^\dagger \delta \cdot |0\rangle_{\mathcal{F}} \otimes |0\rangle_{\mathcal{K}}$ , where  $E_0^\dagger$  denotes the momentum conjugate to  $E_0$ .

Let  $|0\rangle := |0\rangle_{\mathcal{F}} \otimes |0\rangle_{\mathcal{K}}$  and denote by  $\mathcal{F}'$  the Fock representation of  $\mathcal{C}$  on that vector. We extended the chiral algebra  $\mathcal{C}$  by introducing the additional field  $\delta_E(z)$ . With respect to the representation space, that has the effect of introducing an additional state  $\delta|0\rangle$ . Therefore, the new representation space can be identified with  $\mathcal{F}_E := \mathcal{F}' \oplus \mathcal{F}''$ , where  $\mathcal{F}''$  denotes the Fock representation of  $\mathcal{C}$  on  $\delta|0\rangle$ . The deformed fields act between  $\mathcal{F}'$  and  $\mathcal{F}''$ .

#### 4.1 Deformation of the $\eta\xi$ -System - State of the Art

In the following, we denote by  $T^{(a)}$  the energy momentum tensor of the auxiliary  $\eta\xi$ -system and by  $\mathcal{C}^{(a)}$  the respective chiral algebra of fields. The Fock representation space is  $M^+(0) \simeq \mathcal{A}^+$ . The deformation of the  $\eta\xi$ -system as above, where  $\eta$  takes the rôle of  $E$ , is already done in [5]:

$$\begin{aligned} T^{(a)}(z) &\mapsto T_\eta^{(a)}(z) = T^{(a)}(z) + \frac{\beta}{z} \eta(z), \\ \xi(z) &\mapsto \xi_\eta(z) = \xi(z) + \beta \log z \end{aligned} \quad (4.5)$$

and in particular

$$T_\eta^{(a)}|_0 = T_0^{(a)} + \beta \eta_0 . \quad (4.6)$$

The new OPE structure is

$$\begin{aligned} \xi_\eta(z)\delta_\eta(\omega) &= -\log(z-\omega) , \\ T_\eta^{(a)}(z)\Psi_{\eta,\xi}(\omega) &= \frac{0 \cdot \Psi_{\eta,\xi}(\omega) + 1}{(z-\omega)^2} + \frac{\partial_\omega \Psi_{\eta,\xi}(\omega)}{z-\omega} , \end{aligned} \quad (4.7)$$

and  $\Psi_{\eta,\xi}(z)$  is the logarithmic partner of the identity operator  $1 = 1_{\mathcal{A}^+} \otimes 1_{\mathcal{K}}$ . The energy momentum tensor is therefore degenerate on the vacuum vector:

$$\begin{aligned} T_\eta^{(a)}|_0 \cdot |0\rangle &= 0 \cdot |0\rangle , \\ T_\eta^{(a)}|_0 \cdot \xi_0 \delta|0\rangle &= 0 \cdot \xi_0 \delta|0\rangle + |0\rangle . \end{aligned} \quad (4.8)$$

## 4.2 Introducing the Grothendieck-Cousin Operator

The Grothendieck-Cousin operator is mixing holomorphic and anti-holomorphic coordinates. We take this as the starting point for a specification of the deformation just described. The total energy momentum tensor of the  $\eta\xi$ -system is

$$T^{(a)}(z, \bar{z}) = T^{(a)}(z) + \bar{T}^{(a)}(\bar{z}) . \quad (4.9)$$

Instead of introducing an abstract vector space  $\mathcal{K}$ , a sensible way to do a deformation mixing the holomorphic and anti-holomorphic parts is, to consider mappings

$$\begin{aligned} \delta_\eta : \mathcal{C}^{(a)} &\rightarrow \mathcal{C}^{(a)} \otimes \bar{\mathcal{C}}^{(a)} , & \delta_\eta(z) &= 1_{\mathcal{A}^+} \otimes \bar{\xi}_0 - \int^z \eta(\omega) d\omega \otimes 1_{\bar{\mathcal{A}}^+} , \\ \delta_{\bar{\eta}} : \bar{\mathcal{C}}^{(a)} &\rightarrow \mathcal{C}^{(a)} \otimes \bar{\mathcal{C}}^{(a)} , & \delta_{\bar{\eta}}(\bar{z}) &= \xi_0 \otimes 1_{\bar{\mathcal{A}}^+} - 1_{\mathcal{A}^+} \otimes \int^{\bar{z}} \bar{\eta}(\bar{\omega}) d\bar{\omega} \end{aligned} \quad (4.10)$$

and field transformations

$$\delta_{\eta, \bar{\eta}} : \mathcal{C}^{(a)} \otimes \bar{\mathcal{C}}^{(a)} \rightarrow \mathcal{C}^{(a)} \otimes \bar{\mathcal{C}}^{(a)} , \quad f(z, \bar{z}) \mapsto f_{\eta, \bar{\eta}}(z, \bar{z}) =: e^{-q(\delta_\eta(0) \bar{\eta}_0 + \eta_0 \delta_{\bar{\eta}}(0))} : f(z, \bar{z}) . \quad (4.11)$$

This is a specific form of the transformations (4.1) with  $\beta = \bar{\eta}_0$ ,  $\delta = \bar{\xi}_0$  and likewise for the transformation on anti-holomorphic fields. We chose  $\delta$  according to the condition  $[\delta, \beta] = 1_{\mathcal{K}}$ . The position of  $\eta_0$ , playing the rôle of  $\beta$ , is different from (4.1), causing a sign in OPEs.

The mapping above introduces the Grothendieck-Cousin operator

$$T^{(a)}(z, \bar{z}) \mapsto T_\eta^{(a)}(z) + \bar{T}_{\bar{\eta}}^{(a)}(\bar{z}) = \left( T^{(a)}(z) + \frac{q}{z} \eta(z) \bar{\eta}_0 \right) + \left( \bar{T}^{(a)}(\bar{z}) + \frac{q}{\bar{z}} \eta_0 \bar{\eta}(\bar{z}) \right) . \quad (4.12)$$

Indeed, consider the original supersymmetric  $bc$ -system with energy momentum tensor  $T$  as in (3.3). We can now apply the deformation to the auxiliary  $\eta\xi$ -system, as above. This yields  $T(z) \mapsto T_\eta(z) = T(z) + \frac{q}{z} \eta(z) \bar{\eta}_0$ , where we concentrate on the chiral half. If we now rewrite  $\eta$  in terms of the original fields of the full supersymmetric  $bc$ -system and integrate over  $\oint_0 z dz T_\eta(z)$ , we end up with exactly the additional term (A.26):

$$T_0 \mapsto T_{\eta_0} = T_0 - q \oint_{0,0} \Psi_+(z, \bar{z}) \pi(z) \bar{\pi}(\bar{z}) . \quad (4.13)$$

We further have deformations of all other modes and not only of the Hamiltonian. Now again in terms of the bosonized fields:

$$T_n \mapsto T_{\eta_n} = T_n + q \eta_n \bar{\eta}_0 . \quad (4.14)$$

### 4.3 Deformation of the Supercharge

The chiral supercharge gets deformed in the way described above. In terms of the original fields we have

$$\tilde{Q}(z) = Q(z) - \partial_z \psi(z) . \quad (4.15)$$

Its OPE with the new field  $\delta_\eta(z)$  yields

$$\delta_\eta(0)\tilde{Q}(z) = -\frac{\Psi_+(z)}{z} . \quad (4.16)$$

Therefore, we can calculate the deformation of  $\tilde{Q}$  by means of (4.10), ending up with:

$$\tilde{Q}_\eta(z) = \tilde{Q}(z) - \frac{iq}{z} \oint_0 d\bar{z} \Psi_+(z, \bar{z}) \bar{\pi}(\bar{z}) . \quad (4.17)$$

In particular, the zero mode of the supercharge is given by

$$\tilde{Q}_{\eta_0} = \tilde{Q}_0 + \Psi_{+0} \bar{\eta}_0 . \quad (4.18)$$

### 4.4 The Structure of the Space of States

The results of section 4.1 can be carried over to the deformation of the auxiliary  $\eta\xi$ -system just considered. Let  $|0\rangle^{(a)} := |0\rangle_{\mathcal{A}^+} \otimes |\bar{0}\rangle_{\bar{\mathcal{A}}^+}$  and again  $\Psi_{\eta,\xi} := - : \delta_\eta(z) \xi_\eta(z) : .$  The field  $\Psi_{\eta,\xi}$  is the logarithmic partner of  $1^{(a)} := 1_{\mathcal{A}^+} \otimes 1_{\bar{\mathcal{A}}^+} .$  Hence, the energy momentum tensor  $T_\eta^{(a)}$  has a two dimensional representation on the corresponding highest weight vector  $|1\rangle^{(a)} := \xi_0 \bar{\xi}_0 \cdot |0\rangle^{(a)} :$

$$T_\eta^{(a)} \cdot |1\rangle^{(a)} = -|0\rangle^{(a)} . \quad (4.19)$$

Therefore, the logarithmic partners are modelled on  $|1\rangle^{(a)} .$  Nevertheless, that does not lead to a different representation space, since  $\mathcal{A}^+ \simeq M^+(1) \simeq M^+(0) ,$  as we explain in appendix A.1.

For the supersymmetric  $bc$ -system we expect a different solution. The logarithmic partners are supposed to live in  $\mathcal{F}_\times^1 \otimes \bar{\mathcal{F}}_\times^1 ,$  while the original space of states is  $\mathcal{F} := \mathcal{F}_0 \otimes \bar{\mathcal{F}}_0 .$  Indeed, we will find that the zero mode of the deformed energy momentum tensor  $T_\eta(z)$  is a mapping  $T_{\eta_0} : \mathcal{F}_\times^1 \otimes \bar{\mathcal{F}}_\times^1 \rightarrow (\mathcal{F}_\times^1 \otimes \bar{\mathcal{F}}_\times^1) \oplus q (\mathcal{F}_0 \otimes \bar{\mathcal{F}}_0) .$  This affects, however, only a subspace of the full space of states. We will show that the logarithmic extension leads to the Fock representation  $\mathcal{F} \rightarrow \mathcal{F}_{\eta\bar{\eta}} = (\mathcal{F}_0 \oplus \mathcal{F}_\times^1) \otimes (\bar{\mathcal{F}}_0 \oplus \bar{\mathcal{F}}_\times^1) ,$  where the fields, and in particular the energy momentum tensor, are mappings between the summands.

#### 4.4.1 The Space of States for the Supersymmetric $bc$ -System

The extension of the space of states of the full supersymmetric  $bc$ -system is a bit more complicated than for just the  $\eta\xi$ -system. The reason is that the algebra of the auxiliary fermionic fields does not factorize.

For convenience, we define  $N(p, p') := N(p) \otimes \bar{N}(p') .$ <sup>4</sup> If one of its factors, say the first, is  $\bar{N}(p) ,$  we will write  $N(\bar{p}, p')$  and if it is  $N^*(p) ,$  we will denote it by  $N(p^*, p')$  . If both factors are either in the kernel of  $\eta_0$  or of  $\xi_0 ,$  we mark the whole thing by  $\bar{N}(p, p')$  or  $N^*(p, p')$  , respectively.

Logarithmic deformation is applied to the bosonic part of the supersymmetric  $bc$ -system, which is in the representation  $\bar{N}(0, 0) .$  A first peculiarity is that introducing the fields  $\delta_\eta(z)$  and  $\delta_{\bar{\eta}}(\bar{z})$  does enlarge

<sup>4</sup>The overline denotes the kernel of  $\eta_0 ,$  while the bar denotes complex conjugation.

the algebra by the zero modes of  $\xi$  and  $\bar{\xi}$ . Thereby, it gets extended to  $\bar{N}(0,0) \oplus N(0,1^*) \oplus N(1^*,0) \oplus N^*(1,1)$ . This can be understood as follows:

In more detail,  $\bar{N}(0,0) = \left( \bigoplus_{l \in \mathbb{Z}} \mathcal{A}_{\frac{1}{2}}^-(l) \otimes \bar{\mathcal{A}}_{-\frac{1}{2}}^+(l) \right) \otimes \bar{N}(0)$  and the algebra of the auxiliary fermions does not factorize. Therefore, the logarithmic deformation must be performed on the factor  $\bar{\mathcal{A}}_{-\frac{1}{2}}^+(l) \otimes \bar{\mathcal{A}}_{-\frac{1}{2}}^+(l')$  of each summand and not on  $\bar{\mathcal{A}}^+ \otimes \bar{\mathcal{A}}^+$ , as before. In order to do so, we use (4.11) which means that we act with  $\delta_\eta(z)$ ,  $\delta_{\bar{\eta}}(\bar{z})$  and  $\delta_\eta(z)\delta_{\bar{\eta}}(\bar{z})$  on fields. From the point of view of the Fock space, the algebra is thence shifted by  $\xi_0$ ,  $\bar{\xi}_0$  and  $\xi_0\bar{\xi}_0$ . Thereby we arrive at  $\bar{\mathcal{A}}_{-\frac{1}{2}}^+(l) \otimes \bar{\mathcal{A}}_{-\frac{1}{2}}^{+*}(l'-1)$  via  $\delta_\eta(z)$ , and at  $\mathcal{A}_{-\frac{1}{2}}^{+*}(l-1) \otimes \bar{\mathcal{A}}_{-\frac{1}{2}}^+(l')$  and  $\mathcal{A}_{-\frac{1}{2}}^{+*}(l-1) \otimes \bar{\mathcal{A}}_{-\frac{1}{2}}^{+*}(l'-1)$  by means of the other modes, respectively.

In terms of the full theory, the algebra just obtained is  $(\mathcal{F}_0 \oplus \mathcal{F}_0^{1*}) \otimes (\bar{\mathcal{F}}_0 \oplus \bar{\mathcal{F}}_0^{1*})$ . As the CGO is defined on  $\mathbb{C}_\times$ , we apply a coordinate transformation to  $\mathcal{F}_0^1$ , ending up with the result just proposed. Notice, that  $\mathcal{F}_0^1$  and  $\mathcal{F}_\infty^1$  are glued together by means of  $\mathcal{F}_\times^1$  and therefore,  $\mathcal{F}_\infty^1$  is naturally embedded in that fock space.

We can now argue, why the zero mode of the energy momentum tensor  $T_\eta(z)$  of the full supersymmetric  $bc$ -system is a mapping  $T_{\eta_0} : \mathcal{F}_\times^1 \otimes \bar{\mathcal{F}}_\times^1 \rightarrow (\mathcal{F}_\times^1 \otimes \bar{\mathcal{F}}_\times^1) \oplus q(\mathcal{F}_0 \otimes \bar{\mathcal{F}}_0)$ .

In order to be more concrete, we define states

$$|n\rangle_0^{(l)} := \eta_{r_1} \cdots \eta_{r_i} \xi_{k_1} \cdots \xi_{k_j} |0\rangle_{\mathcal{A}^+}, \quad \begin{aligned} r_1 < \cdots < r_i < 0, \quad k_1 < \cdots < k_j < 0, \\ n = \sum |r_i| + |k_j|, \quad l = i - j \end{aligned} \quad (4.20)$$

and

$$|n\rangle_\infty^{(l)} := \eta_{r_1} \cdots \eta_{r_i} \xi_{k_1} \cdots \xi_{k_j} |1\rangle_{\mathcal{A}^+}, \quad \begin{aligned} r_1 < \cdots < r_i < 0, \quad k_1 < \cdots < k_j < 0, \\ n = \sum |r_i| + |k_j|, \quad l = i - j \end{aligned} \quad (4.21)$$

These are basis elements of  $\bar{\mathcal{A}}_{-\frac{1}{2}}^+(l)$  and  $\mathcal{A}_{-\frac{1}{2}}^{+*}(l-1)$ , respectively, for the zero modes of  $\eta$  and  $\xi$  are absent. The algebra of the  $|n\rangle_\infty^{(l)}$  has to be enlarged by the zero mode of  $\eta$ , such that  $\mathcal{A}_{-\frac{1}{2}}^{+*}(l-1) \rightarrow \mathcal{A}_{-\frac{1}{2}}^{+*}(l-1) = \mathbb{C}[|n\rangle_\infty^{(l)}]_{n \in \mathbb{N}} \oplus \eta_0 \cdot \mathbb{C}[|n\rangle_\infty^{(l-1)}]_{n \in \mathbb{N}}$ . Similar definitions hold for the anti-chiral states.

The modes of the deformed chiral energy momentum tensor of the auxiliary  $\eta\xi$ -system are  $T_\eta^{(a)} = T_k^{(a)} + q \eta_k \bar{\eta}_0$ . Their action on  $|n, \bar{n}\rangle_{s, \bar{s}}^{(l, \bar{l})} := |n\rangle_s^{(l)} \otimes |\bar{n}\rangle_{\bar{s}}^{(\bar{l})}$ ,  $s, \bar{s} \in \{0, \infty\}$  is as follows.

For the zero mode we have

$$T_\eta^{(a)} \cdot |n, \bar{n}\rangle_{s, \bar{s}}^{(l, \bar{l})} = n |n, \bar{n}\rangle_{s, \bar{s}}^{(l, \bar{l})} - q \mathcal{N} |n, \bar{n}\rangle_{0,0}^{(l, \bar{l})}, \quad (4.22)$$

where we introduce the shortcut  $\mathcal{N} := (-)^{i+\bar{i}+j+\bar{j}} \delta_{s,\infty}$  and  $\bar{\mathcal{N}} := (-)^{i+\bar{i}+j+\bar{j}} \delta_{\bar{s},\infty}$ . As  $\eta_0^2 = 0$ , the image of the Hamiltonian is in its kernel. Therefore, due to the first Grothendieck-Cousin operator,  $T_{\eta_0} : \mathcal{F}_\times^1 \otimes \bar{\mathcal{F}}_\times^1 \rightarrow (\mathcal{F}_\times^1 \otimes \bar{\mathcal{F}}_\times^1) \oplus q(\mathcal{F}_0 \otimes \bar{\mathcal{F}}_0)$ . On all other states in  $\mathcal{F}_{\eta\bar{\eta}}$ , it is diagonal.

For the other modes with  $k \neq 0$ , we find

$$T_\eta^{(a)} \cdot |n, \bar{n}\rangle_{s, \bar{s}}^{(l, \bar{l})} = T_k^{(a)} \cdot |n, \bar{n}\rangle_{s, \bar{s}}^{(l, \bar{l})} \pm q \eta_k \cdot \bar{\mathcal{N}} |n, \bar{n}\rangle_{s,0}^{(l, \bar{l})} \quad (4.23)$$

and the mapping  $T_{\eta_k}$  is in general not diagonal if the states are in the subspace  $(\mathcal{F}_0 \oplus \mathcal{F}_\times^1) \otimes \bar{\mathcal{F}}_\times^1$ . The sign is plus if  $s = 0$  and minus, otherwise.

## 4.5 Morse-Novikov Theory Included

The results of the last section are unexpected from the point of view of the Morse-Novikov theory on  $\widetilde{LX}$ , that we are investigating. The reason is, that there, we are only treating two sorts of spaces of states. One of them is associated with flow lines that descend from  $\{0\} \in X$  and are described by polynomials of coordinates along  $\widetilde{LX}_{0,n}$ . The other is related to flow lines descending from  $\{\infty\}$ , belonging to  $\widetilde{LX}_{\infty,n}$ . The Fock spaces on which they are modeled are, as we explained,  $\mathcal{F}_0 \otimes \bar{\mathcal{F}}_0$  and  $\mathcal{F}_\infty^1 \otimes \bar{\mathcal{F}}_\infty^1$ , respectively.

The space of states we obtained for the two dimensional field theory by means of the logarithmic deformation is, however, larger. It even includes states in  $(\mathcal{F}_0 \otimes \bar{\mathcal{F}}_x^1) \oplus (\mathcal{F}_x^1 \otimes \bar{\mathcal{F}}_0)$  that have no immediate geometric interpretation. In a way, these spaces are mixing coordinate charts around  $\{0\} \in X$  with charts around  $\{\infty\} \in X$ , introducing an additional non-locality into the two dimensional field theory.

## 5 Discussion and Open Questions

We will discuss our results in three graded steps. Let us start with a summary of the main results.

### Summary

We have deformed the supersymmetric  $bc$ -system with target and domain manifold being  $\mathbb{CP}^1$ , associated to (2.1). The guideline for our calculations was the condition that the zero mode of the energy momentum tensor should acquire additional terms

$$T_0 \mapsto T_0 + \delta_1 + \delta_2 .$$

The argument has been, that the zero mode of the energy momentum tensor should be identified with (part of) the Hamiltonian, the GCOs included, of the Morse-Novikov theory which is associated with the A-model as given by (2.1). From the point of view of Morse-Novikov theory, these terms are an effect of the nontrivial geometry of target space  $\widetilde{L\mathbb{CP}^1}$ .

The method of deformation was a logarithmic deformation of the supersymmetric  $bc$ -system according to [5]. We performed it for the first GCO  $\delta_1$  and started with the  $bc$ -system in the representation  $\mathcal{F}_0$ . Thereby, the space of states was extended to  $(\mathcal{F}_0 \oplus \mathcal{F}_x^1) \otimes (\bar{\mathcal{F}}_0 \oplus \bar{\mathcal{F}}_x^1)$ . The same applies to the second GCO and the full energy momentum tensor obtains extra terms on every level:

$$T_{-n} \mapsto T_{-n} + \eta_{-n} \bar{\eta}_0 + q \tilde{\eta}_{-n} \bar{\tilde{\eta}}_0 , \quad n \geq 0 .$$

The result is an LCFT with logarithmic partners on each level of the energy momentum tensor. The space of states of the Morse-Novikov theory is included as a subspace, it is just  $(\mathcal{F}_0 \otimes \bar{\mathcal{F}}_0) \oplus (\mathcal{F}_\infty^1 \otimes \bar{\mathcal{F}}_\infty^1)$ . In terms of the original fields the energy momentum tensor reads

$$T(z) \mapsto T(z) - \oint_0 d\bar{z} (\Psi_-(z, \bar{z}) + q \Psi_+(z, \bar{z})) \pi(z) \bar{\pi}(\bar{z}) .$$

It is globally defined on  $X$ , where the chart transition has to come along with a change of sign in the A-field.

The supercharge and hence the cohomology is also deformed and yields the globally defined quantity:

$$\tilde{Q}(z) \mapsto \tilde{Q}(z) + \frac{i}{z} \oint_0 d\bar{z} (\Psi_-(z, \bar{z}) - q \Psi_+(z, \bar{z})) \bar{\pi}(\bar{z}) .$$

In short terms: we have shown on the level of the two dimensional supersymmetric  $bc$ -system, i.e. for the corresponding Virasoro algebra, that classically the proposal of Frenkel, Losev and Nekrasov is correct, the model satisfies all properties of an LCFT. If we call it therefore an LCFT we mean that it has an appropriate Virasoro algebra.

## Questions

What are the questions that we could not answer? There are mainly two:

The more important one, which we did not touch, is if the thus deformed  $bc$ -system is an LCFT on quantum level? What is the vacuum expectation value  $\langle T_\eta \rangle$  of the deformed energy momentum tensor and in which vacuum or in how many should we take it? Notice, that we considered the full moduli space of vacuum configurations of the theory with instantons and one of the vacuum configurations is charged and not conformally invariant. As at least one of the GCOs is a mapping from the representation space of the vacuum which is not conformally invariant, we further are not sure in what respect we should call the quantum theory an (L)CFT.

The second, maybe related question results from our calculations. In order to make a logarithmic deformation we bosonized the theory. Thereby we introduced an auxiliary fermionic  $bc$ -system which introduces an additional current and a charge anomaly. That is, however, invisible in the perspective of the original theory. What effect does that have on the quantum level and how is that related with the theory before bosonization?

## Speculations

So far the results and related questions. The motivation for the calculations we have done was, however, the question we put forth in the introduction: “can we understand the attribute of a conformal field theory on curved target space to be a *logarithmic* conformal field theory as an aspect of gravity?” What can we say about that after all?

Of course we can not say anything about that on the level of the fully quantized theory. On the classical level our résumé is, that we can give a positive answer because of the connection of non-unitarity, gravity and the presence of only instantons that we have drawn. Especially for the logarithmically deformed  $bc$ -system it seems that its’ non-unitarity gets a direct interpretation in terms of gravity, as we explained in section 3.3.2. Moreover, non-unitarity seems to be a necessity for the supersymmetric  $bc$ -system to be an (L)CFT: on the level of the Virasoro algebra one can see that a logarithmic deformation of the Hamiltonian of the  $bc$ -system can only be done by either  $\eta_0$  or  $\xi_0$  but not by both. This again mirrors that in the Morse-Novikov theory under consideration only instantons (or if we choose, only anti-instantons) are present.

The solitude of instantons results in a logarithmic deformation of the supersymmetric  $bc$ -system by the GCOs whose presence can equivalently be interpreted as an effect of the non-trivial geometry of target space.



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# A Notations and Conventions

This appendix serves to fix the notations and to define the objects that are important within this paper. We hope further, that it is self-contained enough in order to enable the ambitious reader, who may not be an expert on that subject, to understand the details left out in the main part and to reproduce the calculations.

## A.1 The CFT Conventions

We will treat a special version of the topological  $A$  model derived in [8], consisting of a bosonic scalar field  $x$ , which is a mapping  $x : \Sigma \rightarrow X$  of a Riemannian surface into a CY manifold  $X$ . The embedding  $x$  consequently splits into a holomorphic  $x^a$  and an antiholomorphic part  $x^{\bar{a}}$ . Its dual  $p$  splits into elements of  $\Omega_{\Sigma}^{1,0} \otimes \Omega_X^{1,0}$  and  $\Omega_{\Sigma}^{0,1} \otimes \Omega_X^{0,1}$ , respectively:  $p = p_{az} dz dx^a + p_{\bar{a}\bar{z}} d\bar{z} dx^{\bar{a}}$ . However, we will not consider  $p$  but  $p' := p + \Gamma_b^a \psi^b \pi_a + \Gamma_{\bar{b}}^{\bar{a}} \psi^{\bar{b}} \pi_{\bar{a}}$ , which does not transform like a one form in  $X$ . The scalar  $\psi$  and one form  $\pi$  are the supersymmetric partner fields with the same transformation properties as the bosons. Though they have the wrong statistics, we will call the superpartners “fermions”.

### A.1.1 Basic Fields and the OPE Conventions

The OPE structure for the embedding  $x$  and its conjugate momentum is given as follows:

$$x^a(z)p'_b(\omega) = \frac{i\delta_b^a}{z-\omega}, \quad \psi^a(z)\pi_b(\omega) = \frac{-i\delta_b^a}{z-\omega}. \quad (\text{A.1})$$

★

We will also need a more general notation for OPEs. Therefore, let us introduce the bracket  $[\cdot, \cdot]_n$  according to [5]. For any two conformal fields  $A(z)$  and  $B(\omega)$  the element  $[A, B]_n(\omega)$  denotes a primary field in the OPE

$$A(z)B(\omega) = \sum_{n \geq 1} \frac{[A, B]_n(\omega)}{(z-\omega)^n}. \quad (\text{A.2})$$

From that, we can derive the identity

$$[B, A]_n(z) = (-)^{F_A F_B} \sum_{k \geq n} \frac{(-)^k}{(k-n)!} \partial_z^{k-n} [A, B]_k(z), \quad (\text{A.3})$$

where  $F_A$  denotes the fermion number of  $A$ .

★

The energy momentum tensor and current of interest will be

$$\begin{aligned} T(z) &= i : p'_a(z) \partial_z x^a(z) - \pi_a(z) \partial_z \psi^a(z) : , \\ j(z) &= -i : x^a(z) p'_a(z) + \psi^a(z) \pi_a(z) : . \end{aligned} \quad (\text{A.4})$$

The conformal weights with respect to  $T$  and charges are then

$$\begin{aligned} \Delta_T(x) &= 0 = \Delta_T(\psi), \quad \text{charge: } -1, \\ \Delta_T(p') &= 1 = \Delta_T(\pi), \quad \text{charge: } +1. \end{aligned} \quad (\text{A.5})$$

Further we will need the supercharge, that is given as

$$Q(z) = i : p'(z) \psi(z) : . \quad (\text{A.6})$$

### A.1.2 Chiral Bosonization

Later, we will map that fields to a supersymmetric  $bc$  system. Most of our conventions of that section goes back to [4, 10]. The  $bc$  systems corresponding to the bosons will be labeled by an index  $\epsilon = -$  and the one for the fermions with an index  $\epsilon = +$ . Its relations to the algebra above is given by

$$\begin{aligned} \epsilon = - : \quad x &\mapsto b, \quad ip' \mapsto c \\ \epsilon = + : \quad \psi &\mapsto b, \quad i\pi \mapsto c. \end{aligned} \tag{A.7}$$

★

The  $bc$ -system above, gives rise to a Heisenberg algebra, generated by

$$[c_n, b_m] = \delta_{n,-m}. \tag{A.8}$$

that has irreducible representations  $M^\epsilon(p)$ ,  $p \in \mathbb{Z}$ :

$$b_n |p\rangle = 0, \quad n > -p, \quad c_n |p\rangle = 0, \quad n \geq p. \tag{A.9}$$

As above  $M^\epsilon(p)$  denotes the bosons by the choice of  $+$  and otherwise the fermions. The current

$$j^\epsilon(z) = - : b(z)c(z) := \sum_{n \in \mathbb{Z}} j_n z^{-n-1} \tag{A.10}$$

gives a partition of  $M^\epsilon(p) = \bigoplus_{l \in \mathbb{Z}} M^\epsilon(p)_l$  such that  $|p\rangle \in M^\epsilon(p)_{-ep}$ . The Virasoro algebra acts on  $M^\epsilon(p)$  by

$$T^\epsilon(z) = : \partial_z b(z)c(z) : , \tag{A.11}$$

where normal ordering is defined in the representation of  $M^\epsilon(0)$ .

Let  $u_n(z) := z^{-n+1} \partial_z$  and  $\mathcal{L}_{u_n}$  denote the Lie derivative in direction of  $u_n$ . The modes of the energy momentum tensor  $T_n^\epsilon$  act on the field modes  $b_m$  and  $c_m$  like the operator  $T'_n = \mathcal{L}_{u_n}$  on the differentials  $z^{-m}$  and  $z^{-m-1} dz$ :

$$\begin{aligned} [T_n^\epsilon, c_m] &= -m c_{m+n}, \\ [T_n^\epsilon, b_m] &= -(m+n) b_{m+n}. \end{aligned} \tag{A.12}$$

In particular, the Hamiltonian, i.e. the zero mode of the  $bc$ -system acts therefore like the Lie derivative in direction of  $z \partial_z$ , the generator of the  $\mathbb{C}^\times$ -symmetry on  $\mathbb{CP}^1$ , on each mode.

The central charge of  $T^\epsilon$  is given by  $c^\epsilon = -2\epsilon$ . Therefore, the central charge of the supersymmetric  $bc$ -system is zero. The current is anomalous with background charge  $\mathcal{Q} = \epsilon$ :

$$\begin{aligned} T_n |p\rangle &= 0, \quad n > 0, \quad T_0 |p\rangle = \frac{1}{2} \epsilon p(p-1) |p\rangle, \\ j_n |p\rangle &= 0, \quad n > 0, \quad j_0 |p\rangle = -\epsilon p |p\rangle. \end{aligned} \tag{A.13}$$

In the case of fermions the extremal states are simply related by

$$\begin{aligned} |p+1\rangle &= \xi_{-p} \xi_{-p+1} \cdots \xi_0 |0\rangle, \quad p \geq 0, \\ |p\rangle &= \eta_{-p} \eta_{-p+1} \cdots \eta_{-1} |0\rangle, \quad p < 0 \end{aligned} \tag{A.14}$$

and  $\eta_0$  is mapping  $\mathcal{M}^+(p)_l \rightarrow \mathcal{M}^+(p)_{l+1}$ .

★

Chiral bosonization means that the algebra  $M^\epsilon(p)$  is expressed in terms of another algebra  $\mathcal{A}_{\alpha_0}^\epsilon(h)$ . This is the Heisenberg algebra

$$[J_n, J_m] = \epsilon n \delta_{n, -m} \quad (\text{A.15})$$

of a bosonic current  $J^\epsilon(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}$  with heighest wheight representation given by

$$J_n \nu_h = h \delta_{n,0} \nu_h, \quad n \geq 0, \quad h \in \mathbb{C} \quad (\text{A.16})$$

and with an action of the Virasoro algebra by means of

$$T_{J^\epsilon}(z) = \epsilon \left( \frac{1}{2} : J^\epsilon(z)^2 : + \alpha_0 \partial_z J^\epsilon(z) \right). \quad (\text{A.17})$$

The central charge is  $c_{J^\epsilon} = \epsilon(1 - 12\epsilon\alpha_0^2)$  and the conformal wheight of  $\nu_h$  is  $\Delta_{T_{J^\epsilon}}(\nu_h) = \frac{1}{2}\epsilon h(h - 2\alpha_0)$ . The identification is as follows. Set  $\alpha_0 = -\frac{1}{2}\epsilon$  and introduce a chiral scalar field

$$\phi^\epsilon(z) = \epsilon \int^z J^\epsilon(\omega) d\omega = \epsilon \left( \phi_0 + J_0 \log z - \sum_{n \neq 0} \frac{J_n}{n} z^{-n} \right), \quad [\phi_0, J_n] = -\epsilon \delta_{n,0}. \quad (\text{A.18})$$

Define further vertex operators

$$V^\epsilon(r, z) =: \exp r \phi(z) := e^{\epsilon r \phi_0} z^{\epsilon r \kappa} e^{-\epsilon r \sum_{n < 0} \frac{J_n}{n} z^{-n}} e^{-\epsilon r \sum_{n > 0} \frac{J_n}{n} z^{-n}}, \quad r \in \mathbb{C} \setminus \{0\}. \quad (\text{A.19})$$

Then in the fermionic case

$$\begin{aligned} c(z) &\mapsto V^+(+, z), \\ b(z) &\mapsto V^+(-, z) \end{aligned} \quad (\text{A.20})$$

and  $M^+(p) \simeq \mathcal{A}^+ := \bigoplus_{l \in \mathbb{Z}} \mathcal{A}_{-\frac{1}{2}}^+(l)$  with  $|p\rangle \mapsto \nu_{-p}$ . Notice that  $M^+(p) \simeq M^+(p')$ ,  $\forall p, p'$ . This is not true for the bosons and also the identification is not that simple.

In the bosonic case, the central charge does not fit with the one in the representation  $M^-(p)$ . Therefore, one has to introduce an auxiliary fermionic  $bc$ -system, with  $b := \xi$  of conformal weight zero and  $c := \eta$  of conformal weight one and background charge  $\mathcal{Q}_{\eta\xi} = 1$ . The fields  $b$  and  $c$  of the former bosonic theory are then identified by

$$\begin{aligned} c(z) &\mapsto V^-(+, z) \otimes V_{\eta\xi}^+(+, z), \\ b(z) &\mapsto V^-(-, z) \otimes \partial_z V_{\eta\xi}^+(-, z), \end{aligned} \quad (\text{A.21})$$

This corresponds, on the first sight, to the algebra  $N(p) := \bigoplus_{l \in \mathbb{Z}} \mathcal{A}_{\frac{1}{2}}^-(p+l) \otimes \mathcal{A}_{-\frac{1}{2}}^+(l)$ . Since the zero mode of  $V_{\eta\xi}^+(-, z)$  (or equivalently  $\xi_0$ ) does not appear,  $M^-(p)$  is, however, only isomorphic to the kernel of the mapping  $1 \otimes V_{\eta\xi,0}^+ : N(p) \rightarrow N(p-1)$  (or equivalently the kernel of  $1 \otimes \eta_0$ ). Therefore,  $M^-(p) \simeq \overline{N}(p) := \bigoplus_{l \in \mathbb{Z}} \overline{\mathcal{A}}_{\frac{1}{2}}^-(p+l) \otimes \overline{\mathcal{A}}_{-\frac{1}{2}}^+(l)$ , where the overline denotes the kernel of  $\eta_0$ . This is much nicer and in more detail described in [4].

## A.2 The Chiral De Rham Complex

The chiral de Rham complex is introduced in [13]. It is a sheaf of vertex algebras on a smooth manifold  $X$  with a canonical embedding of the de Rham complex. The starting point are the Heisenberg and Clifford

algebras of the supersymmetric  $bc$ -system. The zero modes  $\{x_0^a, \psi_0^a\}$  and  $\{p'_{a0}, \pi_{a0}\}$  can be identified with geometric objects on  $X$  in the following manner:

bosons:		fermions:	
$x_0^a$	$x^a$	$\psi_0^a$	$dx^a$
$p'_{a0}$	$\partial_a$	$c_{a0}$	$\iota_a$

In that respect, we can define an exterior derivative, which is just the de Rham differential by  $d := Q_0$ , where  $Q_0$  is the supercharge. Chart transitions can be defined on the zero modes and that supplies us with the de Rham complex. The chiral de Rham complex generalizes these chart transitions to all modes and to the fields that can be assigned to them. They read:

$$\begin{aligned}
x &\mapsto \phi_x(z) = g(x)(z) , \\
p' &\mapsto \phi_{p'}(z) = \left( \frac{\partial g^{-1}}{\partial \phi_x} p' + \frac{\partial^2 g^{-1}}{\partial \phi_x^2} \frac{\partial g}{\partial x} \pi \psi \right) (z) , \\
\psi &\mapsto \phi_\psi(z) = \frac{\partial g}{\partial x} \psi(z) , \\
\pi &\mapsto \phi_\pi(z) = \frac{\partial g^{-1}}{\partial \phi_x} \pi(z) .
\end{aligned} \tag{A.22}$$

Especially one can show that, when  $\phi_x = x^{-1}$ , it is only the zero mode  $x_0$  that has to be inverted [8, pg. 91]. The composite fields, like the energy momentum tensor and the supercharge undergo transformations

$$\begin{aligned}
T(z) &\mapsto \phi_T(z) = T(z) , \\
Q(z) &\mapsto \phi_Q(z) = Q(z) - \partial_z \left( \frac{\partial}{\partial \phi_x} \left[ \log \frac{\partial g^{-1}}{\partial \phi_x} \right] \phi_\psi(z) \right) .
\end{aligned} \tag{A.23}$$

### A.2.1 The Fields $\eta$ and $\xi$ in Terms of the Original Fields

The fields  $\eta$  and  $\xi$  can be expressed in terms of the original fields. That has already been calculated in [8] for  $\eta$ . Below, we will summarize how that works and also give the precise formulas for  $\xi$  and the supercharge.

The trick is the following. Start with the original fields in the coordinate chart  $\mathbb{C}_0$ . Apply a coordinate transformation on  $X$ :  $g : X \ni x \mapsto g(x) = \exp(x)$ . All fields transform according to (A.22):

$$\begin{aligned}
\phi_x(z) &= e^{x(z)} , & \phi_{p'}(z) &= e^{-x(z)} (p'(z) + \psi(z)\pi(z)) , \\
\phi_\psi(z) &= e^{x(z)} \psi(z) , & \phi_\pi(z) &= e^{-x(z)} \pi(z) , \\
\phi_T(z) &= T(z) , & \phi_Q(z) &= Q(z) + \partial_z \psi(z) .
\end{aligned} \tag{A.24}$$

Exponentiating  $x$  generalizes its definition to  $x(z) \mapsto x(z) + \hat{\omega} \log z$ . This demands an additional zero mode conjugate to  $\hat{\omega}$ , that we get by substituting  $p'(z) = \partial_z U(z)$ ,  $U(z) = \int^z p'(\omega) d\omega$ .<sup>5</sup> Now, we can bosonize the model in analogy with [8] and thereby obtain

$$\begin{aligned}
\eta(z) &\mapsto i\pi(z) e^{-i \int^z p'(\omega) d\omega} , \\
\xi(z) &\mapsto \psi(z) e^{i \int^z p'(\omega) d\omega} .
\end{aligned} \tag{A.25}$$

The Grothendieck-Cousin operators are obtained after a change to  $\mathbb{C}_\times$ , for they are mappings between states in the chart  $\mathbb{C}_0$  and  $\mathbb{C}_\infty$ , respectively. The corresponding transformation  $\phi_x \mapsto \phi_x^{-1}$  has the effect

<sup>5</sup>This leads to a different representation space, as described for example in [9] and [1]. For our task, that is, however, not of importance.

that each element of  $\{p', \psi, \pi\}$  gets multiplied by  $-1$ . The Grothendieck-Cousin operators we proposed are now written as

$$\begin{aligned}\delta_{(\infty,0),(0,1)} &= -q \oint_{0,0} \Psi_+(z, \bar{z}) \pi(z) \bar{\pi}(\bar{z}) , \\ \delta_{(0,0),(\infty,0)} &= \oint_{0,0} \Psi_-(z, \bar{z}) \psi(z) \bar{\psi}(\bar{z}) ,\end{aligned}\tag{A.26}$$

where

$$\Psi_{\pm}(z, \bar{z}) = \Psi_{\pm}(z) \Psi_{\pm}(\bar{z}) = e^{\pm i \left( \int^z p'(\omega) d\omega + \int^{\bar{z}} \bar{p}'(\bar{\omega}) d\bar{\omega} \right)} .\tag{A.27}$$

The supercharge transforms to

$$\phi_Q(z) \mapsto \tilde{Q}(z) = Q(z) - \partial_z \psi(z) ,\tag{A.28}$$

while the energy momentum tensor is again not affected.

### A.3 Morse-Bott-Novikov Theory on Loop Space

A nice source on that topic is [6]. The case we will need is that  $X$  is a simply connected, compact Kähler manifold. The loop space  $LX$  of  $X$  is the space of smooth maps  $\gamma : S^1 \rightarrow X$ . It is not simply connected. If we want to formulate a Morse theory on loop space, that is a problem because the potential  $v(x)$  is supposed to be a gradient of some function  $f \in C^\infty(X)$ . Therefore, one lifts the model to the universal cover, which is the space

$$\widetilde{LX} := \{(\gamma, \tilde{\gamma}) \mid \gamma \in LX, \tilde{\gamma} : D \rightarrow X \text{ such that } \gamma = \tilde{\gamma}|_{\partial D}\} / \sim ,\tag{A.29}$$

where  $\sim$  denotes equivalence with respect to homotopy. On  $\widetilde{LX}$ ,  $f$  can be defined uniquely according to

$$f(\tilde{\gamma}) = \int_D \tilde{\gamma}^*(\omega_K) - \alpha \int_{\partial D} \tilde{\gamma}^*(H) .\tag{A.30}$$

$\omega_K$  denotes the Kähler two-form and  $H_\alpha$  is a smooth, real valued function on  $\partial D \times X$ , reducing the critical manifold of the Hamiltonian flow equation to isolated points on  $X$ . If  $\alpha = 0$ , we have the situation of a Morse-Bott-Novikov theory, otherwise, it is Morse-Novikov. On  $LX$ ,  $f$  is a multivalued function and the sheets of the covering, i.e.  $\widetilde{LX}_n$ , are counted by  $H_2(X, \mathbb{Z})$ . That appears within the space of states as an equivariance operator  $q^n$ ,  $n \in \mathbb{Z}$ , [8].

Let  $\alpha \neq 0$  and  $x_\mu$  be a critical point of the flow equation  $\phi_v(x, t) = x(t)$ . The descending manifolds  $\widetilde{LX}_{\mu, n}$  are defined as

$$\widetilde{LX}_{\mu, n} := \left\{ \tilde{\gamma} \in \widetilde{LX}_n : \lim_{t \rightarrow -\infty} \phi_v(\tilde{\gamma}, t) = x_\mu \right\} .\tag{A.31}$$

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