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Conformal Field Theory Survival Kit

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ABSTRACT: The bare necessities of conformal field theory are provided such that chances to survive the string theory crash course, held fall 2000 at Hannover University within the German *String Network*, are increased. This is not a comprehensive introduction to the subject, and provides emergency treatment only.
Although there exist many introductions to conformal field theory, this one might in particular be useful for students without too much knowledge in general quantum field theory.

KEYWORDS: Conformal field theory.

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Contents

1. Introduction

Conformal field theory (CFT) constitutes one of the main tools in string theory. This "survival kit" attempts to provide the reader with the most important techniques and results of this fascinating topic for immediate use in the string theory lectures. Since the material presented here is intended to be covered in three ninety minutes lectures only, the scope of these notes is very limited. The interested reader should consult the seminal paper [1] and the reviews [2] for further help and a mroe thorough treatment of the subject. But before we start to talk about CFT as a theory on its own, we will try to explain briefly, why CFT shows up in string theory so prominently.

Historically, string theory evolved out of an attempt to understand the increasing zoo of hadronic resonances in strong interactions during the sixties. One way of a physicist to understand something is to search for a pattern. Plotting the spin of these strong resonances versus their mass squared, one obtaines the so-called Regge trajectories, i.e. straight lines, which brought some order into the zoo. Although this experimental fact is very beautiful, it is very difficult to explain within a quantum field theory, since particles with high spin normally cause serious difficulties to keep the theory unitary, i.e. to assure that probability is conserved.

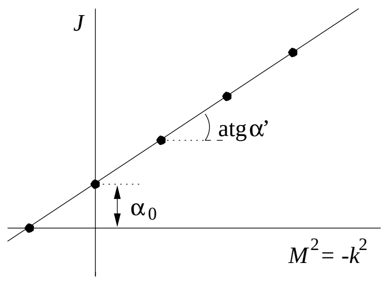

Figure 1: Regge trajectory.

We can illustrate this by considering a four-point amplitude of a real scalar field, say, in the t-channel: $\phi(p_2)\phi(p_3) \longrightarrow \phi(p_1)\phi(p_2)$ with an interchanged particle ψ_J of integer spin J. The interaction term in the Lagrangian for the cubic vertex $g_J\phi\phi\psi_J$ contains therefore J derivatives. If $J = 0$, then the amplitude for the invariant interaction term $g_0\phi\phi\psi_0$ behaves like $A_0 \propto g_0^2(t - M^2)^{-1}$, where M is the mass of the interchanged particle, and g_0 is the coupling constant. Obviously, the amplitude develops a pole when $t = (p_2 + p_3)^2$ is equal to M^2, and it vanishes fro $t \to \infty$, as it should be. However, if $J > 0$, the vertex must be of the form $g_J(\phi \overleftrightarrow{\partial}_{\mu_1} \ldots \overleftrightarrow{\partial}_{\mu_J} \phi)\psi^{(\mu_1,\ldots,\mu_J)}$. According to the Feynman rules, each derivative becomes a momentum factor, and hence the amplitude now goes like $A_J \propto g_J^2(t - M^2)^{-1}s^J$, where the Mandelstam variable $s = (p_1 + p_2)^2$ gives the momentum transfer of the process. More precisely, the amplitude is proportional to the Legendre polynomial $P_J(\theta)$ of the scattering angle θ which, however, happens to be given as $\theta \sim s$ for high energies. Obviously, the amplitude diverges for very high energies. A way out of this dilemma could be that these infinities cancel each other if the full amplitude is a sum over all possible internal states, $A(s,t) \propto \sum_J g_J^2 s^J (t - M^2)^{-1}$. Inspection shows that this could only work if this sum is taken over an infinity of resonances, not only the experimentally observed states on the Regge trajectory. In fact, this idea led to the so-called dual amplitude models, since an automatic consequence of this ansatz was the appearance of a new symmetry, namely the duality of the amplitudes: $A(s,t) = A(t,s)$.

The reader might remind herself that grouping states into multiplets of an assumed underlying symmetry group is another physicist's way to assemble things into a pattern. Better known examples are (iso)spin $SU(2)$ multiplets, flavor $SU(3)$ multiplets etc., which all belong to finite dimensional unitary representations of the corresponding Lie groups. In our case, we want to put an infinity of states into one (unitary) representation of some symmetry group, which hints at the possibility that this group might be infinite-dimensional. Of course, if we view all the states on a Regge trajectory as one representation, it seems natural to consider them as excitations of some fundamental object – the string.

The Regge trajectories have two free parameters, the intercept α_0 with the J-axis, and the slope α'. An early result was that dual amplitudes can only be unitary and analytic, when $\alpha_0 = 1$ or $\alpha_0 = 2$. In the former case, there exists a massless vector particle (a "photon"), in the latter, we have a massless spin two field (a "graviton"). Unfortunately, we also get a spin zero state of *negative* mass squared in both cases, the so-called tachyon.

Further experimental data soon disqualified the Regge trajectories as a valid way to sort the zoo of hadronic resonances, QCD was discovered and established to quite successfully explain strong interactions. But a view people continued to work on models with

Regge trajectory like spectra and their amplitudes, but considering them on a completely different energy scale. In this way, string theory was discovered, since such spectra can easily be understood as excitations of a one-dimensional small but extended object. What made these people pursue this idea was the surprising fact that consistent unitary theories admitting a Regge spectrum do contain a massless spin two state which make them natural candidates for a quantized theory of gravitation. This is so intruiging that other surprises, for example the problem that string theories, as they were called since then, need some extra space-time dimensions in order to be consistent, were accepted with all their consequences instead of being viewed as a strong hint to immediately discard any such crazy theory.

At the end of this introduction, we will comment on how strings, and with them conformal field theory, are a natural consequence of duality of amplitudes. But for the moment we assume that the reader has already heard something about strings. Remembering ordinary particle mechanics, we know that time evolution of a particle generates a world line. In the path integral approach to its quantum theory, we sum over all possible trajectories. The natural action functional of a free relativistic particle of mass m is simply proportional to the length of the world line,

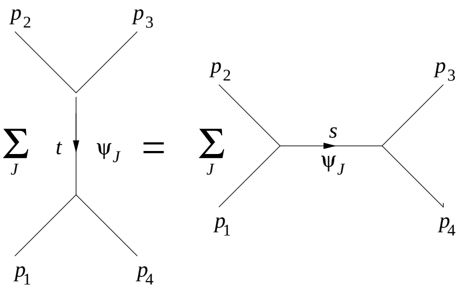

Figure 2: Duality.

$$S[x(\tau)] = -m \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \tag{1.1}$$

with τ the eigen time. Similarly, the most natural quantity to describe the action of a string, a finite one-dimensional object, moving in some flat space-time is the area of the world-surface swept out by it. The reader will learn much about strings in the main lectures of Olaf Lechtenfeld, to which these notes are a mere appendix. If she is impatient, she might consult [5]. We denote with σ and τ the space and time coordinates of the world-sheet, respectively. The embedding of it into a d-dimensional Minkowski space-time $\mathbb{R}^{1,d-1}$ is described by functions $X^\mu(\sigma,\tau)$. The action is then given by the area of the world-sheet with respect to the reduced metric

$$\begin{aligned} \mathrm{d}s^2 &= \eta_{\mu\nu}\mathrm{d}X^\mu\mathrm{d}X^\nu \\ &= \eta_{\mu\nu}\left(\partial_\tau X^\mu\partial_\tau X^\nu\mathrm{d}\tau^2 + \partial_\sigma X^\mu\partial_\sigma X^\nu\mathrm{d}\sigma^2 + 2\partial_\tau X^\mu\partial_\sigma X^\nu\mathrm{d}\sigma\mathrm{d}\tau\right) . \end{aligned} \tag{1.2}$$

Introducing $\xi^0 = \tau$, $\xi^1 = \sigma$, we can write this as

$$\mathrm{d}s^2 = g_{ij}(X)\mathrm{d}\xi^i\mathrm{d}\xi^j . \tag{1.3}$$

With this metric, the action of the string world-sheet $X^\mu(\xi^0,\xi^1)$ is proportional to the area $\int\sqrt{\det g}$,

$$S[X(\xi^i)] = \tfrac{1}{2\pi\alpha'}\int \mathrm{d}\xi^0\mathrm{d}\xi^1\sqrt{(\partial_0 X)^2(\partial_1 X)^2 - (\partial_0 X\cdot\partial_1 X)^2} \tag{1.4}$$

with the usual definition $A \cdot B = \eta_{\mu\nu} A^\mu B^\nu$. The string tension $1/(2\pi\alpha')$ is the analogue of the particle mass. Since we always put $\hbar = c = 1$, its units are mass over length or mass squared.

This action is quite complicated and difficult to handle. Moreover, the embedding mapping X^μ can be arbitrarily complex. Fortunately, there exists a simplification of this action which will be thoroughly derived in the string lectures. As a result, by introducing the auxiliary metric g_{ij} on the two-dimensional word-sheet, an action quadratic in X can be obtained,

$$S[X,g] = -\tfrac{1}{2\pi\alpha'} \int \mathrm{d}^2\xi \sqrt{g} g^{ij} (\partial_i X \cdot \partial_j X)\,. \tag{1.5}$$

The metric on the surface is considered as a new field. Since no derivatives of g_{ij} appear in (1.5), its equations of motions will lead to a constraint on the dynamical field X^μ. We know from general relativity that under a variation $g \mapsto g + \delta g$, the action varies as

$$\delta S = -\tfrac{1}{2\pi\alpha'} \int \mathrm{d}^2\xi \sqrt{g} T^{ij} \delta g_{ij}\,, \tag{1.6}$$

which defines the stress-energy tensor T^{ij}. Therefore, the classical theory defined by the action (1.5) satisfies the equations of motion

$$T^{ij} = (\partial^i X)\cdot(\partial^j X) - \tfrac{1}{2} g^{ij} (\partial^k X)\cdot(\partial_k X) = 0\,, \tag{1.7}$$

$$\frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} \partial^i X^\mu\right) = \Box X^\mu = 0\,. \tag{1.8}$$

Hence, in the presence of the metric g_{ij}, the field X^μ is a free scalar field which does not carry two-dimensional energy or momentum. Writing (1.7) as $(\partial_i X)\cdot(\partial_j X) = \frac{1}{2} g_{ij} (\partial^k X)\cdot(\partial_k X)$ and taking the determinant on each side shows that the action (1.5) is indeed equivalent to the so-called Nambu-Goto action (1.4).

It is instructive to obtain (1.7) via the variation δS with respect to δg_{ij}. Firstly, from $\delta(\delta_i{}^j) = 0 = \delta(g_{ik}g^{kj})$ we find that $\delta g^{ij} = -g^{ij}g^{kl}\delta g_{kl}$. Secondly, the variaton $\delta\sqrt{g}$ can be obtained in a mathematically slightly sloppy way by using $\sqrt{\det g} = \exp(\frac{1}{2}\log\det g)$ which yields $\delta\sqrt{g} = \frac{1}{2}\sqrt{g}\delta\log g$. Now, by definition $\delta\log g = \log\det(g_{ij}+\delta g_{ij}) - \log\det(g_{ij}) = \log(\det(g_{ij}+\delta g_{ij})\det g^{ij})$. With $\log\det g = \mathrm{tr}\log g$ and $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$, we finally arrive at $\delta\sqrt{g} = \frac{1}{2}\sqrt{g}\,\mathrm{tr}(g^{ik}\delta g_{kj} + \mathcal{O}(\delta^2))$. Putting all together, we find up to the factor $-1/(2\pi\alpha')$

$$\begin{aligned}
\delta S &\equiv \tfrac{1}{2} \int \mathrm{d}^2\xi \sqrt{g} T^{ij} \delta g_{ij} \\
&= \int \mathrm{d}^2\xi \left[(\delta\sqrt{g}) g^{ij} (\partial_i X)\cdot(\partial_j X) + \sqrt{g}(\delta g^{ij})(\partial_i X)\cdot(\partial_j X)\right] \\
&= \int \mathrm{d}^2\xi (\partial_i X)\cdot(\partial_j X) \left[g^{kl}\delta g_{lk} \tfrac{1}{2}\sqrt{g} g^{ij} + \sqrt{g}(-g^{ik}g^{jl}\delta g_{kl})\right] \\
&= \int \mathrm{d}^2\xi \sqrt{g} \left[-(\partial^i X)\cdot(\partial^j X) + \tfrac{1}{2} g^{ij} (\partial_k X)\cdot(\partial_l X) g^{kl}\right] \delta g_{ij}\,,
\end{aligned} \tag{1.9}$$

from which (1.7) can be read off.

With the other equation of motion, (1.8), one easily proves the following properties of the stress-energy tensor, namely that it is conserved and traceless. In fact, $\partial_i T^{ij} = -(\Box X)\cdot(\partial^j X) - (\partial^i X)\cdot(\partial_i \partial^j X) + \frac{1}{2}\partial^i((\partial_k X)\cdot(\partial^k X)) = 0$. Also, $T_j^j = -(\partial^j X)\cdot(\partial_j X) + \frac{1}{2}(\partial_k X)\cdot(\partial^k X) g_j^j = (\frac{1}{2}D - 1)(\partial_k X)\cdot(\partial^k X)$, where we have explicitly denoted the dimension $D = 2$ of the world-sheet. Note the remarkable fact that the stress-energy tensor is traceless only for a free scalar field in $D = 2$ dimensions.

The outcome of this is that the free propagation of a string in $\mathbb{R}^{1,d-1}$ is described by a free two-dimensional field theory. But there is more to the equivalence of these two actions. Both actions enjoy several non-trivial symmetries:

1.) Both actions are invariant under arbitrary reparametrizations $\xi^i \mapsto f^i(\xi)$ of the surface, under which the metric transforms as a rank two tensor, $g_{ij} \mapsto (\partial_i f^k(\xi))(\partial_j f^l(\xi))g_{kl}$.

2.) The stress-energy tensor T_{ij} in (1.7) is traceless with respect to g_{ij}, i.e. $g^{ij}T_{ij} = 0$. This constraint is due to local Weyl invariance of the action (1.5), which is invariance under local scalings of the metric, $g_{ij} \mapsto \exp(\phi(\xi^0, \xi^1))g_{ij}$. This is true only in $D = 2$ dimensions, since only then remains the factor $\sqrt{g}g^{ij}$ unchanged.

3.) Classically, we have three symmetries. Besides reparametrization invariance (diffeomorphisms) and local Weyl invariance, there are also the space-time symmetries depending on the isometries of the metric $\eta_{\mu\nu}$. In our standard Minkowski space-time, the action is clearly invariant under the d-dimensional Poincaré group, $X^\mu \mapsto \Lambda^\mu{}_\nu X^\nu + b^\mu$ with $\Lambda^\mu{}_\nu \in SO(1, d-1)$ a Lorentz transformation.

Coordinate transformations which leave the metric invariant up to a local scaling factor preserve all angles and are therefore conformal transformations. As should be clear by now, string theory is described by the (minimal) coupling of a conformally invariant field theory (in our approach of d two-dimensional scalar fields X^μ, $\mu = 0, \ldots, d-1$) to two-dimensional gravity. We have not said anything about the topology of the world-sheet. For simplicity, we assume from now on that the string itself is closed, i.e. forms a small loop (there is more about this to say, which will be said in the string lectures). The simplest topology of a world-sheet of a closed string is a cylinder. In this case, $\sigma = \xi^1$ lives on a circle S^1 while $\tau = \xi^0$ lives on $\mathbb{R}$ or on an intervall $I = [\tau_i, \tau_f]$. The sum of all embeddings of $I \times S^1$ into $\mathbb{R}^{1,d-1}$ describes the propagation of a free string. We can use reparametrization invariance to fix $\sqrt{g}g_{ij} = \eta_{ij}$ in this simple topology. With the Minkowski metric on the cylinder (i.e. in the conformal gauge), the action (1.5) simplifies to

$$S = -\tfrac{1}{2\pi\alpha'} \int \mathrm{d}\tau \mathrm{d}\sigma \left((\partial_\tau X)^2 - (\partial_\sigma X)^2 \right) , \tag{1.10}$$

where the convention $\eta_{\tau\tau} = -\eta_{\sigma\sigma} = 1$ is used.

1.1 CFT on the complex plane

We come now to one of the main tools in two-dimensional CFT which make it to such a powerful instrument for the exploration of string theory. As discussed above, the theory lives (in the simplest case) on the Riemann surface $\mathbb{R} \times S^1$, i.e. a cylinder, where we have taken the times of the initial and final states to be asymptotically in the infinite past and infinite future respectively, $\tau_i = -\infty$, $\tau_f = +\infty$. It is much more convenient to consider the theory on the punctured complex plane $\mathbb{C}^* = \mathbb{C} - \{0\}$, such that we can exploit all the power of complex analysis. This is done as follows:

First, light-cone coordinates (or chiral coordinates) are introduced, $\sigma^+ = \tau + \sigma$ and $\sigma^- = \tau - \sigma$. The metric element then reads $\mathrm{d}s^2 = \mathrm{d}\tau^2 - \mathrm{d}\sigma^2 = \mathrm{d}\sigma^+\mathrm{d}\sigma^-$. Next, τ is mapped via a Wick-rotation to Euclidean time, $\tau \mapsto -\mathrm{i}\tau$ such that $\sigma^+ \mapsto -\mathrm{i}(\tau + \mathrm{i}\sigma) \equiv \mathrm{i}w$ and $\sigma^- \mapsto -\mathrm{i}(\tau - \mathrm{i}\sigma) \equiv \mathrm{i}\bar{w}$. Under Wick rotation, the metric becomes $\mathrm{d}s^2 = -\mathrm{d}w\mathrm{d}\bar{w}$, and the null geodesics are the straight lines $\sigma^\pm = const$, for which $\mathrm{d}s^2 = 0$. We conclude that the causal structure is preserved by any transformation of the form $\sigma^+ \mapsto f(\sigma^+)$, $\sigma^- \mapsto g(\sigma^-)$ where f, g are arbitrary and independent functions. Finally, the complexified coordinates $w, \bar{w}$ are mapped to $\mathbb{C}^*$ by the conformal transformation $z = \mathrm{e}^w$. Therefore, Fourier expansions in $\tau \pm \sigma$, i.e. expansions in e^w and $\mathrm{e}^{\bar{w}}$ become Laurent expansions in $z, \bar{z}$. This will prove extremely useful in what is to come.

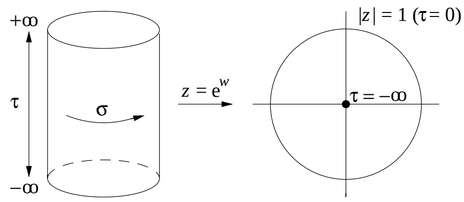

Figure 3: Conformal mapping of the cylinder to $\mathbb{C}^*$.

Infinite past (the lower end of the cylinder) is mapped to the origin of $\mathbb{C}$, and infinite future (the upper end of the cylinder) is mapped to the infinite far of the complex plane. If we would compactify the complex plane to the Riemann sphere, infinite future would go the the added point ∞ on it. Note that light-cone left and right chiral coordinates $\sigma^\pm$ translate into holomorphic and anti-holomorphic variables $z, \bar{z}$ on the Riemann sphere (with both poles removed). A consequence of this description is that the equations of motion (1.8) become $\partial_+\partial_- X^\mu = 0$, i.e.

$$\partial_z \partial_{\bar{z}} X^\mu = \partial\bar{\partial} X^\mu = 0\,. \tag{1.11}$$

The most general solution, which is single valued (i.e. physically observable) is easily written down,

$$X^\mu = q^\mu - \mathrm{i}p^\mu \log|z|^2 + \mathrm{i}\sum_{n\neq 0} \frac{a_n^\mu}{n} z^{-n} + \mathrm{i}\sum_{n\neq 0} \frac{\tilde{a}_n^\mu}{n} \bar{z}^{-n}\,. \tag{1.12}$$

This expression, the string coordinate, has contributions from the location of the center of mass of the string, q^μ, its total momentum p^μ, and oscillator modes a_n^μ and $\tilde{a}_n^\mu$, which describe its left-moving and right-moving excitations respectively. That is more apparent, if we return momentarily to the original coordinates, in which the string coordinate clearly is a Fourier expansion,

$$X^\mu = q^\mu - \mathrm{i}p^\mu(\sigma^+ + \sigma^-) + \mathrm{i}\sum_{n\neq 0} \frac{a_n^\mu}{n} \exp(\mathrm{i}\sigma^+ n) + \mathrm{i}\sum_{n\neq 0} \frac{\tilde{a}_n^\mu}{n} \exp(\mathrm{i}\sigma^- n)\,. \tag{1.13}$$

Since the energy momentum tensor T_{ij} is symmetric and traceless, it has only two non-vanishing components in holomorphic coordinates, namely T_{zz} and $T_{\bar{z}\bar{z}}$. Conservation of energy and momentum yields

$$\bar{\partial} T_{zz} = \partial T_{\bar{z}\bar{z}} = 0\,. \tag{1.14}$$

This means that $T \equiv T_{zz}$ is a holomorphic function on $\mathbb{C}^*$, while $\bar{T} \equiv T_{\bar{z}\bar{z}}$ is anti-holomorphic. Hence, T has a Laurent expansion,

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}\,, \tag{1.15}$$

and analogously for $\bar{T}(\bar{z})$. From now on, many formulæ will only be given for the holomorphic part of the theory, since the anti-holomorphic part is completely analogous. The factor z^{-2} in (1.15) is due to the conformal mapping $z = \mathrm{e}^w$. On the original cylinder, T is a quadratic differential, i.e. $T_{ww}(\mathrm{d}w)^2$ is a scalar. However, the conformal mapping yields $\mathrm{d}w = z^{-1}\mathrm{d}z$ and thus $(\mathrm{d}w)^2 = z^{-2}(\mathrm{d}z)^2$, leading to the additional factor z^{-2}. This holds for a general tensor $T_{w\ldots w\bar{w}\ldots\bar{w}}$ of rank $(j\bar{\jmath})$, which will acquire an overall factor $z^{-j}\bar{z}^{-\bar{\jmath}}$. Since we consider tensors with respect to conformal transformations, $j, \bar{\jmath}$ are also called the left and right chiral conformal weights of the tensor.

The modes L_n have the geometrical meaning to generate infinitesimal conformal transformations through Poisson brackets. Since these are holomorphic functions, a basis for them may be given as

$$z \mapsto z + \varepsilon z^{n+1} = z + \varepsilon v_n^z(z)\,, \tag{1.16}$$

where the vector field $v^z(z)$ may exhibit poles (or zeroes) only at the two points $z = 0$ or $z = \infty$. With $\bar{\partial}T(z) = 0$, clearly also $\bar{\partial}z^nT(z) = 0$, and we can therefore consider $z^nT(z)$ to be the local density of the Noether charge which implements (1.16) on the fields of the theory. Usually, this charge is computed in field theory by integrating over the surface given by $\tau = 0$. In our holomorphic coordinates, this translates into a contour integral

$$L_n = \frac{1}{2\pi\mathrm{i}} \oint_{|z|=1} z^{n+1}T(z)\,. \tag{1.17}$$

Of course, since $T(z)$ is analytic in $\mathbb{C}^*$, L_n is conserved, because it is independent of contour deformations. Actually, since time evolution on the cylinder is equivalent to radial evolution on the z-plane, constant τ surfaces correspond to circles centered at the origin with radii $|z| = \mathrm{e}^\tau$. Analyticity of T implies that

$$L_n = \frac{1}{2\pi\mathrm{i}} \oint_C z^{n+1}T(z) \tag{1.18}$$

for any closed contour C encircling the origin, as long as no sources of energy or momentum are present. Choosing for C a circle $|z| = R$ with a radius $R \neq 1$ yields time-independece. But there is a much larger symmetry at work here, a reflection of conformal invariance of the theory, since L_n is invariant under any homologous deformation of C.

1.2 Invariance under su(1,1)

There are three particular important transformations, which are associated with the generators L_0 and $L_{\pm 1}$. These are infinitesimally given as

$$\begin{aligned} L_{-1}:\ & z \mapsto z + \varepsilon_{-1}\,, \\ L_0:\ & z \mapsto z + \varepsilon_0 z\,, \\ L_1:\ & z \mapsto z + \varepsilon_1 z^2\,, \end{aligned} \tag{1.19}$$

which are infinitesimal translations, dilatations, and so-called special conformal transformations, respectively. Their global versions are $z \mapsto z + b$, $z \mapsto (a/d)z$, and $z \mapsto -z/(cz-1)$, which generate the well known Möbius group

$$z \mapsto \frac{az+b}{cz+d}\,, \quad a,b,c,d \in \mathbb{C}\,, \quad ad - bc = 1\,, \tag{1.20}$$

which are the only conformal automorphisms of the Riemann sphere onto itself. Indeed, there are three complex parameters ε_n, $n = -1, 0, 1$, which allow to impose the condition $ad - bc = 1$. The Möbius group is thus the group $SL(2,\mathbb{C})/\mathbb{Z}_2 = \mathbb{P}SL(2,\mathbb{C})$, i.e. the group of special linear transformation of the projective complex plane. This can be seen as follows: The group $SL(2,\mathbb{C})$ is the set of all 2×2 matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with determinant one, which acts on complex two-dimensional vectors $\left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right)$. Identifying vectors related by an overall complex scale, only the ratio $z = z_1/z_2$ is a free parameter, which indeed transforms as (1.20). This transformation is cleary invariant under multiplying the matrix with an overall factor. The determinant condition can fix this invariance only up to a sign, which explains why we have to divide out a $\mathbb{Z}_2$ symmetry.

Together with the analogous anti-holomorphic relations we learn that $L_0 + \bar{L}_0$ generates time translations $\tau \mapsto \tau + \varepsilon$, and that $L_0 - \bar{L}_0$ generates rotations $\sigma \mapsto \sigma + \varepsilon'$. The full conformal group is $\mathcal{C} = \mathbb{P}SL(2,\mathbb{C}) \times \mathbb{P}SL(2,\mathbb{C})$ which contains as a subgroup $SO(1,3)$. The latter group is what one would naively expect to be the global conformal group on $\mathbb{R}^{1,1}$. $\mathcal{C}$ is twice as large as $SO(1,3)$ in terms of real generators. However, if we impose a reality condition to the independently treated variables $z, \bar{z}$ (such that we either obtain the complex plane or the Minkowski cylinder), the number of generators reduces to that of $SO(1,3)$.

The infinitesimal generators of conformal transformations satisfy the so-called Witt-algebra, which is the algebra of infinitesimal diffeomorphisms on the cricle S^1. Putting $\ell_n = -z^{n+1}\partial_z$, we have

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m}\,. \tag{1.21}$$

We have seen that the conformal group on the Riemann sphere is finite dimensional. However, we found that on the Minkowski cylinder (in chiral $\sigma^\pm$ coordinates, the conformal group actually is $\mathit{Diff}(S^1) \times \mathit{Diff}(S^1)$, and hence infinite-dimensional. Locally, both surfaces admitt an infinite-dimensional algebra. The generators ℓ_n with $n = -1, 0, 1$ are particularly important, since these are the only ones which can be integrated to global transformations on both surfaces. They form the algebra $\mathfrak{su}(2)$ or, equivalently, $\mathfrak{su}(1,1)$.

1.3 The Virasoro algebra

Ultimatively, we wish to work with a quantized theory. Conformal transformations are then generated via commutators with the corresponding Noether charges instead of Poisson brackets. The densities $v^z(z)T_{zz}(z)$ for vector fields $v^z(z)$ yield charges

$$v^z \mapsto L[v] = \frac{1}{2\pi \mathrm{i}} \oint_C v(z)T(z)\,, \tag{1.22}$$

which are well defined since the product of the tensor T_{zz} and the vector v^z is a one-differential over which a contour integral can be taken. This is a representation of the Witt-algebra of vector fields, and the question arises, whether $L[\ell_n] = L_n$ still satisfies (1.21).

One should now recall a strange feature of elementary quantum theory. Physical states correspond to rays in a Hilbert space, not to points, since the phase of the state cannot be observed experimentally. Therefore, a symmetry may be represented on a Hilbert space $\mathcal{H}$ not only linearly, but also projectively, i.e. up to a phase. In the latter case, one says that the symmetry is anomalous, because a projective symmetry is equivalent to a linear symmetry in the central extension of the original symmetry algebra. Indeed, the Witt-algebra admitts precisely one singel central extension, $\dim H^2(\mathit{Diff}(S^1)) = 1$. As a general rule, adding a single central element $\hat{c}$ to the generators L_n, we will be able to work with linear representations. The eigenvalue c of the operator $\hat{c}$ is called the central charge, and labels the representation. Hence, we expect that the representation $L[.]$ has the algebraic structure

$$\big[L[\ell_n], L[\ell_m]\big] = L\left[[\ell_n, \ell_m]\right] + p(n,m)c\mathbb{1} \tag{1.23}$$

with $p(n,m)$ a (complex-valued) function and $\hat{c} = c\mathbb{1}$. The form of $p(n,m)$ can be determined by making use of the antisymmetry of the commutator and the Jacobi identity, which imply

$$\begin{aligned} &p(n,m) = -p(m,n)\,, \\ &(n-m)p(n+m,k) + (m-k)p(m+k,n) + (k-n)p(k+n,m) = 0\,. \end{aligned} \tag{1.24}$$

The general solution to this equation is

$$p(n,m) = (an^3 + bn)\delta_{n+m,0}\,. \tag{1.25}$$

We fix $a = -b$ by the requirement that we want to keep the $\mathfrak{su}(1,1)$ symmetry explicitly, i.e. we want $p(n,m)$ to vanish for $n,m \in \{-1,0,1\}$. We can do this without loss of generality by a linear redifinition of the generators. Hence, we arrive at the celebrated Virasoro algebra,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{\hat{c}}{12}(n^3 - n)\delta_{n+m,0}\,, \tag{1.26}$$

where the choice $a = 1/12$ is conventional. Mathematically, $p(n,m)$ is a so-called co-cycle, and (1.24) are the cocycle conditions. Another way to see why a central extension arises is, to consider the norm of states in the Hilbert space $\mathcal{H}$. We certainly want that there exists a (unique) vacuum state $|0\rangle$ which should be invariant under global conformal transformations, $L_n|0\rangle = 0$ for $n = -1, 0, 1$. Considering the full Virasoro algebra, we cannot expect the vacuum to be invariant under all local conformal transformations, since this would be in contradiction to the algebra. In fact, $L_nL_{-n}|0\rangle = L_{-n}L_n|0\rangle = 0$ implies $0 = [L_n, L_{-n}]|0\rangle = (2nL_0 + \hat{c}/2\binom{n+1}{3})|0\rangle \neq 0$ for $c \neq 0$. To ensure that the vacuum functional $\langle 0|0\rangle$ is invariant under the action of the Virasoro algebra, it suffices to impose the smaller set of conditions

$$L_n|0\rangle = 0 \quad \forall n \geq -1\,. \tag{1.27}$$

Together with (1.15) and (1.26) we can now easily compute the the two-point function $\langle 0|T(z)T(w)|0\rangle = \langle T(z)T(w)\rangle$,

$$\begin{aligned}
\langle T(z)T(w)\rangle &= \langle 0|\sum_n L_n z^{-n-2}\sum_m L_m w^{-m-2}|0\rangle \\
&= \langle 0|\sum_{m,n}(n-m)L_{n+m}z^{-n-2}w^{-m-2}|0\rangle + \frac{c}{2}\sum_{n>1}\binom{n+1}{3}\left(\frac{z}{w}\right)^{-n}(zw)^{-2} \\
&= \frac{c}{2}z^{-4}\sum_{n\geq 0}\binom{n+3}{3}\left(\frac{w}{z}\right)^{n} \\
&= \frac{c}{2}\frac{1}{(z-w)^4}\,.
\end{aligned} \tag{1.28}$$

It follows from this that the central extension of the symmetry algebra amounts to fixing the norm of states in the Hilbert space.

2. Amplitudes

This section is mainly devoted to some issues which arose from the early studies of dual amplitudes and which led to the advent of CFT. It might be helpful to understand how the so-called vertex operators do arise and what kind of objects one ultimatively wants to compute.

2.1 Planar Amplitudes

Let us consider a scattering amplitude of N particles with momenta $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_N$ in d-dimensional Minkowski space with signature $(1, -1, \ldots, -1)$ such that $\boldsymbol{k}^2 = -m^2$. Projecting the scattering process onto a plane such that the external legs do not intersect, we say that two such scattering amplitudes are equivalent under planar ordering, if there exists a cyclic relabelling of the external lines mapping one into the other. For example, the inequivalent planar orderings of a four-point amplitude are $(1,2,3,4)$, $(1,3,2,4)$, $(1,2,4,3)$.

We call a contiguous subset of external lines a planar channel. For example, $(1,2,3)$ is a planar channel in the first of the above planar orderings, and $(1,2)$ is a planar channel in the first and the last ordering. To each planar channel $(i, i-1, \ldots, j) \equiv (i\_j)$ we associate an energy variable $s_{ij} = (\boldsymbol{k}_i + \boldsymbol{k}_{i+1} + \ldots + \boldsymbol{k}_j)^2$ which gives the energy transfer from the planar channel to its complement through an intermediate channel. Given such a planar channel, any other planar channel with at least one external leg in the same position as in the original planar channel is said to overlap. Hence, to each planar channel (i_j) we can associate a whole family of overlapping channels (i'_j') with momentum transfer variables $s_{i'j'}$. Note that the energies of the overlapping channels act as momentum transfers with respect to the original planar channel.

With these definitions at hand, we call an amplitude $A_{\rm planar}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N)$ planar, if and only if

$$A_{\rm planar}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N) = \sum_{\substack{\text{planar}\\ \text{orderings}}} A(\boldsymbol{k}_{i_1},\ldots,\boldsymbol{k}_{i_N})\,, \tag{2.1}$$

where each of the partial amplitudes $A(\boldsymbol{k}_{i_1},\ldots,\boldsymbol{k}_{i_N})$ possesses an infinite number of poles in each planar channel, which lie on Regge trajectories

$$J = \alpha(s) = \alpha' s + \alpha_0 \;\in\; \mathbb{Z}_+\,. \tag{2.2}$$

The residues at the poles are polynomial in the momentum transfer variables of degree at most J.

Applying this axiomatic definition of a planar amplitude to the four-point case, the sum extends over three inequivalent planar orderings,

$$A_{\rm planar}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) = A(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) + A(\boldsymbol{k}_1,\boldsymbol{k}_3,\boldsymbol{k}_2,\boldsymbol{k}_4) + A(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_4,\boldsymbol{k}_3)\,. \tag{2.3}$$

The first term allows the two planar channels (1_2) or (2_3). If we put $s = s_{12}$ and $t = s_{23}$, the required pole structure of the first term dictates that it may only depend on s and t, i.e. $A(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) = A(s,t)$. In a similar way we can conclude that $A(\boldsymbol{k}_1,\boldsymbol{k}_3,\boldsymbol{k}_2,\boldsymbol{k}_4) = A(u,t)$ with $u = s_{13}$, because the second term has the planar channels (1_3) and $(4\_1) \simeq (3\_2)$. Analogously, $A(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_4,\boldsymbol{k}_3) = A(s,u)$.

Duality means that $A(s,s') = A(s',s)$ such that the dual four-point amplitude is $A_{\rm planar}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) = A(s,t) + A(t,u) + A(u,s)$.

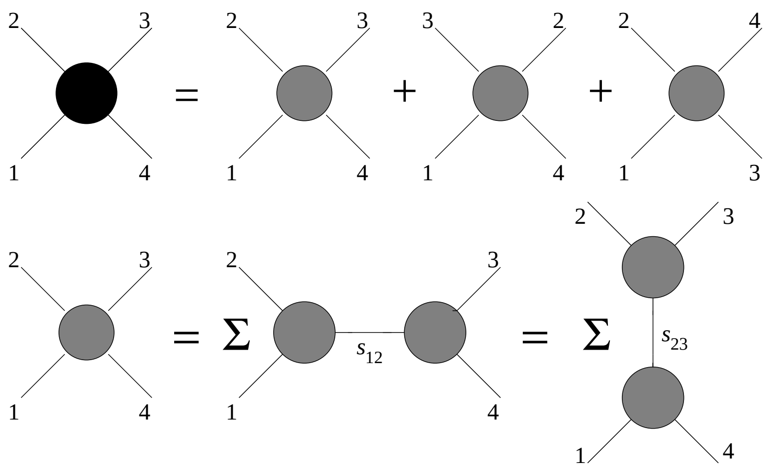

Figure 4: Decomposition of the planar 4-point amplitude into its three different planar orderings, decomposition of a planar ordering into planar channels.

2.2 The Veneziano amplitude

Let us concentrate on one of the partial amplitudes, say $A(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) = A(s,t) = A(t,s)$, which according to our requirements shall be a meromorphic function of s for fixed t and vice versa. A particular simple function satisfying these properties is Euler's beta function,

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} = \sum_{n=0}^{\infty} \frac{1}{u+n}\frac{(-)^n}{n!}\prod_{\ell=1}^{n}(v-\ell) = \sum_{n=0}^{\infty} \frac{1}{v+n}\frac{(-)^n}{n!}\prod_{\ell=1}^{n}(u-\ell)\,, \tag{2.4}$$

where the first series expansion is in u for fixed v, and the second one is in v for fixed u. For fixed u, $B(u,v)$ exhibits poles whenever $v = -n$, $n \in \mathbb{Z}_+$, and the residue is indeed a polynomial in u of degree n. Therefore, the corresponding Regge trajectory is $v = -\alpha(t) = -\alpha' t - \alpha_0$, and similarly $u = -\alpha(s) = -\alpha' s - \alpha_0$. The starting point of string theory, the famous Veneziano amplitude, is then obtained by setting

$$A(s,t) = B(-\alpha(s), -\alpha(t))\,. \tag{2.5}$$

In the days preceeding string theory, physicists developed so-called dual resonance models to explain the proliferation of hadronic resonances. If we believe in the idea of duality, we may try to find microscopical physical systems which explain it. In the early days of dual models, people found that Euler's beta function yields a dual amplitude. With $\alpha(s) = \alpha' s + \alpha_0$, a dual four-point amplitude, the so-called Veneziano amplitude, reads

$$A_{\text{planar}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4) = B(-\alpha(s), -\alpha(t)) + B(-\alpha(t), -\alpha(u)) + B(-\alpha(u), -\alpha(s))\,, \tag{2.6}$$

where s, t, u denote the Mandelstam variables, and where $B(u, v)$ has the integral representation

$$B(-\alpha(s), -\alpha(t)) = \int_0^1 \mathrm{d}z\, z^{-\alpha(s)-1}(1-z)^{-\alpha(t)-1}\,. \tag{2.7}$$

The poles in the s-channel originate from the integration region $z \to 0$, while the region $z \to 1$ is responsible for the poles in the t-channel.

But why is such an amplitude a hint towards string theory? This comes about if we try to interpret the parameter z as some coordinate. It looks as if z is somehow the distance between particles 1 and 2, and $(1 - z)$ is the distance between partilces 2 and 3. One might imagine the external legs as living on a line, i.e. the real axis with coordinates $0, z$, and 1 for particles 1,2, and 3. More generally, let us identify $-\infty$ with $+\infty$ compactifying the real line to a circle S^1 and let us introduce the so-called Koba-Nielsen variables z_i which we associate to every external leg. In this way, an ansatz for a general planar N-point amplitude satisfying the duality property can be found,

$$\begin{aligned} A(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_N) = &\int_{\{z_i > z_j : i > j\}} \prod_{i=1}^{N} \mathrm{d}z_i \delta(z_1 - z_a)\delta(z_{N-1} - z_b)\delta(z_N - z_c) \\ &\times\, (z_b - z_a)(z_c - z_b)(z_c - z_a) \prod_{i=1}^{N-1} (z_{i+1} - z_i)^{\alpha_0 - 1} \prod_{i>j} (z_i - z_j)^{-2\alpha' \boldsymbol{k}_i \cdot \boldsymbol{k}_j}\,. \end{aligned} \tag{2.8}$$

Note that we introduced three marked point z_a, z_b and z_c, whose meaning will become clear below. The region of integration introduces an ordering on the circle, which essentially is a quantum-mechanical "time-ordering".

The Koba-Nielsen variables are the simplest solution to the requirement that a planar amplitude be of the form

$$\int \prod \mu(X_{ij}) \prod_{\substack{\text{planar}\\ \text{channels}}} X_{ij}^{-\alpha(s_{ij})-1}\,, \tag{2.9}$$

such that a variable X_{ij} is associated to each planar channel (i_j) with the property that for $X_{ij} \to 0$ all $X_{i'j'} \to 1$ with (i'_j') an overlapping channel of (i_j). In this way poles never develop in two different channels for the same region of parameters. The measure $\mu(X_{ij})$ is yet unspecified, but should at least reflect the cyclic permutation symmetry of planar amplitudes. An ansatz can be found with the Koba-Nielsen variables by simply putting $X_{ij} = z_i - z_j$.

The amplitude (2.8), resticted to $N = 4$ and $z_a = 0$, $z_b = 1$, $z_c = \infty$ does indeed reproduce (2.6), if the definition of the Mandelstam variables $s_{ij} = (\boldsymbol{k}_i + \boldsymbol{k}_j)^2 = \boldsymbol{k}_i^2 + \boldsymbol{k}_j^2 + 2\boldsymbol{k}_i \cdot \boldsymbol{k}_j$ is used together with momentum conservation, $\sum_i \boldsymbol{k}_i = 0$, and the mass shell condition for the lightest particles on the Regge trajectory (the spin zero particles), $\alpha' \boldsymbol{k}_i^2 + \alpha_0 = 0$. Note that the general ansatz (2.8) is dual only on the mass shell.

A very important feature of (2.8) is that it yields a dual amplitude on the mass shell for arbitrary values of z_a, z_b, and z_c. The reason is that (2.8) is $SL(2,\mathbb{R})$ invariant. This invariance is just invariance under Möbius transformations (1.20) with a, b, c, d real. The infinitesimal transformations, i.e. the ones with a, d close to one and b, c close to zero, are

$$z \mapsto z' = z + \varepsilon(\alpha + \beta z + \gamma z^2)\,, \tag{2.10}$$

from which we find the algebra of the generators to form a sub-algebra of the Witt-algebra (1.21) with $n, m \in \{-1, 0, +1\}$. Explicitly, $\ell_{-1} = -\partial_z$, $\ell_0 = -z\partial_z$, and $\ell_1 = -z^2\partial_z$. Now, the reader might convince herself that under such a transformation

$$\begin{cases} z_i - z_j \mapsto (z_i - z_j)(1 + \varepsilon[\beta + \gamma(z_i + z_j)])\,, \\ \quad \mathrm{d}z_i \mapsto \mathrm{d}z_i(1 + \varepsilon[\beta + 2\gamma z_i])\,. \end{cases} \tag{2.11}$$

It is now convenient to rewrite (2.8) in the following way, where the abbreviation $z_{ij} = z_i - z_j$ has been used:

$$A(\{\boldsymbol{k}_i\}) = \int \mu(\{z_i\}) \prod_{i>j} (z_{ij})^{-2\alpha' \boldsymbol{k}_i \cdot \boldsymbol{k}_j} \prod_i (z_{i+1,i})^{\alpha_0}\,, \tag{2.12}$$

$$\mu(\{z_i\}) = \prod_{i=2}^{N-2} \mathrm{d}z_i (z_{N-1} - z_1)(z_N - z_{N-1})(z_N - z_1) \prod_i (z_{i+1,i})^{-1}\,. \tag{2.13}$$

It follows that the measure $\mu(\{z_i\})$ is $SL(2,\mathbb{R})$ invariant, and z_1, z_{N-1}, z_N are the three fixed variables. A common choice for them is $z_1 \equiv z_a = 0$, $z_{N-1} \equiv z_b = 1$, $z_N \equiv z_c = \infty$. As noted above, this amplitude is dual only on the mass shell, i.e. provided $\alpha' \boldsymbol{k}_i^2 + \alpha_0 = 0$ and $\sum_i \boldsymbol{k}_i = 0$. Therefore, the external spin zero particles have mass $\sqrt{-\alpha_0/\alpha'}$. Since the Regge slope α' is always positive, positive α_0 leads to imaginary mass, i.e. the amplitude yields the scattering of tachyons. Despite problems with a physical interpretation of this, one immediately sees that the value $\alpha_0 = 1$ greatly simplifies (2.8) to

$$\begin{aligned} A(\{\boldsymbol{k}_i\}) = \int_{\{z_i > z_j : i>j\}} \prod_{i=1}^{N} \mathrm{d}z_i \delta(z_1 - z_a)\delta(z_{N-1} - z_b)\delta(z_N - z_c) \\ \times\, (z_b - z_a)(z_c - z_b)(z_c - z_a) \prod_{i>j} (z_i - z_j)^{-2\alpha' \boldsymbol{k}_i \cdot \boldsymbol{k}_j}\,. \end{aligned} \tag{2.14}$$

It is instructive to explicitly check $SL(2,\mathbb{R})$ invariance for this compact form of the planar amplitude. Under $z \mapsto z' = (az + b)/(cz + d)$, we find

$$\begin{cases} z_i - z_j \mapsto (z_i - z_j)[(cz_i + d)(cz_j + d)]^{-1}\,, \\ \quad \mathrm{d}z_i \mapsto \mathrm{d}z_i[(cz_i + d)]^{-2}\,. \end{cases} \tag{2.15}$$

Thus, the net effect of a global Möbius transformation is to multiply the integrand by powers of $D_i = (cz_i + d)$. The measure contributes $(D_i)^{-2}$ for each variable, whereas the kernel contributes the power

$$-2\alpha' \boldsymbol{k}_i \cdot \left(\sum_{j \neq i} \boldsymbol{k}_j \right) \overset{\substack{\text{momentum}\\ \text{conservation}}}{=} -2\alpha' \boldsymbol{k}_i \cdot (-\boldsymbol{k}_i) \overset{\substack{\text{mass}\\ \text{shell}}}{=} +2\,. \tag{2.16}$$

That proves $SL(2,\mathbb{R})$ invariance of the Veneziano amplitude. Besides the simplification, the choice $\alpha_0 = 1$ has other interesting consequences. First of all, the Regge trajectory now contains not only the unpleasant tachyon, but also a massless vector, which indicates the existence of a gauge symmetry. Indeed, the modified amplitude (2.15) is invariant not only under $SL(2,\mathbb{R})$, but under the whole group $\mathit{Diff}(S^1)$. Fixing $\alpha_0 = 1$ thus enhances the symmetry of the scattering amplitudes to the full infinite-dimensional Witt-algebra (1.21) with generators $\ell_n = -z^n \partial_z$ for $n \in \mathbb{Z}$.

Duality, i.e. crossing symmetry between s- and t-channels has given us an $SL(2,\mathbb{R})$ symmetry. Due to quantum mechanics, no positions for the external particles arise, since we worked with well-defined momenta $\boldsymbol{k}_i$. But duality is somehow incorporated into the $SL(2,\mathbb{R})$ symmetry acting on the Koba-Nielsen variables defined on the comactified real line. What physics is behind all this?

2.3 Vertex operators

Our aim is now to construct a quantum-mechanical system whose vacuum expectation values (a.k.a. correlation functions) reproduce the dual amplitudes. Thus, we would like to think of $A(\{\boldsymbol{k}_i\})$ as the expectation values of some operators representing the particles in some Fock space. In quantum mechanics, scattering amplitudes are usually described in terms of the S-matrix operator. Hence, we seek so-called *vertex operators* $V(\boldsymbol{k}, z)$ in some Hilbert space whose vacuum expectation values could yield the dual amplitudes

$$A(\{\boldsymbol{k}_i\}) = \int \prod_{i=1}^{N} \mathrm{d}z_i \langle 0 | V(\boldsymbol{k}_N, z_N) \ldots V(\boldsymbol{k}_1, z_1) | 0 \rangle\,. \tag{2.17}$$

To construct such vertex operators, we first introduce a bosonic Fock space with a vacuum $|0\rangle$ and an infinity of simple harmonic oscillators, i.e. an infinity of pairs of creation and annihilation operators $a_n^{\mu\dagger}$, a_n^{μ} with $n \geq 0$ labelling the "modes" and $\mu = 0, \ldots, D-1$. As usual, we require that the vacuum be a highest-weight state, i.e.

$$a_n^{\mu} |0\rangle = 0 \quad \forall n \geq 1, \mu\,, \tag{2.18}$$

and that the oscillators satisfy standard commutation relations

$$[a_n^{\mu}, a_m^{\nu\dagger}] = -n \delta_{n,m} \eta^{\mu\nu}\,, \tag{2.19}$$

with all other commutators vanishing. We further assume hermiticity such that under conjugation $|0\rangle^\dagger = \langle 0|$ and $a_n^{\mu\dagger} = a_{-n}^{\mu}$.

A basis for the Hilbert space of states is the set of all monomials in creation operators, $a_{n_1}^{\mu_1\dagger}\dots a_{n_k}^{\mu_k\dagger}|0\rangle$. We have a natural gradation on this space by the particle number. So, $a_n^{\nu\dagger}|0\rangle$ is a basis for one particle states, and $a_m^{\mu\dagger}a_n^{\nu\dagger}|0\rangle$ for two-particle states, and so on. Actually, the space generated this way is not a Hilber space, since its norm is not positive definite. For instance,

$$\left|a_n^{\mu\dagger}|0\rangle\right|^2 = \langle 0|a_n^{\mu}a_n^{\mu\dagger}|0\rangle = \langle 0|[a_n^{\mu},a_n^{\mu\dagger}]|0\rangle = -n\eta^{\mu\mu}\langle 0|0\rangle = -n\eta^{\mu\mu}\,. \tag{2.20}$$

Since time has the opposite signature in the metric than space, the states $a_n^{0\dagger}|0\rangle$ have negative norm for $n\geq 1$. This is similar to gauge theories, where the timelike component A^0 of the vector field has negative norm. The way out of this is to decouple such unphysical states such that computable observables do not violate conservation of probability.

Another issue to take care of concerns the zero mode operators. So far $a_0^{\mu}=a_0^{\mu\dagger}\equiv p^{\mu}$ commutes with everything. It is necessary to introduce its conjugate operator such that the only non-vanishing commutator is $[p^{\mu},q^{\nu}]=\mathrm{i}\eta^{\mu\nu}$. To complete our choice of polarization (that $|0\rangle$ be a highest-weight state), we require that the vacuum carry zero momentum, i.e. $p_0^{\mu}|0\rangle=0$.

We are now able to introduce the equivalent of plane waves, namely the "moving ground states"

$$|0;\boldsymbol{k}\rangle \equiv \exp(\mathrm{i}k_{\mu}q^{\mu})|0\rangle \tag{2.21}$$

with momentum k^{μ}, since $p^{\mu}|0;\boldsymbol{k}\rangle = k^{\mu}|0;\boldsymbol{k}\rangle$. We may apply a Lorentz boost to bring these states to a stand-still, and therefore we should view the operators q^{μ} and $p^{\mu}=a_0^{\mu}$ as the center-of-mass position and momentum. The whole "Hilbert" space is thus spanned by

$$\mathcal{H} = \mathrm{span}\left\{(\prod_{i\in I}a_{n_i}^{\mu_i\dagger})|0;\boldsymbol{k}\rangle : \mathbb{N}\supset I=\{n_1,\dots n_k\}, n_{i+1}\geq n_i\right\}\,. \tag{2.22}$$

The following picture emerges: The Hilbert space is labelled by the center-of-mass momentum and by the occupation numbers of the Fourier modes (with $n\geq 1$). The bosonic Fock space we just have constructed is the Fock space of a free open string, where each independent mode of vibrational excitation is quantized as a free oscillator.

In mathematical terms, the Fock space decomposes into a direct sum of Verma modules. To each value $\boldsymbol{k}\in S^{D-1}$ we associate a higehst-weight state $|0;\boldsymbol{k}\rangle$ on which a Verma module of some infinite-dimensional Lie algebra is built. We can now proceed to identify the Fubini-Veneziano vertex operators $V(\boldsymbol{k},z)$ in this Fock space. Let us first consider the string coordinate

$$\begin{aligned} X^{\mu}(z) &= p^{\mu} - 2\mathrm{i}\alpha' p^{\mu}\mathrm{log}z + X_-^{\mu}(z) + X_+^{\mu}(z)\,, \\ X_-^{\mu}(z) &= \quad \mathrm{i}\sqrt{2\alpha'}\sum_{n\geq 1}\frac{a_n^{\mu}}{n}z^{-n}\,, \\ X_+^{\mu}(z) &= -\mathrm{i}\sqrt{2\alpha'}\sum_{n\geq 1}\frac{a_n^{\mu\dagger}}{n}z^{-n}\,. \end{aligned} \tag{2.23}$$

Note that, up to a different normalization explicitly incorporating the string tension α', this expression is quite – but not completely! – similar to (1.12). Obviously, the Koba-Nielsen variable acts as a local coordinate on a string with respect to which a Fourier decomposition of its vibrational modes is made. We have the following vacuum conditions for the positive and negative modes,

$$X_-^\mu(z)|0\rangle = \langle 0|X_+^\mu(z) = 0\,. \tag{2.24}$$

The reader should work out the following statements explicitly. Using the standard commutation relations of the oscillator modes, one easily shows that

$$[X_-^\mu(z), X_+^\mu(w)] = 2\alpha' \sum_{m,n\geq 1} \frac{1}{mn}[a_n^\mu, a_m^{\nu\dagger}]z^{-n}w^{-m} = -2\alpha' \sum_{n\geq 1}\frac{1}{n}\left(\frac{w}{z}\right)^2 = 2\alpha'\log\left(1-\frac{w}{z}\right)\,. \tag{2.25}$$

The result is a function (not an operator) such that we can apply the Baker-Hausdorff formula, i.e. $\exp(A)\exp(B) = \exp(B)\exp(A)\exp(\frac{1}{2}[A,B])$, to find

$$\exp(\mathrm{i}(k_1)_\nu X_+^\nu(z))\exp(\mathrm{i}(k_2)_\mu X_-^\mu(w)) = (w-z)^{-2\alpha'\boldsymbol{k}_1\cdot\boldsymbol{k}_2}w^{2\alpha'\boldsymbol{k}_1\cdot\boldsymbol{k}_2}\exp(\mathrm{i}(k_2)_\mu X_-^\mu(w))\exp(\mathrm{i}(k_1)_\nu X_+^\nu(z))\,. \tag{2.26}$$

The operators $V_{\neq}(\boldsymbol{k},z) = \exp(\boldsymbol{k}\cdot\boldsymbol{X}_-(z))\exp(\boldsymbol{k}\cdot\boldsymbol{X}_+(z))$ can be seen to have the expectation values

$$\langle 0|V_{\neq}(\boldsymbol{k}_N,z_N)\ldots V_{\neq}(\boldsymbol{k}_1,z_1)|0\rangle = \prod_{i>j}(z_i-z_j)^{-2\alpha'\boldsymbol{k}_i\cdot\boldsymbol{k}_j}\prod_i (z_i)^{2\alpha'\boldsymbol{k}_i\cdot\hat{\boldsymbol{K}}_i} \tag{2.27}$$

with $\hat{\boldsymbol{K}}_i = \sum_{j\leq i}\boldsymbol{k}_j$. So far, The second factor spoils this to be of the required form. But we have neglected the zero-modes. In fact, with the definition

$$V(\boldsymbol{k},z) = \exp(k_\mu X_-^\mu(z))\exp[\mathrm{i}k_\mu(q^\mu - 2\mathrm{i}\alpha' p^\mu \log z)]\exp(k_\nu X_+^\nu(z)) \tag{2.28}$$

the spoiling factor is cancelled and momentum conservation is automatically implemented leading to the desired vacuum expectation values

$$\langle 0|V(\boldsymbol{k}_N,z_N)\ldots V(\boldsymbol{k}_1,z_1)|0\rangle = \prod_{i>j}(z_i-z_j)^{-2\alpha'\boldsymbol{k}_i\cdot\boldsymbol{k}_j}\prod_i (z_i)^{\alpha'\boldsymbol{k}_i^2}\,. \tag{2.29}$$

Note that the second term is now solely due to the non-trivial commutator of the zero-mode operators p^μ and q^μ, and is not associated to an ordering problem between different vertex operators.

The above definition of our vertex operators can be written in a very simple form with the use of normal ordering. It is nothing else than the normal ordered exponential of a free field, namely

$$V(\boldsymbol{k},z) = :\!\exp(\mathrm{i}k_\mu X^\mu(z))\!:\,, \tag{2.30}$$

which is just the common normal ordering prescription to move all annihilators right to the creators and to move momentum right to position, exactly as in (2.28). In the following, all such expressions will be understood as implicitly normal ordered, even if the $:\ldots:$ is omitted. With these vertex operators we find the correct integral kernel as

$$\langle 0|\frac{V(\boldsymbol{k}_N,z_N)}{(z_N)^{\alpha'\boldsymbol{k}_N^2}}\ldots\frac{V(\boldsymbol{k}_1,z_1)}{(z_1)^{\alpha'\boldsymbol{k}_1^2}}|0\rangle = \prod_{i>j}(z_i-z_j)^{-2\alpha'\boldsymbol{k}_i\cdot\boldsymbol{k}_j}\,. \tag{2.31}$$

As they are defined, the vertex operators do not carry any free space-time indices, and are thus scalar (spin zero) operators. If the corresponding states lie on a Regge trajectory, we must have $\alpha' \boldsymbol{k}_i^2 + \alpha_0 = 0$, such that we can indeed reproduce the amplitudes as vacuum expectation values of a product of vertex operators,

$$A(\{\boldsymbol{k}_i\}) = \int \left(\prod_{i=1}^{N} \frac{\mathrm{d}z_i}{z_i^{\alpha_0}} \right) \langle 0 | V(\boldsymbol{k}_N, z_N) \ldots V(\boldsymbol{k}_1, z_1) | 0 \rangle \,. \tag{2.32}$$

This expression is $SL(2,\mathbb{R})$ invariant by construction, and becomes invariant under the full $\mathit{Diff}(S^1)$, i.e. arbitrary reparametrizations of the z-coordinate for the special value $\alpha_0 = 1$. Clearly, for this choice of α_0 the measure $\mathrm{d}z/z$ is reparametrization invariant. It is nice to write $z = \mathrm{e}^{\theta}$. Then, keeping in mind that the integration measure includes cyclic ordering $z_i > z_j$ for $i > j$, i.e. the step function $\Theta(\theta_i - \theta_j)$, we see that θ is something like a time coordinate, and the amplitude can be seen as expectation value of a time ordered product of operators which create states with momentum $\boldsymbol{k}_i$ at times θ_i.

2.4 The Virasoro-Shapiro amplitude

The planar amplitudes are not the only ones satisfying the duality conditions. If the additional requirement that poles may only develop in one channel at a time is dropped, Veneziano and Shapiro found a non-planar solution to duality. Let us concentrate on the four-point amplitude $A(s,t,u)$, depending on the three Mandelstam variables $s = (\boldsymbol{k}_1 + \boldsymbol{k}_2)^2$, $t = (\boldsymbol{k}_2 + \boldsymbol{k}_3)^2$, and $u = (\boldsymbol{k}_3 + \boldsymbol{k}_1)^2$. These are related via $s + t + u = \sum_{i=1}^{4} m_i^2$, and for the sake of simplicity we restrict ourselves to the case that all masses are equal, $\boldsymbol{k}_i^2 = -m^2$. Again, states on Regge trajectories are labelled by $\alpha(s) = \alpha' s + \alpha_0$. Then, Virasoro's proposal was

$$A(s,t,u) = \frac{\Gamma(-\frac{1}{2}\alpha(s))\Gamma(-\frac{1}{2}\alpha(t))\Gamma(-\frac{1}{2}\alpha(u))}{\Gamma(-\frac{1}{2}[\alpha(s)+\alpha(t)])\Gamma(-\frac{1}{2}[\alpha(t)+\alpha(u)])\Gamma(-\frac{1}{2}[\alpha(u)+\alpha(s)])} \,. \tag{2.33}$$

This amplitude is non-planar, but still satisfies duality, $A(s,t,u) = A(t,s,u) = A(u,t,s) = A(s,u,t) = A(t,u,s) = A(u,s,t)$. Thus, the amplitude is invariant under permutations of the Mandelstam variables which means that it is invariant under cyclic relabellings of the external legs. Again, this amplitude has an infinity of poles, and their residues are polynomials of bounded degree. To see this, it is best to use the integral representation

$$A(s,t,u) = \int \mathrm{d}^2 z |z|^{-\alpha(s)-2} |1-z|^{-\alpha(t)-2} \,. \tag{2.34}$$

This expression is remarkably close to the integral representation of the four-point Veneziano amplitude (2.6 and 2.7), except that now z is a complex variable, and that the exponents differ by one. The physical interpretation of this is that this amplitude refers to a closed string, while the Veneziano amplitude came from an open string.

In the planar (Veneziano) case, $z \in \mathbb{R}$ and $\alpha_0 = 1$ was a distinguished value for which a massless vector (a photon) appeared in the spectrum enhancing the $SL(2,\mathbb{R})$ symmetry of the vacuum to $\mathit{Diff}(S^1)$. In the non-planar (Virasoro) case, $z \in \mathbb{C}$ and $\alpha_0 = 2$ is a special value for which a massless two-tensor (a graviton, a dilaton, and a Kalb-Ramond field) arises. The symmetry of the vacuum is now $SL(2,\mathbb{C})$, which for $\alpha_0 = 2$ enlarges to $\mathit{Diff}(S^1) \times \mathit{Diff}(S^1)$. The $SL(2)$ symmetry enables us in both cases to fix three of the coordinates to arbitrarily chosen values, usually 0, 1, and ∞.

In the planar case, we constructed vertex operators $V(\boldsymbol{k}, z) = {:}\exp(\mathrm{i}k_\mu X^\mu(z)){:}$ with the string coordinate (skipping the normalization with α')

$$X^\mu(z) = q^\mu - \mathrm{i}p^\mu \log z + \mathrm{i} \sum_{n \neq 0} \frac{a_n}{n} z^{-n} \,. \tag{2.35}$$

In the non-planar case, we can proceed in a similar fashion in order to construct vertex operators which reproduce the dual amplitudes in terms of vacuum expectation values. However, we find that they are now of the form $V(\boldsymbol{k}, z, \bar{z}) = {:}\exp(\mathrm{i}k_\mu X^\mu(z,\bar{z})){:}$ with the string coordinate of a closed string given by

$$X^\mu(z,\bar{z}) = q^\mu - \mathrm{i}p^\mu \log z\bar{z} + \mathrm{i} \sum_{n \neq 0} \frac{a_n}{n} z^{-n} + \mathrm{i} \sum_{n \neq 0} \frac{\bar{a}_n}{n} \bar{z}^{-n} \,. \tag{2.36}$$

This is precisely the expression (1.12) we found earlier, where we derived it from the action for the string world-sheet (provided we identify $\tilde{a}_n$, so far considered to be independent of the a_n, with $\bar{a}_n$). Were it not for the zero mode, we would have the decomposition $X^\mu(z,\bar{z}) = X^\mu(z) + X^\mu(\bar{z})$ into holomorphic and anti-holomorphic parts. At least the oscillator modes are doubled such that we will have two copies of the Virasoro algebra as symmetries.

3. CFT proper

We will now detach ourselves from any string theoretic motivations and consider CFT solely on its own. As mentioned in section 1.1, we work on the complex plane (or Riemann sphere) with the holomorphic coordinate z and the holomorphic differential or one-form $\mathrm{d}z$. A field $\Phi(z)$ is called a *conformal* or *primary* field of *weight* h, if it transforms under holomorphic mappings $z \mapsto z'(z)$ of the coordinate as

$$\Phi_h(z)(\mathrm{d}z)^h \mapsto \Phi_h(z')(\mathrm{d}z')^h = \Phi_h(z)(\mathrm{d}z)^h\,. \tag{3.1}$$

In case that the conformal weight h is not a (half-)integer, it is better to write this as

$$\Phi_h(z) \mapsto \Phi_h(z') = \Phi_h(z)\left(\frac{\partial z'(z)}{\partial z}\right)^{-h}\,. \tag{3.2}$$

One should keep in mind that all formulæ here have an anti-holomorphic counterpart. Since a primary field factorizes into holomorphic and antiholomorphic parts, $\Phi_{h,\bar{h}}(z,\bar{z}) = \Phi_h(z)\Phi_{\bar{h}}(\bar{z})$, in most cases, we can skip half of the story. Infinitesimally, if $z'(z) = z+\varepsilon(z)$ with $\bar{\partial}\varepsilon = 0$, the transformation of the field is

$$\Phi_h(z')(\mathrm{d}z')^h = \left(\Phi_h(z) + \varepsilon(z)\partial_z\Phi_h(z) + \ldots\right)(\mathrm{d}z)^h\left(1+\partial_z\varepsilon(z)\right)^h\,. \tag{3.3}$$

Therefore, the variation of the field with respect to a holomorphic coordinate transformation is

$$\delta\Phi_h(z) = \left(\varepsilon(z)\partial + h(\partial\varepsilon(z))\right)\Phi_h(z)\,. \tag{3.4}$$

Since this transformation is supposed to be holomorphic in $\mathbb{C}^*$, it can be expanded as a Laurent series,

$$\varepsilon(z) = \sum_{n\in\mathbb{Z}}\varepsilon_n z^{n+1}\,. \tag{3.5}$$

This suggests to take the set of infinitesimal transformations $z \mapsto z' = z + \varepsilon_n z^{n+1}$ as a basis from which we find the generators of this reparametrization symmetry by considering $\Phi_h \mapsto \Phi_h + \delta_n\Phi_h$ with

$$\delta_n\Phi_h(z) = \left(z^{n+1}\partial + h(n+1)z^n\right)\Phi_h(z)\,. \tag{3.6}$$

The generators are thus the generators of the already encountered Witt-algebra (1.21), $\ell_n = -z^{n1+}\partial$.

We are interested in a quantized theory such that conformal fields become operator valued distributions in some Hilber space $\mathcal{H}$. We therefore seek a representation of $\ell_n \in$ *Diff* (S^1) by some operators $L_n \in \mathcal{H}$ such that

$$\delta_n\Phi_h(z) = [L_n, \Phi_h(z)]\,. \tag{3.7}$$

We already have done this in section 1.3, where we discovered the Virasoro algebra (1.26), $[L_n, L_m] = (n-m)L_{n+m} + \frac{\hat{c}}{12}(n^3-n)\delta_{n+m,0}$. We once more remark that $\mathfrak{sl}(2)$ is a subalgebra of $Diff(S^1)$ which is independent of the central charge c. So, we start with considering the consequences of just $SL(2,\mathbb{C})$ invariance on correlation functions of primary conformal fields of the form

$$G(z_1,\ldots,z_N) = \langle 0|\Phi_{h_N}(z_N)\ldots\Phi_{h_1}(z_1)|0\rangle\,. \tag{3.8}$$

We immediately can read off the effect on primary fields from (3.6), which is $\delta_{-1}\Phi_h(z) = \partial\Phi_h(z)$, $\delta_0\Phi_h(z) = (z\partial + h)\Phi_h(z)$, and $\delta_1\Phi_h(z) = (z^2\partial + 2hz)\Phi_h(z)$.

3.1 Conformal Ward identities

Global conformal invariance of correlation functions is equivalent to the statement that $\delta_i G(z_1,\ldots,z_N) = 0$ for $i \in \{-1,0,1\}$. Since δ_i acts as a (Lie-) derivative, we find the following differential equations for correlation functions $G(\{z_i\})$,

$$\begin{cases} 0 = \sum_{i=1}^{N}\partial_{z_i}G(z_1,\ldots,z_N)\,, \\ 0 = \sum_{i=1}^{N}(z\partial_{z_i} + h_i)G(z_1,\ldots,z_N)\,, \\ 0 = \sum_{i=1}^{N}(z^2\partial_{z_i} + 2h_i z_i)G(z_1,\ldots,z_N)\,, \end{cases} \tag{3.9}$$

which are the so-called *conformal Ward identites*. The general solution to these three equations is

$$\langle 0|\Phi_{h_N}(z_N)\ldots\Phi_{h_1}(z_1)|0\rangle = F(\{\eta_k\})\prod_{i>j}(z_i - z_j)^{\mu_{ij}}\,, \tag{3.10}$$

where the exponents $\mu_{ij} = \mu_{ji}$ must satisfy the conditions

$$\sum_{j\neq i}\mu_{ij} = -2h_i\,, \tag{3.11}$$

and where $F(\{\eta_k\})$ is an arbitray function of any set of $N-3$ independent harmonic ratios (a.k.a. crossing ratios), for example

$$\eta_k = \frac{(z_1 - z_k)(z_{N-1} - z_N)}{(z_k - z_N)(z_1 - z_{N-1})}\,,\qquad k = 2,\ldots N-2\,. \tag{3.12}$$

The above choice is conventional, and maps $z_1 \mapsto 0$, $z_{N-1} \mapsto 1$, and $z_N \mapsto \infty$. This remaining function cannot be further determined, because the harmonic ratios are already $SL(2,\mathbb{C})$ invariant, and therefore any function of them is too. This confirms that $\mathfrak{sl}(2)$ invariance allows us to fix (only) three of the variables arbitrarily. If we compare this general form (3.10) with the Veneziano amplitudes built from vertex operators as in (2.31), we find that $\mu_{ij} = -2\alpha'\boldsymbol{k}_i\cdot\boldsymbol{k}_j$ and $h_i = \alpha'\boldsymbol{k}_i^2$. Thus, we arrive at the very nice result that the vertex operators $V(\boldsymbol{k}, z)$ in (2.30) are primary conformal fields of weight $\alpha'\boldsymbol{k}^2 = \alpha_0$. Since the integral over the Koba-Nielsen variables, i.e. the coordinates of the primary fiels, is well-defined only for $\alpha_0 = 1$, we learn that a string amplitude (of tachyons) is related to a particular simple conformal field theory where all involved conformal fields have conformal scaling weight one.

Let us rewrite the conformal Ward identities (3.9) as

$$0 = \langle(\delta_i \Phi_{h_N}(z_N))\Phi_{h_{n-1}}(z_{N-1})\ldots\Phi_{h_1}(z_1)\rangle + \langle(\Phi_{h_N}(z_N)(\delta_i\Phi_{h_{n-1}}(z_{N-1}))\ldots\Phi_{h_1}(z_1)\rangle \\ + \ldots + \langle(\Phi_{h_N}(z_N)\Phi_{h_{n-1}}(z_{N-1})(\delta_i\Phi_{h_1}(z_1))\rangle\,, \tag{3.13}$$

where $\delta_i\Phi_h(z) = [L_i, \Phi_h(z)]$ for $i \in \{-1,0,1\}$. We assume that the in-vacuum is $SL(2,\mathbb{C})$ invariant, i.e. that $L_i|0\rangle = 0$ for $i \in \{-1,0,1\}$. Then (3.13) is nothing else than $\langle 0|L_i\left(\Phi_{h_N}(z_N)\ldots\Phi_{h_1}(z_1)\right)|0\rangle$ from which it follows that $\langle 0|L_i$ must be states orthogonal to (and hence decoupled from) any other state in the theory for $i \in \{-1,0,1\}$.

In a well-defined quantum field theory, we have an isomorphism between the fields in the theory and states in the Hilbert space $\mathcal{H}$. This isomorphism is particularly simple in CFT and induced by

$$\lim_{z\to 0}\Phi_h(z)|0\rangle = |h\rangle\,, \tag{3.14}$$

where $|h\rangle$ is a highest-weight state of the Virasoro algebra. Indeed, since $[L_n, \Phi_h] = (z^{n+1}\partial + h(n+1)z^n)\Phi_h$, we find with the highest-weight property of the vacuum $|0\rangle$ as in (1.27) that for all $n > 0$

$$L_n|h\rangle = \lim_{z\to 0} L_n\Phi_h(z)|0\rangle = \lim_{z\to 0}[L_n, \Phi_h(z)]|0\rangle = \lim_{z\to 0}\left(z^{n+1}\partial + (n+1)hz^n\right)\Phi_h(z)|0\rangle \\ = 0\,. \tag{3.15}$$

Furthermore, $L_0|h\rangle = h|h\rangle$ by the same consideration. Thus, primary fields correspond to highest-weight states. In particular, our vertex operators $V(\boldsymbol{k}, z)$ correspond to the highest-weight states $|0,\boldsymbol{k}\rangle \equiv \lim_{z\to 0} V(\boldsymbol{k}, z)|0\rangle$. Since $z \to 0$ in $\mathbb{PC}^1$ corresponds to $\tau \to -\infty$ on the cylinder, i.e. the world sheet, the above field-state isomorphism is precisely what physics would suggest to us.

A nice exercise is to apply the conformal Ward identities to a two-point function $G = \langle\Phi_h(z)\Phi_{h'}(w)$. The constraint from L_{-1} is that $(\partial_z + \partial_w)G = 0$, meaning that $G = f(z-w)$ is a function of the distance only. The L_0 constraint then yields a linear ordinary differential equation, $((z-w)\partial_{z-w} + (h+h'))f(z-w) = 0$, which is solved by $const\cdot(z-w)^{-h-h'}$.

Finally, the L_1 constraint yields the condition $h = h'$. However, we should be carefull here, since this does not necessarily imply that the two fields have to be identical. Only their conformal weights have to coincide. In fact, we will encounter examples where the propagator $\langle h|h'\rangle = \lim_{z\to\infty}\langle 0|z^{2h}\Phi_h(z)\Phi_{h'}(0)|0\rangle$ is not diagonal. Therefore, if more than one field of conformal weight h exists, the two-point functions aquire the form $\langle\Phi_h^{(i)}(z)\Phi_{h'}^{(j)}(w)\rangle = (z-w)^{-2h}\delta_{h,h'}D_{ij}$ with $D_{ij} = \langle h;i|h;j\rangle$ the propagator matrix. The matrix D_{ij} then induces a metric on the space of fields. In the following, we will assume that $D_{ij} = \delta_{ij}$ except otherwise stated.

It is worth noting that the conformal Ward identites (3.9) allow us to fix the two- and three-point functions completely upto constants. In fact, the two-point functions are simply given by

$$\langle\Phi_h(z)\Phi_{h'}(w)\rangle = \frac{\delta_{h,h'}}{(z-w)^{2h}}\,, \tag{3.16}$$

where we have taken the freedom to fix the normalization of our primary fields. The three-point functions turn out to be

$$\langle\Phi_{h_i}(z_i)\Phi_{h_j}(z_j)\Phi_{h_k}(z_k)\rangle = \frac{C_{ijk}}{(z_{ij})^{h_i+h_j-h_k}(z_{ik})^{h_i+h_k-h_j}(z_{jk})^{h_j+h_k-h_i}}\,, \tag{3.17}$$

where we again used the abbreviation $z_{ij} = z_i - z_j$. The constants C_{ijk} are not fixed by $SL(2,\mathbb{C})$ invariance and are called the *structure constants* of the CFT. Finally, the four-point function is determined upto an arbitrary function of one crossing ratio, usually chosen as $\eta = (z_{12}z_{34})/(z_{24}z_{13})$. The solution for μ_{ij} is no longer unique for $N \geq 4$, and the customary one for $N = 4$ is $\mu_{ij} = H/3 - h_i - h_j$ with $H = \sum_{i=1}^{4} h_i$, such that the four-point functions reads

$$\langle\Phi_{h_4}(z_4)\Phi_{h_3}(z_3)\Phi_{h_2}(z_2)\Phi_{h_1}(z_1)\rangle = \prod_{i>j}(z_{ij})^{H/3-h_i-h_j}F(\frac{z_{12}z_{34}}{z_{24}z_{13}})\,. \tag{3.18}$$

Note again that $SL(2,\mathbb{C})$ invariance cannot tell us anything about the function $F(\eta)$, since η is invariant under Möbius transformations.

3.2 Virasoro representation theory: Verma modules

We already encoutered highest-weight states, which are the states corresponding to primary fields. On each such highest-weight state we can construct a *Verma module* $V_{h,c}$ with respect to the Virasoro algebra Vir by applying the negative modes L_n, $n < 0$ to it. Such states are called *descendant* states. In this way our Hilbert space decomposes as

$$\begin{aligned} \mathcal{H} &= \textstyle\bigoplus_{h,\bar{h}} V_{h,c} \otimes V_{\bar{h},c}\,, \\ V_{h,c} &= \mathrm{span}\left\{(\textstyle\prod_{i\in I} L_{-n_i}|h\rangle : \mathbb{N} \supset I = \{n_1,\ldots n_k\}, n_{i+1} \geq n_i\right\}\,, \end{aligned} \tag{3.19}$$

where we momentarily have sketched the fact that the full CFT has a holomorphic and an anti-holomorphic part. Note also, that we indicate the value for the central charge in the Verma modules. We have so far chosen the anti-holomorphic part of the CFT to be simply a copy of the holomorphic part, which guarantees the full theory to be local. However, this is not the only consistent choice, and heterotic strings are an example where left and right chiral CFT definitely are very much different from each other.

A way of counting the number of states in $V_{h,c}$ is to introduce the *character* of the Virasoro algebra, which is a formal power series

$$\chi_{h,c}(q) = \mathrm{tr}_{V_{h,c}} q^{L_0-c/24}\,. \tag{3.20}$$

For the moment, we consider q to be a formal variable, but we will later interpret it in physical terms, where it will be defined by $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ with a complex parameter τ living in the upper half plane, i.e. $\Im\mathrm{m}\,\tau > 0$. The meaning of the constant term $-c/24$ will also become clear further ahead.

The Verma module possesses a natural gradation in terms of the eigen value of L_0, which for any descendant state $L_{-\{n\}}|h\rangle \equiv L_{-n_1}\ldots L_{-n_k}|h\rangle$ is given by $L_0 L_{-\{n\}}|h\rangle =$

$(h + |\{n\}|)|h\rangle \equiv (h + n_1 + \ldots + n_k)|h\rangle$. One calls $|\{n\}|$ the level of the descendant $L_{-\{n\}}|h\rangle$. The first descendant states in $V_{h,c}$ are easily found. At level zero, there exists of course only the highest-weight state itself, $|h\rangle$. At level one, we only have one state, $L_{-1}|h\rangle$. At level two, we find two states, $L_{-1}^2|h\rangle$ and $L_{-2}|h\rangle$. In general, we have

$$\begin{aligned} V_{h,c} &= \bigoplus\nolimits_N V_{h,c}^{(N)} \,, \\ V_{h,c}^{(N)} &= \mathrm{span}\left\{ L_{-\{n\}}|h\rangle : |\{n\}| = N \right\} \,, \end{aligned} \tag{3.21}$$

i.e. at each level N we generically have $p(N)$ linearly independent descendants, where $p(N)$ denotes the number of partitions of N into positive integers. If all these states are physical, i.e. do not decouple from the spectrum, we easily can write down the character of this highest-weight representation,

$$\chi_{h,c}(q) = q^{h-c/24} \prod_{n\geq 1} \frac{1}{1-q^n} \,. \tag{3.22}$$

To see this, the reader should make herself clear that we may act on $|h\rangle$ with any power of L_{-m} independently of the powers of any other mode $L_{-m'}$, quite similar to the Fock space of oscillators (2.22). A closer look reveals that (3.20) is indeed formally equivalent to the partition function of an infinite number of oscillators with energies $E_n = n$. The expression (3.22) contains the generating function for the numbers of partitions, since expanding it in a power series yields

$$\begin{aligned} &\prod_{n\geq 1} (1-q^n)^{-1} = \sum_{N\geq 0} p(N) q^N \\ &= 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + 22q^8 + 30q^9 + 42q^{10} + \ldots \,. \end{aligned} \tag{3.23}$$

3.3 Virasoro representation theory: Null vectors

The above considerations are true in the generic case. But if we start to fix our CFT by a choice of the central charge c, we have to be carefull about the question whether all the states are really linearly independent. In other words: May it happen that for a given level N a particular linear combination

$$|\chi_{h,c}^{(N)}\rangle = \sum_{|\{n\}|=N} \beta^{\{n\}} L_{-\{n\}}|h\rangle \equiv 0\,? \tag{3.24}$$

With this we mean that $\langle\psi|\chi_{h,c}^{(N)}\rangle = 0$ for all $|\psi\rangle \in \mathcal{H}$. To be precise, this statement assumes that our space of states admitts a sesqui-linear form $\langle.|.\rangle$. In most CFTs, this is the case, since we can define asymptotic out-states (the $\tau \to +\infty$ limit on the cylinder) by

$$\langle h| \equiv \lim_{z\to\infty} \langle 0|\Phi_h(z) z^{2h} \,. \tag{3.25}$$

This definition is forced by the requirement to be compatible with $SL(2,\mathbb{C})$ invariance of the two-point function (3.16). We then have $\langle h'|h\rangle = \delta_{h',h}$. The exponent z^{2h} arises due to the conformal transformation $z \mapsto z' = 1/z$ we implicitly have used. We further assume the hermiticity condition $L_{-n}^\dagger = L_n$ to hold.

The hermiticity condition is certainly fulfilled for unitary theories. We already know from the calculation (1.28) of the two-point function $\langle T(z)T(w)\rangle$ that necessarily $c \geq 0$ for unitary theories. Otherwise, $\|L_{-n}|0\rangle\|^2 = \langle 0|L_n L_{-n}|0\rangle = \langle 0|[L_n, L_{-n}]|0\rangle = \frac{1}{12}c(n^3-n)\langle 0|0\rangle$ would be negative for $n \geq 2$. Moreover, redoing the same calculation for the highest-weight state $|h\rangle$ instead of $|0\rangle$, we find $\|L_{-n}|h\rangle\|^2 = \left(\frac{1}{12}c(n^3-n)+2nh\right)\langle h|h\rangle$. The first term dominates for large n such that again c must be non-negative, if this norm should be positive definite. The second term dominates for $n = 1$, from which we learn that h must be non-negative, too. To summarize, unitary CFTs necessarily require $c \geq 0$ and $h \geq 0$, where the theory is trivial for $c = 0$ and where $h = 0$ implies that $|h = 0\rangle = |0\rangle$ is the (unique) vacuum.

To answer the above question, we consider the $p(N) \times p(N)$ matrix $K^{(N)}$ of all possible scalar products $K^{(N)}_{\{n'\},\{n\}} = \langle h|L_{\{n'\}}L_{-\{n\}}|h\rangle$. This matrix is hermitean by definition. If this matrix has a vanishing or negative determinant, then it must possess an eigen vector (i.e. a linear combination of level N descendants) with zero or negative norm, respectively. The converse is not necessarily true, such that a positive determinant could still mean the presence of an even number of negative eigen values. For $N = 1$, this reduces to the simple statement $\mathrm{det}K^{(1)} = \langle h|L_1L_{-1}|h\rangle = \|L_{-1}|h\rangle\|^2 = \langle h|2L_0|h\rangle = 2h\langle h|h\rangle = 2h$, where we used the Virasoro algebra (1.26). Thus, there exists a null vector at level $N = 1$ only for the vacuum highest-weight representation $h = 0$.

We note a view points concerning the general case. Firstly, due to the assumption that all highest-weight states are unique (i.e. $\langle h'|h\rangle = \delta_{h',h}$), it follows that it suffices to analyze the matrix $K^{(N)}$ in order to find conditions for the presence of null states. Note that scalar products $\langle h|L_{\{n'\}}L_{-\{n\}}|h\rangle$ are automatically zero for $|\{n'\}| - |\{n\}| \neq 0$ due to the highest-weight property. Secondly, using the Virasoro algebra (1.26), each matrix element can be reduced to a polynomial function of h and c. This must be so, since the total level of the descendant $L_{\{n'\}}L_{-\{n\}}|h\rangle$ is zero such that use of the Virasoro algebra allows to reduce it to a polynomial $p_{\{n'\},\{n\}}(L_0, \hat{c})|h\rangle$. It follows that $K^{(N)}_{\{n'\},\{n\}} = p_{\{n'\},\{n\}}(h, c)$.

It is an extremely useful exercise to work out the level $N = 2$ case by hand. Since $p(2) = 2$, The matrix $K^{(2)}$ is the 2×2 matrix

$$K^{(2)} = \begin{pmatrix} \langle h|L_2L_{-2}|h\rangle & \langle h|L_2L_{-1}L_{-1}|h\rangle \\ \langle h|L_1L_1L_{-2}|h\rangle & \langle h|L_1L_1L_{-1}L_{-1}|h\rangle \end{pmatrix} . \tag{3.26}$$

The Virasoro algebra reduces all the four elements to expressions in h and c. For example, we evaluate $L_1L_1L_{-2}|h\rangle = L_1[L_1, L_{-2}]|h\rangle = 3L_1L_{-1}|h\rangle = 6L_0|h\rangle$ etc., such that we arrive at

$$K^{(2)} = \begin{pmatrix} 4h + \frac{1}{2}c & 6h \\ 6h & 4h + 8h^2 \end{pmatrix} \langle h|h\rangle . \tag{3.27}$$

For $c, h \gg 1$, the diagonal dominates and the eigen values are hence both positive. The determinant is

$$\mathrm{det}K^{(2)} = 2h\left(16h^2 + 2(c-5)h + c\right)\langle h|h\rangle^2 . \tag{3.28}$$

At level $N = 2$, there are three values of the highest weight h,

$$h \in \left\{0, \tfrac{1}{16}(5 - c \pm \sqrt{(c-1)(c-25)})\right\} , \tag{3.29}$$

where the matirx $K^{(2)}$ develops a zero eigen value. Note that one finds two values $h_\pm$ for each given central charge c, besides the value $h = 0$ which is a remnant of the level one null state. The corresponding eigen vector is easily found and reads

$$|\chi^{(2)}_{h_\pm,c}\rangle = \left(\tfrac{2}{3}(2h_\pm + 1)L_{-2} - L_{-1}^2\right)|h_\pm\rangle . \tag{3.30}$$

This can be generalized. The reader might occupy herself some time with calculating the null states for the next few levels. Luckily, there exist at least general formulæ for the zeroes of the so-called Kac determinant $\det K^{(N)}$, which are curves in the (h, c) plane. Reparametrizing with some hind-sight

$$c = c(m) = 1 - 6\frac{1}{m(m+1)}\,, \quad \text{i.e.} \quad m = -\tfrac{1}{2}\left(1 \pm \sqrt{\frac{c-25}{c-1}}\right)\,, \tag{3.31}$$

one can show that the vanishing lines are given by

$$\begin{aligned} h_{p,q}(c) \;&=\; \frac{\left((m+1)p - mq\right)^2 - 1}{4m(m+1)} \\ &=\; -\tfrac{1}{2}pq + \tfrac{1}{24}(c-1) + \tfrac{1}{48}\left((13 - c \mp \sqrt{(c-1)(c-25)})p^2 + (13 - c \pm \sqrt{(c-1)(c-25)})q^2\right)\,. \end{aligned} \tag{3.32}$$

Note that the two solutions for m lead to the same set of h-values, since $h_{p,q}(m_+(c)) = h_{q,p}(m_-(c))$. With this notation for the zeroes, the Kac determinant can be written upto a constant α_N of combinatorical origin as

$$\det K^{(N)} = \alpha_N \prod_{pq \le N} (h - h_{p,q}(c))^{p(n-pq)} \propto \det K^{(N-1)} \prod_{pq = N} (h - h_{p,q}(c))\,, \tag{3.33}$$

where we have set $\langle h|h\rangle = 1$, and where $p(n)$ denotes again the number of partitions of n into positive integers.

A deeper analysis not only reveals null states, where the scalar product would be positive semi-definite, but also regions of the (h, c) plane where negative norm states are present. A physical sensible string theory should possess a Hilbert space of states, i.e. the scalar product should be positive definite. Therefore, an analysis which regions of the (h, c) plane are free of negative-norm states is a very important issue in string theory. As a result, for $0 \le c < 1$, only the discrete set of points given by the values $c(m)$ with $m \in \mathbb{N}$ in (3.31) and the corresponding values $h_{p,q}(c)$ with $1 \le p < m$ and $1 \le q < m+1$ in (3.32) turns out to be free of negative-norm states. In the string theory lectures, the reader will learn that the region $c \ge 25$ is particularly interesting, and that indeed $c = 26$ admitts a positive definite Hilbert space.

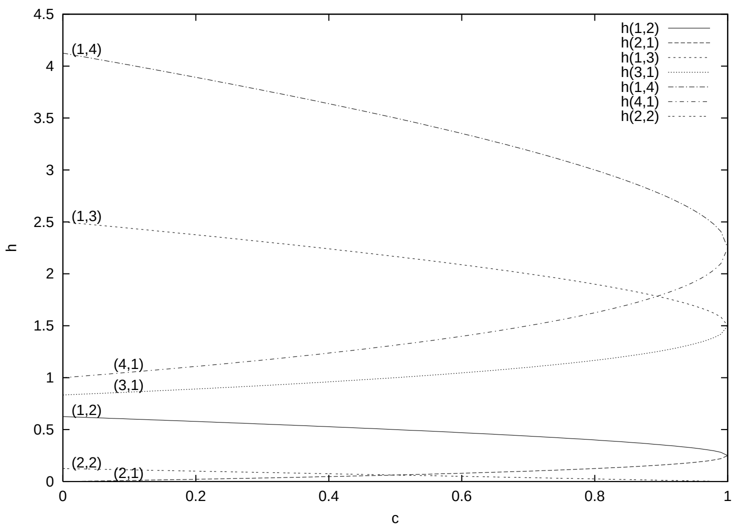

Figure 5: The first few of the lines $h_{p,q}(c)$ where null states exist. They are also the lines where the Kac determinant has a zero, indicating a sign change of an eigenvalue.

To complete our brief discussion of Virasoro representation theory, we note the following: If null states are present in a given Verma module $V_{h,c}$, they are states which are orthogonal to all other states. It follows, that they, and all their descendants, decouple from the other states in the Verma module. Hence, the correct representation module is the irreducible sub-module with the ideal generated by the null state divided out, or more precisley, with the maximal proper sub-module divided out, i.e.

$$V_{h_{p,q}(c),c} \longrightarrow M_{h_{p,q}(c),c} = V_{h_{p,q}(c),c} \Big/ \mathrm{span}\{|\chi^{(N)}_{h_{p,q}(c),c}\rangle \equiv 0\}\ , \tag{3.34}$$

or mathematically more rigorously, $M_{h_{p,q}(c),c}$ is the unique sub-module such that

$$V_{h_{p,q}(c),c} \longrightarrow M'_{h_{p,q}(c),c} \longrightarrow M_{h_{p,q}(c),c} \tag{3.35}$$

is exact for all M'. Due to the state-field isomorphism, it is clear that this decoupling of states must reflect itself in partial differential equations for correlation functions, since descendants of primary fields are made by acting with modes of the stress energy tensor on them. These modes, as we have seen, are represented as differential operators. The precise relationship will be worked out further below. Thus, null states provide a very powerful tool to find further conditions for expectation values. They allow us to exploit the infinity of local conformal symmetries as well, and under special circumstances enable us – at least in principle – to compute *all* observables of the theory.

3.4 Descendant fields and Operator product expansion

As we associated to each highest-weight state a primary field, we may associate to each descendant state a descendant field in the following way: A descendant is a linear combination of monomials $L_{-n_1}\dots L_{-n_k}|h\rangle$. The modes L_n were extracted from the stress-energy tensor via a contour integration (1.18). This suggests to create the descendant field $\Phi_h^{(-n_1,\dots,-n_k)}(z)$ by a successive application of contour integrations

$$\begin{aligned}&\Phi_h^{(-n_1,\dots,-n_k)}(z) = \\ &\oint_{C_1}\frac{\mathrm{d}w_1}{(w_1-z)^{n_1-1}}T(w_1)\oint_{C_2}\frac{\mathrm{d}w_2}{(w_2-z)^{n_2-1}}T(w_2)\dots\oint_{C_k}\frac{\mathrm{d}w_k}{(w_k-z)^{n_k-1}}T(w_k)\Phi_h(z)\,,\end{aligned} \tag{3.36}$$

where from now on we include the prefactors $\frac{1}{2\pi\mathrm{i}}$ into the definition of $\oint \mathrm{d}z$. The contours C_i all encircle z and C_i completely encircles C_{i+1}, in short $C_i \succ C_{i+1}$.

There is only one problem with this definition, namely that it involves products of operators. In quantum field theory, this is a notoriously difficult issue. Firstly, operators may not commute, secondly, and more seriously, products of operators at equal points are not well-defined unless normal ordered. As we defined (3.36), we took care to respect "time" ordering, i.e. radial ordering on the complex plane. In order to evaluate equal-time commutators, we define for operators A, B and arbitrary funtions f, g the densities

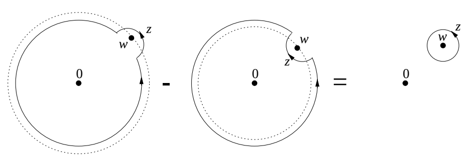

Figure 6: Contour deformation for OPE calculations.

$$A_f = \oint_0 \mathrm{d}z f(z)A(z)\,,\quad B_g = \oint_0 \mathrm{d}w g(w)B(w)\,, \tag{3.37}$$

where the contours are circles around the origin with radii $|z| = |w| = 1$. Then, the equal-time commutator of these objects is

$$[A_f, B_g]_{\text{e.t.}} = \oint_{C_1} \mathrm{d}z f(z)A(z) \oint_{C_2} \mathrm{d}w g(w)B(w) - \oint_{C_2} \mathrm{d}w g(w)B(w) \oint_{C_1} \mathrm{d}z f(z)A(z)\,, \tag{3.38}$$

where we took the freedom to deform the contours in a homologous way such that radial ordering is kept in both terms. As indicated in the figure five, both terms together result in the following expression,

$$[A_f, B_g]_{\text{e.t.}} = \oint_0 \mathrm{d}w g(w) \oint_w \mathrm{d}z f(z)A(z)B(w) \tag{3.39}$$

with the conour around w as small as we wish. The inner integration is thus given by the singularities of the operator product expansion (OPE) of $A(z)B(w)$. We suppose that products of operators have an asymptotic expansion for short distances of their arguments. The singular part of this short-distance expansion determines via contour integration the corresponding equal-time commutators. For example, with

$$T_\varepsilon = \oint_0 \mathrm{d}z \varepsilon(z)T(z) \tag{3.40}$$

as the general version of (1.18) for $\varepsilon(z) = z^{n+1}$, we recognize immediately $\delta_\varepsilon \Phi_h(w) = (\varepsilon\partial_w + h(\partial_w\varepsilon))\Phi_h(w) = [T_\varepsilon, \Phi_h(w)]$. If this is to be reproduced by an OPE, it must be of the form

$$T(z)\Phi_h(w) = \frac{h}{(z-w)^2}\Phi_h(w) + \frac{1}{(z-w)}\partial_w\Phi_h(w) + \textit{regular terms}\,. \tag{3.41}$$

To see this, one essentially has to apply Cauchy's integral formula $\oint \mathrm{d}z f(z)(z-w)^{-n} = \frac{1}{(n-1)!}\partial^{n-1}f(w)$. Of course, we may also attempt to find the OPE of the stress-energy tensor with itself from the Virasoro algebra (1.26) in the same way, which yields

$$T(z)T(w) = \frac{c/2}{(z-w)^4}\mathbb{1} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T(w) + \textit{regular terms}\,. \tag{3.42}$$

The reader is encouraged to verify that the above OPE does indeed yield the Viraosro algebra (1.26), if substituted into (3.39).

Note that $T(z)$ is not a proper primary field of weight two due to the term involving the central charge. Since $T(z)$ behaves as a primary field under L_i, $i \in \{-1, 0, 1\}$ meaning that it is a weight two tensor with respect to $SL(2,\mathbb{C})$, it is called quasi-primary. One important consequence of this is that the stress-energy tensor on the complex plane and the original stress energy tensor on the cylinder differ by a constant term. Indeed, remembering that the transfer from the complexified cylinder coordinate w to the complex plane coordinate z was given by the conformal map $z = \mathrm{e}^w$, one obtaines

$$T_{\text{cyl}}(w) = z^2 T(z) - \frac{c}{24}\mathbb{1}\,, \quad \text{i.e.} \quad (L_n)_{\text{cyl}} = L_n - \frac{c}{24}\delta_{n,0}\,. \tag{3.43}$$

This explains the appearance of the factor $-c/24$ in the definition (3.20) of the Virasoro characters.

The structure of OPEs in CFT is fixed to some degree by two requirements. Firstly, the OPE is not a commutative product, but it should be associative, i.e. $(A(x)B(y))C(z) = A(x)(B(y)C(z))$. The motivation for this presumption comes from the duality properties of string amplitudes. Duality is crossing symmetry in CFT correlation functions, which can be seen to be equivalent to associativity of the OPE. For example, one may evaluate a four-point function in several regions, where different pairs of coordinates are taken close together such that OPEs can be applied. Secondly, the OPE must be consistent with global conformal invariance, i.e. it must respect (3.16), (3.17), and (3.18). This fixes the OPE to be of the following generic form,

$$\Phi_{h_i}(z)\Phi_{h_j}(w) = \sum_k \frac{C_{ij}^k}{(z-w)^{h_i+h_j-h_k}}\Phi_{h_k}(w) + \ldots\,, \tag{3.44}$$

where the structure constants are identical to the structure constants which appeared in the three-point functions (3.17). Note that due to our normalization of the propagators (two-point functions), raising and lowering of indices is trivial (unless the two-point functions are non-trivial, i.e. $D_{ij} \neq \delta_{ij}$).

We can divide all fields in a CFT into a few classes. First, there are the primary fields Φ_h corresponding to highest-weight states $|h\rangle$ and second, there are all their Virasoro descendant fields $\Phi_h^{(-\{n\})}$ corresponding to the descendant states $L_{-\{n\}}|h\rangle$ given by (3.36). For instance, the stress energy tensor itself is a descendant of the identity, $T(z) = \mathbb{1}^{(-2)}$. We further divide descendant fields into two sub-classes, namely fields which are quasi-primary, and fields which are not. Quasi-primary fields transform conformally covariant for $SL(2,\mathbb{C})$ trnasformations only.

General local conformal transformations are implemented in a correlation function by simply inserting the Noether charge, which yields

$$\delta_\varepsilon\langle 0|\Phi_{h_N}(z_N)\ldots\Phi_{h_1}(z_1)|0\rangle = \langle 0|\oint \mathrm{d}z\varepsilon(z)T(z)\Phi_{h_N}(z_N)\ldots\Phi_{h_1}(z_1)|0\rangle\,, \tag{3.45}$$

where the contour encircles all the coordinates z_i, $i = 1,\ldots,N$. This contour can be deformed into the sum of N small contours, each encircling just one of the coordinantes, which is a standard technique in complex analysis. That is equivalent to rewriting (3.45) as

$$\sum_i\langle 0|\Phi_{h_N}(z_N)\ldots(\delta_\varepsilon\Phi_{h_i}(z_i))\ldots\Phi_{h_1}(z_1)|0\rangle = \sum_i\langle 0|\Phi_{h_N}(z_N)\ldots\left(\oint_{z_i}\mathrm{d}z\varepsilon(z)T(z)\Phi_{h_i}(z_i)\right)\ldots\Phi_{h_1}(z_1)|0\rangle\,. \tag{3.46}$$

Since this holds for any $\varepsilon(z)$, we can proceed to a local version of the equality between the right hand sides of (3.45) and (3.46), yielding

$$\langle 0|T(z)\Phi_{h_N}(z_N)\ldots\Phi_{h_1}(z_1)|0\rangle = \sum_i\left(\frac{h_i}{(z-z_i)^2} + \frac{1}{(z-z_i)}\partial_{z_i}\right)\langle 0|\Phi_{h_N}(z_N)\ldots\Phi_{h_1}(z_1)|0\rangle\,. \tag{3.47}$$

This identity is extremely usefull, since it allows us to compute any correlation function involving descendant fields in terms of the corresponding correlation function of primary fields. For the sake of simplicity, let us consider the correlator $\langle 0|\Phi_{h_N}(z_N)\ldots\Phi_{h_1}(z_1)\Phi_h^{(-k)}(z)|0\rangle$ with only one descendant field involved. Inserting the definition (3.36) and using the conformal Ward identity (3.47), this gives

$$\oint\frac{\mathrm{d}w}{(w-z)^{k-1}} \tag{3.48}$$

$$\times\left[\langle 0|T(z)\Phi_{h_N}(z_N)\ldots\Phi_{h_1}(z_1)\Phi_h(z)|0\rangle - \sum_i\left(\frac{h_i}{(w-z_i)^2} + \frac{1}{(w-z_i)}\partial_{z_i}\right)\langle 0|\Phi_{h_N}(z_N)\ldots\Phi_{h_1}(z_1)\Phi_h(z)|0\rangle\right]\,.$$

The contour integration in the first term encircles all the coordinates z and z_i, $i = 1, \ldots, N$. Since there are no other sources of poles, we can deform the contour to a circle around infinity by pulling it over the Riemann sphere accordingly. The highest-weight property $\langle 0|L_k = 0$ for $k \leq 1$ ensures that the integral around $w = \infty$ vanishes. The other terms are evaluated with the help of Cauchy's formula to

$$\mathcal{L}^i_{-k} \equiv -\oint_{z_i} \frac{\mathrm{d}w}{(w-z)^{k-1}} \left(\frac{h_i}{(w-z_i)^2} + \frac{1}{(w-z_i)} \partial_{z_i} \right) = \frac{(k-1)h_i}{(z_i - z)^k} + \frac{1}{(z_i-z)^{k-1}} \partial_{z_i} \,. \tag{3.49}$$

Going through the above small-print shows that a correlation function involving descendant fields can be expressed in terms of the correlation function of the corresponding primary fields only, on which explicitly computable partial differential operators act. Collecting $\mathcal{L}_{-k} = \sum_i \mathcal{L}^i_{-k}$ yields a partial differential operator (which implicitly depends on z) such that

$$\langle 0|\Phi_{h_N}(z_N) \ldots \Phi_{h_1}(z_1) \Phi^{(-k)}_h(z)|0\rangle = \mathcal{L}_{-k} \langle 0|\Phi_{h_N}(z_N) \ldots \Phi_{h_1}(z_1)\Phi_h(z)|0\rangle \,, \tag{3.50}$$

where this operator $\mathcal{L}_{-k}$ has the explicit form

$$\mathcal{L}_{-k} = \sum_{i=1}^{N} \left(\frac{(k-1)h_i}{(z_i-z)^k} + \frac{1}{(z_i-z)^{k-1}} \partial_{z_i} \right) \tag{3.51}$$

for $k > 1$. Due to the global conformal Ward identities, the case $k = 1$ is much simpler, being just the derivative of the primary field, i.e. $\mathcal{L}_{-1} = \partial_z$. Thus, correlators involving descendant fields are entirely expressed in terms of correlators of primary fields only. Once we know the latter, we can compute all correlation functions of the CFT.

On the other hand, if we use a descendant, which is a null field, i.e.

$$\chi^{(N)}_{h,c}(z) = \sum_{|\{n\}|=N} \beta^{\{n\}} \Phi_h^{-\{n\}}(z) \tag{3.52}$$

with $|\chi^{(N)}_{h,c}\rangle$ orthogonal to all other states, we know that it completely decouples from the physical states. Hence, every correlation function involving $\chi^{(N)}_{h,c}(z)$ must vanish. Hence, we can turn things around and use this knowledge to find partial differential equations, which must be satisfied by the correlation function involving the primary $\Phi_h(z)$ instead. For example, the level $N = 2$ null field yields according to (3.30) the equation

$$\left(\tfrac{2}{3}(2h_\pm + 1)\mathcal{L}_{-2} - \partial_z^2\right) \langle 0|\Phi_{h_N}(z_N) \ldots \Phi_{h_1}(z_1)\Phi_{h_\pm}(z)|0\rangle = 0 \tag{3.53}$$

with $h_\pm$ given by the non-trivial values in (3.29).

A particular interesting case is the four-point function. The three global conformal Ward identites (3.9) then allow us to express derivatives with respect to z_1, z_2, z_3 in terms of derivatives with respect to z. Every new-comer to CFT should once in her life go through this computation for the level two null field: If the field $\Phi_h(z)$ is degenerate of level two, i.e. possesses a null field at level two, we can reduce the partial differential equation (3.53) for $G_4 = \langle\Phi_{h_3}(z_3)\Phi_{h_2}(z_2)\Phi_{h_1}(z_1)\Phi_h(z)\rangle$ to an ordinary Riemnann differential equation

$$\begin{aligned}0 &= \left(\frac{3}{2(2h+1)}\partial_z^2 - \sum_{i=1}^{3}\left(\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i}\partial_{z_i}\right)\right) G_4 \\ &= \left(\frac{3}{2(2h+1)}\partial_z^2 + \sum_{i=1}^{3}\left(\frac{1}{z-z_i}\partial_z - \frac{h_i}{(z-z_i)^2}\right) + \sum_{i<j}\frac{h+h_i+h_j-\varepsilon_{ij}^k h_k}{(z-z_i)(z-z_j)}\right) G_4\,.\end{aligned} \tag{3.54}$$

This can be brought into the well-known form of the Gauss hypergeometric equation by extracting a suitable factor $x^p(1-x)^q$ from G_4 with x the crossing ratio $x = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$. Using the general ansatz (refeq:4pt), we first rewrite the four-point function for the particular choice of coordinates $z_3 = \infty$, $z_2 = 1$, and $z_1 = 0$ (i.e. $z \equiv x$) in the following form, where we renamed $h = h_0$ to allow consistent labelling:

$$\begin{aligned}\langle\Phi_{h_3}(\infty)\Phi_{h_2}(1)\Phi_{h_1}(0)\Phi_{h_0}(z)\rangle &= z^{p+\mu_{01}}(1-z)^{q+\mu_{20}}F(z)\,, \\ \mu_{ij} &= (h_0+h_1+h_2+h_3)/3 - h_i - h_j\,, \\ p &= \tfrac{1}{6} - \tfrac{2}{3}h_0 - \mu_{01} - \tfrac{1}{6}\sqrt{r_1}\,, \\ q &= \tfrac{1}{6} - \tfrac{2}{3}h_0 - \mu_{01} - \tfrac{1}{6}\sqrt{r_2}\,, \\ r_i &= 1 - 8h_0 + 16h_0^3 + 48h_ih_0 + 24h_i\,.\end{aligned} \tag{3.55}$$

The remaining function $F(z)$ then is a solution of the hypergeometric system ${}_2F_1(a,b;c;z)$ given by

$$\begin{aligned}0 &= \big(z(1-z)\partial_z^2 + [c-(a+b+1)z]\partial_z - ab\big)\,F(z)\,, \\ a &= \tfrac{1}{2} - \tfrac{1}{6}\sqrt{r_1} - \tfrac{1}{6}\sqrt{r_2} - \tfrac{1}{6}\sqrt{r_3}\,, \\ b &= \tfrac{1}{2} - \tfrac{1}{6}\sqrt{r_1} - \tfrac{1}{6}\sqrt{r_2} + \tfrac{1}{6}\sqrt{r_3}\,, \\ c &= 1 - \tfrac{1}{3}\sqrt{r_1}\,.\end{aligned} \tag{3.56}$$

The general solution is then a linear combination of the two linearly independent solutions ${}_2F_1(a,b;c;z)$ and $z^{1-c}{}_2F_1(a-c+1,b-c+1;2-c;z)$. Which linear combination one has to take is determined by the requirement that the full four-point function involving holomorphic and anti-holomorphic dependencies must be single-valued to represent a physical observable quantity. For $|z| < 1$, the hypergeometric function enjoys a convergent power series expansion

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}\,, \quad (x)_n = \Gamma(x+n)/\Gamma(x)\,, \tag{3.57}$$

but it is a quite interesting point to note that the integral representation has a remarkably similarity to our expressions of dual string-amplitudes encountered in section two, namely

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \mathrm{d}t\, t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,, \tag{3.58}$$

which, of course, is no accident. However, we must leave this issue to the curiosity of the reader, who might browse through the literature looking for the keyword *free field construction*.

A further consequence of the fact, that descendants are entirely determined by their corresponding primaries is that we can refine the structure of OPEs. Let us assume we want to compute the OPE of two primary fields. The right hand side will possibly involve both, primary and descendant fields. Since the coefficients for the descendant fields are fixed by local conformal covariance, we may rewrite (3.44) as

$$\Phi_{h_i}(z)\Phi_{h_j}(w) = \sum_{k,\{n\}} \mathcal{C}_{ij}^{k}\beta_{ij}^{k,\{n\}}\,(z-w)^{h_k+|\{n\}|-h_i-h_j}\,\Phi_{h_k}^{(-\{n\})}(w)\,, \tag{3.59}$$

where the coefficients β are determined by conformal covariance. Note that we have skipped the anti-holomorphic part, although an OPE is in general only well-defined for fields of the full theory, i.e. for fields $\Phi_{h,\bar{h}}(z,\bar{z})$. An exception is the case where all conformal weights satisfy $2h\in\mathbb{Z}$, since then holomorphic fields are already local.

Finally, we can explain how associativity of the OPE and crossing symmetry are related. Let us consider a four-point function $G_{ijkl}(z,\bar{z}) = \langle 0|\phi_l(\infty,\infty)\phi_k(1,1)\phi_j(z,\bar{z})\phi_i(0,0)|0\rangle$. There are three different regions for the free coordinate z, for which an OPE makes sense, corresponding to the contractions $z\to 0 : (i,j)(k,l)$, $z\to 1 : (k,j)(i,l)$, and $z\to\infty : (l,j)(k,i)$. In fact, these three regions correspond to the s, t, and u channels. Duality states, that the evaluation of the four-point function should not depend on this choice. Absorbing all descendant contributions into functions $\mathcal{F}$ called *conformal blocks*, duality imposes the conditions

$$\begin{aligned} G_{ijkl}(z,\bar{z}) &= \sum_m C_{ij}^m C_{mkl}\mathcal{F}_{ijkl}(z|m)\bar{\mathcal{F}}_{ijkl}(\bar{z}|m) \\ &= \sum_m C_{jk}^m C_{mli}\mathcal{F}_{ijkl}(1-z|m)\bar{\mathcal{F}}_{ijkl}(1-\bar{z}|m) \\ &= \sum_m C_{jl}^m C_{mki} z^{-2h_j}\mathcal{F}_{ijkl}(\frac{1}{z}|m)\bar{z}^{-2\bar{h}_j}\bar{\mathcal{F}}_{ijkl}(\frac{1}{z}|m)\,, \end{aligned} \tag{3.60}$$

where m runs over all primary fields which appear on the right hand side of the corresponding OPEs. The carefull reader will have noted that these last equations were written down in terms of the full fields in the so-called *diagonal* theory, i.e. where $\bar{h}=h$ for all fields. This is one possible solution to the physical requirement that the full correlator be a single-valued analytic function. Under certain circumstances, other solutions, so-called non-diagonal theories, do exist.

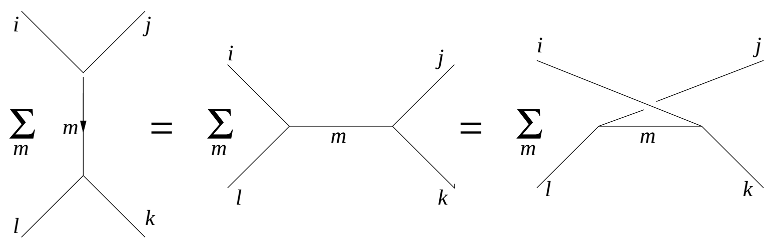

Figure 7: The three different ways to evaluate a four-point amplitude, i.e. s- t- and u-channels.

In the full theory, with left- and right-chiral parts combined, the OPE has the folowing structure, where the contributions from descendants have been made explicit:

$$\begin{aligned} &\Phi_{h_i,\bar{h}_i}(z,\bar{z})\Phi_{h_j,\bar{h}_j}(w,\bar{w}) = \\ &\sum_{k,\{n\}}\sum_{\bar{k},\{\bar{n}\}} \mathcal{C}_{ij}^{k}\beta_{ij}^{k,\{n\}}\mathcal{C}_{\bar{\imath}\bar{\jmath}}^{\bar{k}}\beta_{\bar{\imath}\bar{\jmath}}^{\bar{k},\{\bar{n}\}}\,(z-w)^{h_k+|\{n\}|-h_i-h_j}(\bar{z}-\bar{w})^{\bar{h}_k+|\{\bar{n}\}|-\bar{h}_i-\bar{h}_j}\,\Phi_{h_k,\bar{h}_k}^{(-\{n\},-\{\bar{n}\})}(w,\bar{w})\,. \end{aligned} \tag{3.61}$$

Correlation functions in the full CFT should be single valued in order to represent observables, i.e. physical measurable quantities. This imposes further restrictions on the particular linear combinations of the conformal blocks $\mathcal{F}_{ijkl}(z|m)$ in (3.60). In most CFTs, the diagonal combination $\bar{h}=h$ is a solution, but it is easy to see, that the monodromy of a field $\Phi_{h,\bar{h}}(z,\bar{z})$ under $z\mapsto e^{2\pi i}z$ yields the less restrictive condition $h-\bar{h}\in\mathbb{Z}$, such that off-diagonal solutions can be possible.

The success story of CFT is much rooted in the following observation first made by Belavin, Polyakov and Zamolodchikov [1]: If an OPE of two primary fields $\Phi_i(z)\Phi_j(w)$ is considered, which both are degenerated at levels N_i and N_j respectively, then the right hand side will only involve contributions from primary fields, which *all* are degenerate at a certain levels $N_k \leq N_i + N_j$. In particular, the sum over conformal families k on the right hand side is then always finite, and so is the set of conformal blocks one has to know. In particular, the set of degenerate primary fields (and their descendants) forms a closed operator algebra. For example, considering a four-point function where all four fields are degenerate at level two, we find only two conformal blocks for each channel, which precisely are the hypergeometric functions computed above and their analytic continuations. Even more remarkably, for the special values $c(m)$ in (3.31) with $m \in \mathbb{N}$, there are only *finitely* many primary fields with conformal weights $h_{p,q}(c)$ with $1 \leq p < m$ and $1 \leq q < m+1$ given by(3.32). All other degenerate primary fields with weights $h_{p,q}(c)$ where p or q lie outside this range turn out to be null fields within the Verma modules of the descendants of these former primary fields. Hence, such CFTs have a finite field content and are actually the "smallest" CFTs. This is why they are called *minimal models*. Unfortunately, they are not very useful for string theory, but turn up in many applications of statistical physics [4].

4. The free Boson

One particularly important CFT is the theory of massless scalar fields in two dimensions. We already encountered the string embedding map X^μ, which turned out to be such a field. We forget now about the space-time index μ, and call the scalar field ϕ. The action reads in complex coordinates

$$S \propto \int \mathrm{d}^2 z {:}\partial\phi\bar{\partial}\phi{:}\,, \tag{4.1}$$

leading to the equation of motion $\partial\bar{\partial}\phi = 0$. A general formal solution to this is the holomorphic Laurent ansatz

$$j(z) \equiv \partial\phi(z) = -\mathrm{i}\sum_{n\in\mathbb{Z}} a_n z^{-n-1}\,, \tag{4.2}$$

where $a_0 \equiv p$ if the reader wishes to compare this with (1.12), which now would read as

$$\phi(z,\bar{z}) = q - \mathrm{i}p\log z\bar{z} + \mathrm{i}\sum_{n\neq 0}\frac{a_n}{n}z^{-n} + \mathrm{i}\sum_{n\neq 0}\frac{\tilde{a}_n}{n}\bar{z}^{-n}\,. \tag{4.3}$$

After quantization of the theory, the canonical commutators are

$$[a_n, a_m] = n\delta_{n+m,0}\,, \quad [a_n, \tilde{a}_m] = 0\,, \quad [\tilde{a}_n, \tilde{a}_m] = n\delta_{n+m,0}\,, \quad [q,p] = \mathrm{i}\,. \tag{4.4}$$

In order to evaluate the OPE of two scalar fields, we have to keep normal ordering in mind, i.e. we always shift p to the right of q, and a_n to the right of a_m whenever $n > m$. With these conventions, we get

$$\phi(z,\bar{z})\phi(w,\bar{w}) = {:}\phi(z,\bar{z})\phi(w,\bar{w}){:} - \mathrm{i}[p,q]\log z\bar{z} + \left[\mathrm{i}\sum_{n>0}\frac{a_n}{n}z^{-n}, \mathrm{i}\sum_{m<0}\frac{a_m}{m}w^{-m}\right] + c.c. \tag{4.5}$$

Using the commutation relations we find that the oscillator terms are reduced to the c-function $-\sum_{n>0}\sum_{m<0}\frac{1}{nm}nz^{-n}w^{-m}\delta_{n+m,0}$ and analogously for the anti-holomorphic terms, such that we get for the right hand side

$${:}\phi(z,\bar{z})\phi(w,\bar{w}){:} - \log z\bar{z} + \sum_{n>0}\frac{1}{n}\left(\frac{w}{z}\right)^n + \sum_{n>0}\frac{1}{n}\left(\frac{\bar{w}}{\bar{z}}\right)^n\,,$$

which converges for $|z| > |w|$. Since the product was radially ordered, this is satisfied such that we find the OPE to be

$$\phi(z,\bar{z})\phi(w,\bar{w}) = \underbrace{\log(z-w) + \log(\bar{z}-\bar{w})}_{\text{singular part}} + \underbrace{:\phi(z,\bar{z})\phi(w,\bar{w}):}_{\text{regular part}} \,. \tag{4.6}$$

The commutators thus yield the singular part of the OPE for $z \to w$, as they should, while the regular part is given by the normal ordered product.

This OPE is not what we usually expect in a CFT, but neither is the scalar field a good conformal field, since it cannot be factorized entirely into holomorphic and anti-holomorphic part. However, its derivatives are proper (anti-)holomorphic conformal fields, and it is very easy to find the OPE for $\partial\phi$ from the above expression by differentiation,

$$\partial\phi(z)\partial\phi(w) = \frac{-1}{(z-w)^2} + :\partial\phi(z)\partial\phi(w):. \tag{4.7}$$

Since vacuum expectation values of normal ordered products vanish by definition, the expectation value of $\langle\partial\phi(z)\partial\phi(w)\rangle$ is immediately read off from (4.7) to be $-1/(z-w)^2$.

Earlier we found the classical energy momentum tensor (1.7), which in our current notation reads $T(z) = -\frac{1}{2}\partial\phi(z)\partial\phi(z)$. This has to be improved in the quantized theory by normal ordering. Thus, we define

$$T(z) \equiv -\tfrac{1}{2}:\partial\phi(z)\partial\phi(z): = -\tfrac{1}{2}\lim_{z\to w}\left(\partial\phi(z)\partial\phi(z) - \frac{-1}{(z-w)^2}\right). \tag{4.8}$$

The carefull reader might query at this moment whether we have not mixed different notions of normal ordering here. In fact, we have, but they are known to coincide for the case of the free boson. With the above definition for $T(z)$, the energy of the vacuum is put to zero.

We are now in the position to compute further OPEs. Let us start with the OPE of $T(z)$ with $\partial\phi(w)$. Since $T(z)$ is already normal ordered, we only have to worry about ordering between $T(z)$ and $\partial\phi(w)$. This is easily achieved with Wick's theorem, which in this case simply amounts to

$$-\tfrac{1}{2}:\partial\phi(z)\partial\phi(z):\partial\phi(w) = -\tfrac{1}{2}\left(:\partial\phi(z)\partial\phi(z)\partial\phi(w): + 2\partial\phi(z)\frac{-1}{(z-w)^2}\right). \tag{4.9}$$

The factor of two in the last term is due to combinatorics: there are two ways to contract. The field in the last term has yet to be expressed at the coordinate w, which simply means a Taylor expansion $\partial\phi(z) = \partial\phi(w) + (z-w)\partial^2\phi(w) + \frac{1}{2}(z-w)^2\partial^3\phi(w) + \ldots$, and finally yields

$$T(z)\partial\phi(w) = \frac{1}{(z-w)^2}\partial\phi(w) + \frac{1}{(z-w)}\partial(\partial\phi(w)) + \textit{regular terms}\,. \tag{4.10}$$

Comparing with (3.41), we conclude that $\partial\phi(w)$ is a primary conformal field with weight $h = 1$.

In the same way, we can compute the OPE of the energy momentum tensor with itself and compare it with (3.42). It is left to the industrious reader to verify that the result is

$$T(z)T(w) = \frac{1/2}{(w-w)^4}\mathbb{1} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial T(w) + reg.\ terms\,, \qquad (4.11)$$

which identifies the central charge of the free boson theory to be $c = 1$. It emphasizes that the central charge is a quantum effect, since it arose due to normal ordering, and we can rightly guess that the central charge of the bosonic string theory is equal to the number of free bosons X^μ, i.e. $c = D$.

The above exercise is also a good training in the application of Wick's theorem. The OPE of the stress-energy tensor with itself yields an expression of four fields, paired into two normal ordered products. In total, that allows four simple contractions and two double contractions:

$$\begin{aligned}
T(z)T(w) &= \tfrac{1}{4}{:}\partial\phi(z)\partial\phi(z){:}{:}\partial\phi(w)\partial\phi(w){:} \\
&= \tfrac{1}{4}{:}\partial\phi(z)\partial\phi(z)\partial\phi(w)\partial\phi(w){:} + 4\tfrac{1}{4}{:}\partial\phi(z)\partial\phi(w){:}\frac{-1}{(z-w)^2} + 2\tfrac{1}{4}\left(\frac{-1}{(z-w)^2}\right)^2 \\
&= regular + \sum_n \frac{1}{n!}(z-w)^n{:}\partial^n(\partial\phi(w))\partial\phi(w){:}\frac{-1}{(z-w)^2} + \frac{1/2}{(z-w)^4}\mathbb{1} \\
&= \frac{1/2}{(z-w)^4}\mathbb{1} + -\tfrac{1}{2}{:}\partial\phi(w)\partial\phi(w){:}\frac{2}{(z-w)^2} + -\tfrac{1}{2}\partial{:}\phi(w)\partial\phi(w){:}\frac{1}{(z-w)^1} + regular\,,
\end{aligned} \qquad (4.12)$$

which yields (4.11). In the last line, the Taylor expansion of the field $\partial\phi(z)$ around w was evaluated for the singular terms only, and $2{:}A\partial A{:} = \partial{:}AA{:}$ was used. As a further check, we can extract the Virasoro algebra from this OPE. To do so, we write

$$\begin{aligned}
[L_n, L_m] &= \left(\oint dz \oint dw - \oint dw \oint dz\right) z^{n+1}T(z)w^{m+1}T(w) \\
&= \oint dw \oint_w dz z^{n+1}w^{m+1}T(z)T(w) \\
&= \oint dw \oint_w dz \left(\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w)\right) \\
&= \oint dw w^{m+1}\left(\frac{c}{2}\frac{1}{3!}(\partial^3 z^{n+1}) + 2\frac{1}{1!}(\partial^1 z^{n+1})T(w) + z^{n+1}\partial T(w)\right)_{z=w} \\
&= \oint dw w^{m+1}\left(\frac{c}{12}(n+1)n(n-1)w^{n-2} + 2(n+1)w^n T(w) + w^{n+1}\partial T(w)\right) \\
&= \oint dw \left(\frac{c}{12}(n+1)n(n-1)w^{n+m-1} + 2(n+1)w^{n+m+1}T(w) + w^{n+m+2}\partial T(w)\right) \\
&= \oint dw \left(\frac{c}{12}(n+1)n(n-1)w^{n+m-1} + 2(n+1)w^{n+m+1}T(w) - (2+n+m)w^{n+m+1}T(w)\right) \\
&= \frac{c}{12}(n^3-n)\delta_{n+m,0} + (n-m)L_{n+m}\,,
\end{aligned} \qquad (4.13)$$

where we have used partial integration (there are no boundary terms here!) in the penultimate line. The second line is of course due to the deformation of integration contours trick already discussed in section 3.4 (cf. figure five).

4.1 Vertex operators revisited

The free boson CFT does contain other primary fields, namely the vertex operators. The scalar field ϕ in (4.3) is not a good conformal field, but its derivatives were. Another way to get rid off the unpleasant logarithm is to exponentiate the field. However, we have to normal order this expression. So, we define

$$V_k(z,\bar{z}) = :\!\exp(\mathrm{i}k\phi(z,\bar{z}))\!:\,, \tag{4.14}$$

and compute its OPE with the energy momentum tensor. To do so, we use that the expectation value $\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle = -\log|z-w|^2$ as can be seen from the OPE (4.6). With this, and Wick's theorem to contract products of normal ordered quantities into fully normal ordered quantities times expectation values in the usual way,

$$\begin{aligned} T(z)V_k(w,\bar{w}) &= -\tfrac{1}{2}:\!\partial\phi(z)\partial\phi(z)\!:\sum_{n=0}^{\infty}\frac{1}{n!}(\mathrm{i}k)^n:\!\phi^n\!: \\ &= -\tfrac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}(\mathrm{i}k)^n\frac{n(n-1)}{(z-w)^2}:\!\phi^{n-2}\!: + \sum_{n=0}^{\infty}\frac{1}{n!}(\mathrm{i}k)^n\frac{n}{(z-w)}:\!\partial\phi(z)\phi^{n-1}\!: \\ &= \frac{k^2/2}{(z-w)^2}V_k(w,\bar{w}) + \frac{1}{(z-w)}\partial_w V_k(w,\bar{w}) + reg.\ terms\,. \end{aligned} \tag{4.15}$$

The same calculation goes through for $\bar{T}(\bar{z})$. Therefore, the vertex operators are primary fields with conformal weights $h = \bar{h} = \frac{k^2}{2}$. Note that V_k has the same conformal weight as V_{-k}.

The two-point function of two such vertex operators can be found in many ways, e.g. by exploiting $SL(2,\mathbb{C})$ invariance, and turns out as

$$\langle V_k(z,\bar{z})V_{k'}(w,\bar{w})\rangle = (z-w)^{-2\frac{k^2}{2}}(\bar{z}-\bar{w})^{-2\frac{k^2}{2}}\delta_{k+k',0} = |z-w|^{-k^2}\delta_{k+k',0}\,. \tag{4.16}$$

More generally, n-point functions of arbitrary vertex operators of the free bosonic CFT can all be computed with Wick's theorem, and the result is quite simple, namely

$$\langle\prod_i V_{k_i}(z_i,\bar{z}_i)\rangle = \prod_{i>j}|z_j - z_i|^{k_i k_j}\delta_{\sum_i k_i,0}\,, \tag{4.17}$$

provided $|z_i| > |z_j|$ for $i < j$. Thus, these n-point functions are trivially zero unless the "charge" balance $\sum_i k_i = 0$ is kept, i.e. total momentum is conserved.

The condition $\sum_i k_i = 0$ comes from the existence of a conserved charge. Actually, the operator $j(z) = \mathrm{i}\partial\phi(z)$ is a conserved current with zero mode $a_0 = p$, as can be infered from its mode expansion $j(z) = pz^{-1} + \sum_{n\neq 0} a_n z^{-n-1} \equiv \sum_n a_n z^{-n-1}$. Since the vacuum was defined in such a way that $\langle 0|p = p|0\rangle = 0$, it follows that

$$\begin{aligned} 0 &= \langle 0|\overset{(\leftarrow)}{p}\prod_i V_{k_i}(z_i,\bar{z}_i)|0\rangle \\ &= \langle 0|\overset{(\rightarrow)}{p}\left(\prod_i V_{k_i}(z_i,\bar{z}_i)\right)|0\rangle \\ &= \sum_j \langle 0|\prod_{i>j}V_{k_i}(z_i,\bar{z}_i)(pV_{k_j}(z_i,\bar{z}_i))\prod_{i<j}V_{k_i}(z_i,\bar{z}_i)|0\rangle \\ &= \sum_j (k_j)\langle 0|\prod_i V_{k_i}(z_i,\bar{z}_i)|0\rangle\,, \end{aligned} \tag{4.18}$$

where we have indicated the direction in which p is applied. That is in fact consistent with the operator produkt expansion of the CFT. Due to global conformal invariance, the only non-vanishing one-point function must be of a field of zero conformal weight (which in general is the identity). The OPE of two vertex operators can be calculated with Wick's theorem or by inserting the mode expansion of the free field and using the Baker-Hausdorff formula. It yields

$$\begin{aligned} V_{k'}(z,\bar{z})V_k(w,\bar{w}) &= :\exp(\mathrm{i}k'\phi(z,\bar{z})): :\exp(\mathrm{i}k\phi(w,\bar{w})): \\ &= (z-w)^{\frac{1}{2}(k+k')^2-\frac{1}{2}k^2-\frac{1}{2}k'^2}(\bar{z}-\bar{w})^{\frac{1}{2}(k+k')^2-\frac{1}{2}k^2-\frac{1}{2}k'^2} :\exp(\mathrm{i}(k+k')\phi(w,\bar{w})): + \ldots \\ &= |z-w|^{2kk'}V_{k+k'}(w,\bar{w}) + \ldots \end{aligned} \tag{4.19}$$

as its leading term. Therefore, contracting all fields via successive OPEs will finally result in the vertex operator $V_K(0,0)$, $K=\sum_i k_i$, which must be of conformal weight zero. (The OPE can only be applied for short distances. However, global conformal invariance always admitts to achieve this situation by a global translation of all points $z_i \mapsto z_i + Z$ with $|Z| \gg 1$ and a following inversion.)

We learn from this that the two-point function is non-zero only for $k' = -k$, meaning that the correct definition of the in- and out-states is

$$|k\rangle = V_k(0,0)|0\rangle\,, \quad \langle k| = \lim_{z\to\infty}\langle 0|(V_k(z,\bar{z}))^\dagger z^{k^2}\bar{z}^{k^2} = \lim_{z\to\infty}\langle 0|V_{-k}(z,\bar{z})z^{k^2}\bar{z}^{k^2}\,, \tag{4.20}$$

such that $\langle k'|k\rangle = \delta_{k,k'}$. One says that the field $V_{-k}(z,\bar{z})$ is the *conjugate* field of $V_k(z,\bar{z})$. Note that $h(k) = \frac{1}{2}k^2 = h(-k)$ such that conjugate fields have the same conformal weights, as it must be.

4.2 Chiral bosons

The free boson could not be split into holomorphic and anti-holomorphic parts, i.e. into left and right chiral components. However, we could generalize its mode expansion to

$$\Phi(z,\bar{z}) = q_L + q_R - \mathrm{i}(p_L \log z + p_R \log\bar{z}) + \mathrm{i}\sum_{n\neq 0}\left(\frac{a_n}{n}z^{-n} + \frac{\tilde{a}_n}{n}\bar{z}^{-n}\right)\,, \tag{4.21}$$

introducing left and right chiral momenta and center-of-mass coordinates. A consistent choice for the commutators is $[q_L,p_L] = [q_R,p_R] = \mathrm{i}$, with commutators mixing left and right chiral parts vanishing. In this way, we obtain a field which can be split into left and right chiral components, $\Phi(z,\bar{z}) = \Phi_L(z) + \Phi_R(\bar{z})$, with

$$\Phi_L(z) = q_L - \mathrm{i}p_L\log z + \mathrm{i}\sum_{n\neq 0}\frac{a_n}{n}z^{-n}\,, \tag{4.22}$$

and analogously for $\Phi_R(\bar{z})$. These modifications do not change any of our previously obtained results that depended only on $\partial\phi$ or $\bar{\partial}\phi$. But in addition, we can now introduce chiral vertex operators, e.g. $V_{k,L}(z) = :\exp(\mathrm{i}k\Phi_L(z)):$ which can be seen to have conformal weight $h = \frac{1}{2}k^2$ and $\bar{h} = 0$.

To understand the meaning of our modifications, we momentarily translate back to cylinder coordinates, which gives us

$$\Phi(\xi^0,\xi^1) = q + 2p\xi^0 + L\xi^1 + oscillators\,, \tag{4.23}$$

where $p_L = p + \frac{1}{2}L$ and $p_R = p - \frac{1}{2}L$. The extra ξ^1 is new, and naively violates our requirement that Φ be periodic in the space direction, i.e. $\Phi(xi^0, xi^1 + 2\pi) = \Phi(xi^0, xi^1)$. We can repair this by imposing a new symmetry on the field, namely $\Phi = \Phi + 2\pi L$, which must hold for all eigen values of the operator L as well as for all integer linear combinations of such eigen values. If Φ is to possess a non-trivial dependence on ξ^1, then the eigen values of L must be quantized on a lattice Λ with $\dim\Lambda = D$, the number of free bosons in the theory.

The existence of such a lattice has a natural interpretation in closed string theory. It means that the target space (to which the string coordinate maps) is not Euclidean or Minkowski flat, but is compactified on a D-dimensional torus. The string may then wind several times around any of the compactified dimensions before it closes.

Interestingly, the same lattice description can be found by looking at the chiral vertex operators. Obviously, vertex operators $V_{\boldsymbol{k},L}(z) = \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\Phi}_L(z))$ have integer conformal weigths for $\boldsymbol{k}^2 \in 2\mathbb{Z}$. One can prove that the OPE of these vertex operators is

$$V_{\boldsymbol{k},L}(z)V_{\boldsymbol{k}',L}(w) = (z-w)^{\boldsymbol{k}\cdot\boldsymbol{k}'}V_{\boldsymbol{k}+\boldsymbol{k}',L}(w) + reg.\ terms\,, \tag{4.24}$$

which forms a closed operator algebra only if with $\boldsymbol{k}$ and $\boldsymbol{k}'$ also $\boldsymbol{k}'' = \boldsymbol{k} + \boldsymbol{k}'$ satisfies the condition that its square is an even integer. If the set of all $\boldsymbol{k}$, for which a chiral vertex operator with integer conformal weight exists, forms an even lattice Λ, then the operator algebra closes and yields an extended chiral symmetry algebra, i.e. an algebra of chiral local single-valued fields. Chiral local fields are candidates for symmetry generators, since they admitt well-defined Noether charge densities. Thus, they admitt additional quantum numbers.

We know that our theory contains states $|\boldsymbol{p}_L, \boldsymbol{p}_R\rangle$ which are highest-weight states with respect to the Virasoro algebra. Skipping normal ordering signs, these states are created by applying vertex operators $V_{\boldsymbol{p}_L,\boldsymbol{p}_R}(z,\bar{z}) = \exp(\mathrm{i}\boldsymbol{p}_L \cdot \boldsymbol{\Phi}_L(z))\exp(\mathrm{i}\boldsymbol{p}_R \cdot \boldsymbol{\Phi}_R(z))$ to the vacuum and letting $z, \bar{z}$ both tend to zero, such that $\exp(\mathrm{i}\boldsymbol{p}_L \cdot \boldsymbol{\Phi}_L(0))\exp(\mathrm{i}\boldsymbol{p}_R \cdot \boldsymbol{\Phi}_R(0))|0\rangle = \exp(\mathrm{i}\boldsymbol{p}_L \cdot \boldsymbol{q}_L + \mathrm{i}\boldsymbol{p}_R \cdot \boldsymbol{q}_R)|0\rangle$. Locality with respect to the chiral vertex operators of the extended chiral algebra requires that $\boldsymbol{k} \cdot \boldsymbol{p}_L \in \mathbb{Z}$ and $\boldsymbol{k} \cdot \boldsymbol{p}_R \in \mathbb{Z}$. Thus, the only possible momenta that the theory admitts must be elements of the dual lattice Λ^*, which is defined as $\Lambda^* = \{\boldsymbol{p} : \boldsymbol{p} \cdot \boldsymbol{k} \in \mathbb{Z} \;\; \forall \boldsymbol{k} \in \Lambda\}$.

4.3 OPEs and path integrals

Our discussion of the free massless scalar field theory started from classical consideration and went on towards a quantized theory. During this procedure, we were a bit sloppy translating identities valid for classical fields into operator identities. With the latter we mean equations written down for operatorsm, which are implicitly understood to hold only when evaluated within expectation values. In the path integral formalism, an expectation value is defined by

$$\langle F[\phi]\rangle = \int (\mathcal{D}\phi)\exp(-S[\phi])F[\phi] \tag{4.25}$$

where $F[\phi]$ is an arbitrary functional of the field ϕ, and $S[\phi] = \frac{1}{2\pi\alpha'}\int \mathrm{d}^2z\partial\phi\bar{\partial}\phi$. The operator equations of motion are then found by the variation principle $\delta\langle F[\phi]\rangle = 0$ in the following way:

$$\begin{aligned} 0 &= \int (\mathcal{D}\phi)\frac{\delta}{\delta\phi(z,\bar{z})}\exp(-S)F[\phi] \\ &= -\int (\mathcal{D}\phi)\exp(-S)\frac{\delta S}{\delta\phi(z,\bar{z})}F[\phi] \\ &= -\langle \frac{\delta S}{\delta\phi(z,\bar{z})}F[\phi]\rangle \\ &= \tfrac{1}{\pi\alpha'}\langle \partial\bar{\partial}\phi(z,\bar{z})F[\phi]\rangle \, , \end{aligned} \tag{4.26}$$

provided that $F[\phi]$ does not contain an insertion at the point $(z,\bar{z})$. Since this holds for any such insertion, the condition $\langle\partial\bar{\partial}\phi(z,\bar{z})F[\phi]\rangle = 0$ is usually refered to in the form of the operator equation of motion $\partial\bar{\partial}\phi(z,\bar{z}) = 0$.

What happens if $F[\phi]$ does contain such an insertion at a point coincident with $(z,\bar{z})$? The calculation is as above, except that the variation is applied to $F[\phi]$ as well. Therefore, if we take $F[\phi] = \phi(z',\bar{z}')G[\phi]$, we find

$$\begin{aligned} 0 &= \int (\mathcal{D}\phi)\frac{\delta}{\delta\phi(z,\bar{z})}\left(\exp(-S)\phi(z',\bar{z}')G[\phi]\right) \\ &= -\int (\mathcal{D}\phi)\exp(-S)\left(\tfrac{1}{\pi\alpha'}\partial\bar{\partial}\phi(z,\bar{z})\phi(z',\bar{z}') + \delta^2(z-z',\bar{z}-\bar{z}')\right)G[\phi] \\ &= \tfrac{1}{\pi\alpha'}\langle\partial\bar{\partial}\phi(z,\bar{z})\phi(z',\bar{z}')G[\phi]\rangle + \langle\delta^2(z-z',\bar{z}-\bar{z}')G[\phi]\rangle \, , \end{aligned} \tag{4.27}$$

provided again that all other insertions $G[\phi]$ are located away from $(z,\bar{z})$. it follows that the equation of motion holds except at coincident points. Thus, we find the operator equation of motion

$$\partial_z\partial_{\bar z}\phi(z,\bar z)\phi(z',\bar z') = -\pi\alpha'\delta^2(z-z',\bar z-\bar z')\,. \tag{4.28}$$

So far, we have not thought about normal ordering, which we should do if we consider products of operators at coinciding points. Usually, the normal ordered product is required to behave exactly like a classical object, i.e. we want to recover the equation

$$\partial_z\partial_{\bar z}{:}\phi(z,\bar z)\phi(z',\bar z'){:} = 0\,. \tag{4.29}$$

Recalling that $\partial\bar\partial\log|z|^2 = 2\pi\delta^2(z,\bar z)$, one immediately sees that this can be satisfied with the definition

$${:}\phi(z,\bar z)\phi(z',\bar z'){:} = \phi(z,\bar z)\phi(z',\bar z') + \tfrac{\alpha'}{2}\log|z-z'|^2\,, \tag{4.30}$$

which we may turn around to read off the OPE of two scalar fields. As it happens, this definition of normal ordering yields exactly the same result as our earlier prescription defined in terms of oscillator modes, (4.6). The reader should keep in mind, that this is not necessarily the case.

5. Ghost systems

Another very important family of CFTs are the so-called ghost systems. Mathematically, they are the CFT description of the complex analysis of j-differentials. Thus, one starts with considering a pair of anti-commuting fields $b(z)$ and $c(z)$ with conformal weights j and $1-j$ respectively. Indeed, $b^{(j)} = b(z)(\mathrm{d}z)^j$ and $c^{(1-j)} = c(z)(\mathrm{d}z)^{1-j}$ are invariant under conformal transformations provided $b(z)$ transforms as $b(z') = b(z)(\mathrm{d}z'/\mathrm{d}z)^{-j}$ and analogously for $c(z)$.

Altough we will see in a moment that the resulting CFT is not unitary, it possesses a natural scalar product defined via

$$\langle b^{(j)}, c^{(1-j)}\rangle = \oint b(z)c(z)\mathrm{d}z = \oint b(z)(\mathrm{d}z)^j c(z)(\mathrm{d}z)^{1-j}\,. \tag{5.1}$$

If $2j\in\mathbb{Z}$, these fields make sense as chiral fields, meaning that they behave benign under the monodromy $z\mapsto \mathrm{e}^{2\pi\mathrm{i}}z$, aquiering nothing more than a sing (for j half-integer). Under these circumstances, they possess a mode expansion

$$b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-j}\,,\quad \text{i.e.}\quad b_n = \oint \mathrm{d}z\, b(z) z^{n+1}\,, \tag{5.2}$$

and analogously for $c(z)$. Since the fields are anti-commuting their modes satisfy the relations

$$\{b_m, c_n\} = \delta_{m+n,0}\,, \tag{5.3}$$

with all other anti-commutators vanishing.

Let us first extract some general information such as the equations of motion. The action of the bc system is given by

$$S = \tfrac{1}{2\pi}\int \mathrm{d}^2 z\, b(z)\bar{\partial}c(z)\,, \tag{5.4}$$

which is conformally invariant by construction due to $j + (1-j) = 1$. The operator equations of motion may be obtained in the same way as in section 4.3 without any complications, and are

$$\begin{cases} \bar{\partial}c(z) = \bar{\partial}b(z) = 0\,, \\ \bar{\partial}b(z)c(z') = 2\pi\delta^2(z-z',\bar{z}-\bar{z}')\,, \\ \bar{\partial}b(z)b(z') = \bar{\partial}c(z)c(z') = 0\,. \end{cases} \tag{5.5}$$

Since we have not yet fixed j and therefore do not know whether we have a well-defined mode expansion, we define normal ordering by requiering that normal ordered objects behave classically. Recalling that $\bar{\partial}z^{-1} = \partial\bar{z}^{-1} = 2\pi\delta^2(z,\bar{z})$, we find that the normal ordered product $:bc:$ must read

$$:b(z)c(z'): = b(z)c(z') - (z-z')^{-1}\,. \tag{5.6}$$

Again, we may turn this around to identifiy the sinigular part of the corresponding OPE. Combinatorically, normal ordering for the ghost system is much the same as for the free scalar field, i.e. goes with Wick's theorem, except that interchanging two fields may result in sign flips. Therefore, when contracting two fields, one should first anti-commute them until they are next to each other, where each anti-commutation flips the sign. We thus obtain the following OPEs, where $x \sim y$ means that x is equal to y upto regular terms:

$$\begin{array}{ll} b(z)c(w) \sim \dfrac{1}{z-w}\,, & c(z)b(w) \sim \dfrac{1}{z-w}\,, \\ b(z)b(w) = \mathcal{O}(z-w)\,, & c(z)c(w) = \mathcal{O}(z-w)\,. \end{array} \tag{5.7}$$

Note that there are two sign flips in the second OPE, one from anti-commuting, and one due to $z \leftrightarrow w$. The last both OPEs are actually not only holomorphic, but they have a zero due to anti-symmetry (Pauli principle: expectation values with two identical fermions at the same place must vanish).

The stress energy tensor is obtained via Noether's theorem with respect to world sheet transformations $\delta z = \varepsilon(z)$, under which $\delta b = (\varepsilon\partial + j(\partial\varepsilon))b$ and $\delta c = (\varepsilon\partial + (1-j)(\partial\varepsilon))c$, such that

$$T(z) = (1-j):(\partial b)c: - j:b(\partial c):\,, \quad \bar{T}(\bar{z}) = 0\,. \tag{5.8}$$

The interested reader should work our the OPE of $T(z)$ with the fields $b(w)$ and $c(w)$ to verify that they have the expected form (3.41). Also, the OPE of $T(z)$ with $T(w)$ is not hard to work out, it has the standard form (3.42) and it reveals the conformal anomaly to be

$$c = c_{bc} = -2(6j^2 - 6j + 1) < 0 \;\text{ for }\; j \in \mathbb{R} - [\tfrac{1}{2}(1-\tfrac{1}{\sqrt{3}}), \tfrac{1}{2}(1+\tfrac{1}{\sqrt{3}})]\,, \tag{5.9}$$

which is clearly negative for all (half)-integer j except $j = \frac{1}{2}$. Obviously, this CFT is purely holomorphic (or actually meromorphic). Of course, there exists a completely analogous anti-holomorphic CFT with action $S = \frac{1}{2\pi}\int d^2z\bar{b}\partial\bar{c}$. But as it stands, this is a theory which is completely left-chiral, the right-chiral part being the trivial CFT with $c = 0$.

The bc system admitts a *ghost number* symmetry $\delta b = -i\varepsilon b$, $\delta c = i\varepsilon c$. It stems from the global $U(1)$ symmetry of the action under the transformation $b(z) \mapsto \exp(-i\alpha(z))b(z)$, $c(z) \mapsto \exp(i\alpha(z))c(z)$ for arbitrary holomorphic $\alpha(z)$. The corresponding Noether current is simply $j(z) = -{:}bc{:}(z)$. Thus we may expect to have a quantum number with respect to the corresponding conaserved Noether charge, the ghost number. Again, it is defined for the left-chiral sector, and an analogous definition holds for the right-chiral sector, both being separately conserved. If one computes the OPE of T with j, one finds that

$$T(z)j(w) \sim \frac{1-2j}{(z-w)^3} + \frac{1}{(z-w)^2}j(w) + \frac{1}{(z-w)}\partial_w j(w)\,, \tag{5.10}$$

meaning that $j(w)$ is not a primary conformal field. Under conformal mappings, $j(w)$ thus transforms as

$$\delta j(w) = \left(-\varepsilon(w)\partial_w - (\partial_w\varepsilon(w)) + \tfrac{1}{2}(2j-1)\partial_w^2\right) j(w)\,. \tag{5.11}$$

One particular case is $j = 1-j$, i.e. $j = \frac{1}{2}$. The central charge (5.9) is then $c = 1$. It is customary, to use the notion $b = \psi$, $c = \bar{\psi}$ in this case. It is then easy to see that this CFT can be split into two identical copies by writing $\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$ and $\bar{\psi} = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2)$, such that

$$S = \tfrac{1}{4\pi}\int d^2z\left(\psi_1\bar{\partial}\psi_1 + \psi_2\bar{\partial}\psi_2\right)\,, \tag{5.12}$$

$$T = -\tfrac{1}{2}\left(\psi_1\partial\psi_1 + \psi_2\partial\psi_2\right)\,. \tag{5.13}$$

Each of the ψ_i theories has central charge $c = \frac{1}{2}$, and can be recognized as the CFT of a free fermion. This theory corresponds to the case $m = 3$ in (3.31) and is the first non-trivial example of a so-called *minimal model*, which are CFTs with only finitely many Virasoro conformal families (primaries with all their descendants). It will not concern us further, but it should at least be noted that it possesses only three primary fields of conformal weights $h_{1,1} = h_{2,3} = 0$, $h_{1,2} = h_{2,2} = \frac{1}{16}$, and $h_{2,1} = h_{1,3} = \frac{1}{2}$ according to (3.32), which prefectly coincides with the two order parameters of the two-dimensional Ising model (plus the identity), the spin σ and the energy ϵ, and their critical exponents. Another important value is $j = 2$, for which we get $c_{bc} = -26$, and which is important in bosonic string theory.

5.1 $\beta\gamma$ systems

We can redo everything from the last section with a pair of fields $\beta(z)$ and $\gamma(z)$, which behave exactly as in the bc system except that they are *commuting*. The only differences are that the OPEs now read $\beta(z)\gamma(w) \sim -(z-w)^{-1}$ and $\gamma(z)\beta(w) \sim (z-w)^{-1}$. Note the differing sign. The stress energy tensor looks exactly as in the bc system, but the different sign under commutation yields now the central charge $c_{\beta\gamma} = 2(6j^2 - 6j + 1) = -c_{bc}$. The system with $j = \frac{3}{2}$ has $c_{\beta\gamma} = 11$ is important for the superstring.

5.2 Mode expansions

We will assume for now that $j \in \mathbb{Z}$. Then we have well-defined mode expansions (5.2), i.e.

$$b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-j}\,, \quad c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n-(1-j)}\,. \tag{5.14}$$

The anti-commutators can be obtained from the OPE, and turn out to be $\{b_m, c_n\} = \delta_{m+n,0}$ with all other anti-commutators vanishing. It seems sensible to impose highest-weight conditions, and to consider states which are annihilated by all modes b_n and $c_{n'}$ with $n, n' > 0$. But what about the zero modes? It turns out that we have now pairs $|+\rangle$, $|-\rangle$ of highest-weight states with the properties

$$\begin{cases} b_0|-\rangle = 0\,, & b_0|+\rangle = |-\rangle\,, \\ c_0|-\rangle = |+\rangle\,, & c_0|+\rangle = 0\,, \\ b_n|-\rangle = b_n|+\rangle = c_n|-\rangle = c_n|+\rangle\,, & n > 0\,. \end{cases} \tag{5.15}$$

We may construct Verma modules on these highest-weight states by acting with the modes b_{-n} and $c_{-n'}$ with $n > 0$. We now have to fix notation by convention, saying that b_0 be an annihilator, and that c_0 be a creator. This singles out $|-\rangle$ as the ghost vacuum $|0\rangle^{(-)}$. Note, however, that for consistency we must require that ${}^{(+)}\langle 0| = {}^{(-)}\langle 0|c_0$ be the correct out-vacuum such that ${}^{(+)}\langle 0|0\rangle^{(-)} = 1$. In this way we guarantee that the conditions defining the in-vacuum $|0\rangle^{(-)}$ are dual to those defining the out-vacuum ${}^{(+)}\langle 0|$. However, this is a further example for the situation that the "metric on field space", the two-point structure constants $\langle\alpha|\beta\rangle = D_{\alpha\beta}$ is not diagonal.

Let us now introduce a grading or particle number operator, the ghost number operator N_g for the resulting Fock space. We define its action on the vacua as $N_g|0\rangle^{(\mp)} = \mp\frac{1}{2}|0\rangle^{(\mp)}$, and further define that it counts the modes as $N_g(b_n) = -b_n$ and $N_g(c_n) = +c_n$. This definition is cooked up in such a way that the scalar product (5.1) is non-vanishing only if the total ghost number is zero. For instance, ${}^{(-)}\langle 0|0\rangle^{(-)} = 0$ since the total ghost number is $N_g = -1$. Indeed, $|0\rangle^{(-)} = b_0|0\rangle^{(+)}$, and since $b_0^\dagger = b_0$, we see that ${}^{(-)}\langle 0|b_0 = 0$.

Next, we consider the mode expansion of $T(z)$. Since the stress energy tensor is made up from the bc system, its Virasoro modes will have the form

$$L_m \propto \sum_{n\in\mathbb{Z}} (mj - n){:}b_n c_{m-n}{:} + \delta_{m,0}\mathcal{N}_{bc}\,, \tag{5.16}$$

where there might be an additional term due to normal ordering, which can only be a constant since the anti-commutators are c-numbers. Note that this is mode normal ordering, i.e. normal ordering of creation operators left to annihilation operators, which should not be confused with field normal ordering. The constant $\mathcal{N}_{bc}$ is easily computed by checking the consistency condition that

$$2L_0|-\rangle = [L_1, L_{-1}]|-\rangle = (jb_0c_1)((1-j)b_{-1}c_0)|-\rangle = j(1-j)|-\rangle \stackrel{!}{=} 0\,. \tag{5.17}$$

Thus, we learn that $\mathcal{N}_{bc} = \frac{1}{2}j(1-j)$ amd hence

$$L_m = \sum_{n\in\mathbb{Z}}(mj-n){:}b_n c_{m-n}{:} + \tfrac{1}{2}j(1-j)\delta_{m,0}\,. \tag{5.18}$$

The non-vanishing constant $\mathcal{N}_{bc}$ hints at the fact that mode normal ordering and field normal ordering are not identical in the ghost system. One can show that the difference amounts to

$$({:}b(z)c(z'){:})_{\text{field ordering}} - ({:}b(z)c(z'){:})_{\text{mode ordering}} = \frac{1}{(z-z')}\left(\left(\frac{z}{z'}\right)^{1-j} - 1\right)\,. \tag{5.19}$$

The reader should convince herself that the corresponding ordering constant $\mathcal{N}_\phi$ in the free bosonic CFT is zero, i.e. that the Virasoro modes are given simply by

$$L_m = \tfrac{1}{2}\sum_{n\in\mathbb{Z}}{:}a_{m-n}a_n{:} \tag{5.20}$$

without an additional term $\mathcal{N}_\phi\delta_{n,0}$. This can be done in complete analogy to the ghost system, i.e. by checking that $L_0|0\rangle = \frac{1}{2}[L_1, L_{-1}]|0\rangle = 0$. The fact that there is no ordering constant is coincident with the fact that mode normal ordering and field normal ordering are equivalent for the free bosonic theory.

Let us return to the ghost number current $j = -{:}bc{:}$ with its charge

$$N_g = \tfrac{1}{2\pi\mathrm{i}}\int_0^{2\pi}\mathrm{d}w\, j_{\text{cyl}}(w) = \sum_{n>0}(c_{-n}b_n - b_{-n}c_n) + c_0 b_0 - \frac{1}{2}\,, \tag{5.21}$$

which indeed satisfies $[N_g, b_n] = -b_m$ and $[N_g, c_n] = +c_n$. It therefore counts the number of c excitations minus the number of b excitations of a given state. The constant is necessary to reproduce our definition of the action of N_g on the ground states $N_g|\mp\rangle = \mp\frac{1}{2}|\mp\rangle$.

Note that we have defined the ghost number for the physically relevant cylinder (the string world-sheet). Since the ghost current is not a primary field, the translation to the complex plane has to be performed carefully. Recalling that $z = \mathrm{e}^w$, we find

$$(\partial_z w)j_{\text{cyl}}(w) = j(z) + (j-\tfrac{1}{2})(\partial_z^2 w)/(\partial_z w) = j(z) + (j-\tfrac{1}{2})z^{-1}\,. \tag{5.22}$$

This is quite similar to the effect that the zero mode of the Virasoro algebra, L_0, receives a shift by $-c/24$ when we map the theory from the cylinder to the complex plane. Thus, the ghost number also receives a shift, namely $N_{g,\text{plane}} = \oint \mathrm{d}z\, j(z) = N_g + Q_j$ with $Q_j = j - \frac{1}{2}$. The above definitions led to unusual vacuum states, which are *not* the $SL(2,\mathbb{C})$-invariant vacua introduced earlier. This disadvantage is the price paid for treating the ghost system in a way where ordering prescriptions are more or less independent of the spin j of the system.

5.3 Ghost number and zero modes

The above approach is sometimes not useful, especially if a particular ghost system is considered. Then, it is more natural to use the $SL(2,\mathbb{C})$-invariant vacuum. Let us now be specific and put $j = 2$. For this value, the bc system thus consists out of a spin-two field and a vector field, and has central charge $c = -26$. The string lectures will tell us, that this ghost system is particularly important for the bosonic string.

The mode expansions read in this specific case simply

$$b(z) = \sum_n b_n z^{-n-2}\,, \quad c(z) = \sum_n c_n z^{-n+1}\,. \tag{5.23}$$

We now wish to reproduce the canonical field normal ordering by a mode normal ordering prescription. The natural way to do this for a chiral local field $\Phi_h(z)$, $2h \in \mathbb{Z}$, with mode expansion $\Phi_h(z) = \sum_n \phi_n z^{-n-h}$ is to call all modes with $n > -h$ annihilators, and all other modes creators, i.e. by imposing highest weight conditions $\phi_n|0\rangle = 0$ for $n > -h$. In our example, we thus would like to impose

$$b_n|0\rangle = 0 \ \forall\, n \geq -1\,, \quad c_n|0\rangle = 0 \ \forall\, n \geq 2\,. \tag{5.24}$$

In this way, the vacuum $|0\rangle$ is indeed the $SL(2,\mathbb{C})$-invariant vacuum. The corresponding conditions for the out-vacuum then read

$$\langle 0|b_{-n} = 0 \ \forall\, n \geq -1\,, \quad \langle 0|c_{-n} = 0 \ \forall\, n \geq 2\,. \tag{5.25}$$

But now, we have to keep in mind that the modes b_{-n} are conjugate to the modes c_n, since we have the canonical commutation relations $\{b_n, c_m\} = \delta_{n+m,0}$. Both highest-weight conditions together tell us that the three modes b_{-1}, b_0, b_0 are annihilators in both directions, i.e. they annihilate to the right as well as to the left. On the other hand, the three modes c_{-1}, c_0, c_1 are creators in both directions, i.e. they neither annihilate to the right nor to the left.

As a consequence, we find that $\langle 0|0\rangle = \langle 0|\{b_0, c_0\}|0\rangle = 0$. Even more strangely, also $\langle 0|c_i|0\rangle = 0$ for $i \in \{-1,0,1\}$. In fact, the first non-vanishing expression is $\langle 0|c_{-1}c_0c_1|0\rangle$, i.e. we need at least three c-modes. One sees this by inserting a one in the form $1 = \{b_i, c_{-i}\}$ for $i \in \{-1,0,1\}$. For example, $\langle 0|c_0c_1|0\rangle = \langle 0|\{b_1, c_{-1}\}c_0c_1|0\rangle = 0$. Of course, this does not anylonger work for the correlator $\langle 0|c_{-1}c_0c_1|0\rangle$, since we are forced to insert the one as $1 = \{b_n, c_{-n}\}$ with $n > 1$, which does not annihilate anymore. The three c-modes are necessary to eat up the three zero modes of the field $b(z)$. One might hide them in a redefinition of the out-vacuum as $\langle\tilde{0}| = \langle 0|c_{-1}c_0c_1$ such that $\langle\tilde{0}|0\rangle = 1$.

We therefore find that the ghost system correlators can only be non-zero, if the total ghost number, i.e. the number of c-fields minus the number of b-fields is exactly three, $N_g = \#c - \#b = 3$. The reader should note that this differs from our discussion in the preceeding section, since we made a different choice of vacuum. The vacuum used now is the physical vacuum.

5.4 Correlation functions

The above discussion can immediately applied to calculate correlation functions of the bc ghost system. We already know that, for instance, $\langle c(z)c(w)\rangle = 0$. The first non-trivial correlator is

$$\begin{aligned}\langle c(z_1)c(z_2)c(z_3)\rangle &= \langle 0|\sum_n\sum_m c_{-n}z_1^{+n+1}c_{n-m}z_2^{-(n-m)+1}c_m z_3^{-m+1}|0\rangle \\ &= \sum_{n\leq -1}\sum_{m\leq 1}\langle 0|c_{-n}z_1^{+n+1}c_{n-m}z_2^{-(n-m)+1}c_m z_3^{-m+1}|0\rangle\,, \end{aligned} \tag{5.26}$$

where we inserted the mode expansion and used the highest-weight condition of the vacuum states. There are only two summations here, since the total level (with respect to the L_0 grading) must be zero, which fixes the mode of the third field, if the modes of the other two fields are given. Since all the modes c_k anti-commute with each other, it is easy to see that the only non-vanishing choices are $m,n\in\{-1,0,1\}$. This leads to the six terms

$$\begin{aligned}\langle 0|\,\big(c_{-1}c_1c_0z_1^2z_3 \;&+\; c_{-1}c_0c_1z_1^2z_2 + c_0c_{-1}c_1z_1z_2^2 \\ &+\; c_0c_1c_{-1}z_1z_3^2 + c_1c_{-1}c_0z_2^2z_3 + c_1c_0c_{-1}z_2z_3^2\big)\,|0\rangle \\ &= \langle 0|c_{-1}c_0c_0\left(-z_1^2z_3 + z_1^2z_2 - z_1z_2^2 + z_1z_3^2 + z_2^2z_3 - z_2z_3^2\right)|0\rangle\,, \end{aligned} \tag{5.27}$$

where the signs come from anti-commuting the modes. Collecting terms results in the simple expression

$$\langle c(z_1)c(z_2)c(z_3)\rangle = (z_1-z_2)(z_1-z_3)(z_2-z_3)\,, \tag{5.28}$$

which indeed satisfies the Pauli principle. In the same manner, all correlation functions can be obtained. Firstly, it is clear that an arbitrary correlation function must have first order zeroes for each pair of coordinates, where two c-fields coinicde. The same is true for each pair of coordinates, where two b-fields approach each other. Only when a c-field approaches a b-field, the singular OPE (5.7) will lead to a first order pole. The only non-trivial feature is that the number of c-fields must exceed the number of b-fields by precisely three. Thus, in all generality we find

$$\langle 0|\prod_{i=1}^{p}c(z_i)\prod_{j=1}^{q}b(w_j)|0\rangle = \prod_{i<i'}(z_i-z_{i'})\prod_{j<j'}(w_j-w_{j'})\prod_{i,j}(z_i-w_j)^{-1}\delta_{p,q+3}\,. \tag{5.29}$$

6. N=1 supersymmetric CFT

The bosonic string, although a very nice toy model, is not very suitable to describe the physics of our universe. This is partially due to the existence of the unphysical tachyons. However, there exist other string theories, which make use of the principle of supersymmetry to arrive at a particle spectrum which is closer to phenomenology. The principles and basics of supersymmetry will be explained in Jan Plefkas lectures, and are therefore not to be found here.

Supersymmetry can be introduced to the string in two different ways, namely as supersymmetry in the target space (the space-time we live in), or supersymmetry on the world sheet, spanned by the moving string. CFT lives on this world sheet, and hence string people consider supersymmetric CFTs. We will collect here a few basics on the simplest supersymmetric CFT.

6.1 Fermionic Currents

We already might have developed a feeling that integer conformal weights are something special, since fields with integer conformal weights are local and single valued seperately in their holomorphic as well as their anti-holomorphic part. In fact, we can take only one such part and still have a perfectly well behaved conformal field. An example for this is the stress-energy tensor, which has scaling dimension two (although it is not a conformal field, but only quasi-primary). Let us assume that we have such a *chiral* field $J(z)$ with conformal weight $h \in \mathbb{Z}_+$. Such a field has a mode expansion

$$J(z) = \sum_r z^{-r-h} J_r \,, \quad J_r = \oint \mathrm{d}z z^{r+h-1} J(z) \,. \tag{6.1}$$

Since the field is supposed to be conformal (or primary), its OPE with the stressr-energy tensor is of the form (3.41). Extracting the commutator of the modes J_r with the Virasoro generators from it yields

$$[L_n, J_r] = (h(h-1) - r) J_r \,. \tag{6.2}$$

It follows that acting with J_r om a state decreases the conformal weight of this state by r. Thus, the modes respect the L_0 grading, and can be added to the symmetry algebra. If the commutators $[J_r, J_s]$ are known, they, the Virasoro algebra and (6.2) form a so-called *extended chiral symmetry algebra*, and $J(z)$ is called a *current*.

Physicists tend to treat fermions on the same footing as bosons, in particular when supersymmetry is involved. It is therefore desirable to check whether we can relax the conditions for chiral fields a bit and allow the conformal weight h to be half-integer. We immediately are then faced with the question over which range the mode index r should run. Two possibilities easily come to mind, namely $r \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$, i.e. r integer or r half-integer. Plugging this into (6.1), we see that under $z \mapsto \mathrm{e}^{2\pi\mathrm{i}} z$, the current $J(z)$ transforms in the following way:

$$J(\mathrm{e}^{2\pi\mathrm{i}} z) = \sum_{r \in \mathbb{Z} + \epsilon\frac{1}{2}} (\mathrm{e}^{2\pi\mathrm{i}} z)^{-r-h} J_r = -(-)^\epsilon J(z) \tag{6.3}$$

for $\epsilon = 0, 1$ and h half-integer. Therefore, half-integer modes lead to periodic boundary conditions, while integer modes yield anti-periodic boundary conditions. Anti-periodic boundary conditions mean that the field introduces a branch cut of order two in the complex plane. Remembering that we originally come from the cylinder as the string world sheet, we see that the monodromy refers to the boundary conditions with respect to our compactified space coordinate σ.

However, the reader should keep the following in mind: If we go back from the complex plane to the cylinder via a conformal transformation (remember that $z = \mathrm{e}^w$), we aquire a factor $(\frac{\partial \log(z)}{\partial z})^{-h}$ such that

$$J_{\mathrm{cyl}}(w) = z^h J(z) \,. \tag{6.4}$$

If the conformal weight h is half-integer, we learn that the periodicity changes. Thus, it is better to call the choice of the modes by a name in order to avoid confusion when talking about periodicity. Half-integer modes define the so-calle *Neveu-Schwarz* sector of the CFT, while integer modes define the *Ramond* sector of the theory. The following picture emerges:

	Modes	Plane $\mathbb{C}$	Cylinder $\mathbb{R} \times S^1$
Neveu-Schwarz	$r \in \mathbb{Z} + \frac{1}{2}$	periodic	anti-periodic
Ramond	$r \in \mathbb{Z}$	anti-periodic	periodic

6.2 The N=1 Algebra

The simplest supersymmetric CFT is generated by the $N = 1$ superconformal algebra. This algebra consists of the Virasoro generators of the stress-energy tensor $T(z)$, and an additional supersymmetric partner $T_F(z)$ to it, which has conformal weight $h = \frac{3}{2}$. Together, they form a closed operator algebra, meaning that the singular parts of their OPEs only involve (derivatives of) the fields T and T_F. The complete set of OPEs reads

$$\begin{aligned} T(z)T(w) &\sim \frac{c/2}{(z-w)^4}1\!\!1 + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w)\,, \\ T(z)T_F(w) &\sim \frac{3/2}{(z-w)^2}T_F(w) + \frac{1}{z-w}\partial T_F(w)\,, \\ T_F(z)T_F(w) &\sim \frac{c/6}{(z-w)^3}1\!\!1 + \frac{1/2}{z-w}\partial T(w)\,. \end{aligned} \tag{6.5}$$

Note that T_F is indeed a primary field of weight $\frac{3}{2}$ with respect to the stress-energy tensor. Supersymmetry associates to each bosonic field a fermionic partner and vice versa. The lectures on supersymmetry have explained how field theories and in particular their actions can be written down in a manifestly supersymmetric way with the help of Grassmann variables θ, $\theta^2 = 0$. It follows from simple dimensional reasoning that the scaling dimension of θ is $\frac{1}{2}$ such that the scaling dimensions of a field and its super-partner differ by $\frac{1}{2}$. Introducing $\mathcal{T}(z,\theta) = T(z) + \theta T_F(z)$ as the full super-field, the above three OPEs could be collected in one manifestly supersymmetric OPE. From the above OPEs, the algebra of the commutators of the modes can be extracted. The modes of T_F are traditionally called G_r where according to our above discussion r may be integer or half-integer. The algebra also closes within the set of modes L_n and G_r, and reads

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \tfrac{1}{12}\hat{c}(n^3 - n)\delta_{n+m,0}\,, \\ [L_n, G_r] &= (\tfrac{1}{2}n - r)G_{n+r}\,, \\ \{G_r, G_s\} &= 2L_{r+s} + \tfrac{1}{3}\hat{c}(r^2 - \tfrac{1}{4})\delta_{r+s,0}\,. \end{aligned} \tag{6.6}$$

The last line involves an anti-commutator, because the corresponding OPE is odd under the exchange $z \leftrightarrow w$. Of course, this is as it should be, since T_f is a fermionic field.

Another noteworthy feature is that the field T_F has a zero mode in the Ramond sector, G_0 with $\{G_0, G_0\} = 2L_0 - \frac{1}{12}\hat{c}$. This implies immediately, that a state annihilated by G_0 must have conformal weight $h = \frac{c}{24}$, i.e. $G_0|h\rangle_R = 0 \Longleftrightarrow h = \frac{c}{24}$. On the other hand, if a state is not annihilated by G_0, then we find with $G_0|h\rangle_R = g|h\rangle_R$ that $\{G_0, G_0\}|h\rangle_R = 2g^2|h\rangle_R = (2L_0 - \frac{c}{12})|h\rangle_R$ which implies $h = \frac{c}{24} + g^2 > \frac{c}{24}$. Thus, in the Ramond sector states not annihilated by G_0 appear in pairs of opposite fermion number, i.e. $|h-\rangle_R$ and $|h+\rangle_R = G_0|h-\rangle_R$. They have the same conformal weigth, sine $[L_0, G_0] = 0$.

Considering now the Neveu-Schwarz sector, one should look at the particular anti-commutation relation

$$\{G_r, G_{-r}\} = 2L_0 + \frac{1}{3}\hat{c}(r^2 - \frac{1}{4})\,. \tag{6.7}$$

Since r is half-integer, we always have $(r^2 - \frac{1}{4}) \geq 0$. Thus, in a unitary theory with $c > 0$ and $h \geq 0$, the left-hand side is positive or zero, the latter occuring for $r = \frac{1}{2}$ and $h = 0$ only. If the left-hand side is positive, we see that $|G_{-r}|h\rangle|^2 > 0$ for ground states, meaning that excitations of ground states have positive norm. There is a unique ground state which is annihilated by $G_{-\frac{1}{2}}$, namely the vacuum. Of course, all highest-weight states are annihilated by modes G_r with $r \geq \frac{1}{2}$.

7. Modular Invariance

So far, we have considered CFT on the simplest possible worldsheet, the cylinder, which we have mapped by a conformal transformation to the punctured complex plane. In string theory, the cylinder is the world sheet of one freely moving non-interacting closed string. Interaction of several strings, as will be explained in the string lectures, yields world sheets which might be any Riemann surface. It is intuitive to use the genus of the Riemann surface as an order count, since it directly corresponds to the loop order of the Feynmann diagram of the low-energy effective field theory, where the extent of the string becomes invisible. So, to zero-th order, we have a Riemann sphere with a number of tubes attached, one for each string which interacts with the others. To first order, we find a torus, again with a number of tubes attached, and so on.

The tubes of the incoming and outgoing strings, if these are considered to be otherwise non-interacting, can be thought of asymptotically as infinitely long and infinitely thin spikes. In effect, these tubes can be replaced by punctures of the Riemann surface, where an appropriate vertex operator carrying the right momentum and quantum numbers is placed. What remains is the non-trivial topology of the Riemann surface.

Up to now, we have described a CFT algebraically by a set of highest-weight states $|h, \bar{h}\rangle = \Phi_{h,\bar{h}}(0,0)|0\rangle$, on which the left- and right chiral Virasoro algebra acts. The question which naturally arises is which combinations of such ground states actually occur in the CFT. If we know this, we have a complete characterization of the physical states in the theory, namely all the admissible ground states plus all their descendants created by the generators of the Virasoro algebras, minus all null states.

Crossing symmetry, or equivalently duality, has already given us some constraints, but these were constraints for the complex plane only. Do different Riemann surfaces yield different constraints? And is it possible to have a theory consistent on any arbitrary Riemann surface? The answer to both questions is yes, and we will sketch a bit of the answer in the following. As a general result, one can show for a large class of CFTs that crossing symmetry of correlators on the complex plane and modular invariance of the partition function on the torus is sufficient to make the theory consistent on arbitrary Riemann surfaces. This is one of the motivations why modular invariance on the torus is often considered to be a fundamental requirement for CFT.

Interestingly, also condensed matter physicists are very fond of modular invariance. To understand this, first note that we usually consider CFTs in complex variables and, thus, automatically as Euclidean field theory. Time is then commonly interpreted as temperature, and partition functions are well defined objects. Since CFT in string theory is often considered in its Euclidean from, the following motivation is also helpful for the understanding of modular invariance in string theory. Now, let us conformally map the complex plane (with variable z) with the origin deleted onto a strip of width L (with variable u). This map is given by the exponential $z = \exp(2\pi \mathrm{i} u/L)$. It is a well known technique in statistical physics to consider the system on a periodic strip, here with width L, and to introduce the transfer matrix

$$\mathcal{T} = \exp\left\{-\frac{2\pi}{L}(L_0 + \bar{L}_0 - \frac{c}{12})\right\} .$$

Here $L_0 + \bar{L}_0$ serves as Hamiltonian, since this linear combination generates time translations.[1] The additional term involving the central charge comes from the used conformal map. This map is not one-to-one, and introduces a conformal anomaly. The reader might convince herself first that the stress energy tensor on the strip is related to the one on the plane via

$$T_{\mathrm{strip}}(u) = -(2\pi/L)^2 \left[T_{\mathrm{plane}}(z)z^2 - \frac{1}{24}c\right] ,$$

and then that with $\langle T_{\mathrm{plane}}(z)\rangle = 0$ one must have $\langle T_{\mathrm{strip}}(u)\rangle = \frac{1}{24}c(2\pi/L)^2$. Hence, the above mentioned shift in the transfer matrix.

The OPE of the stress energy tensor with itself tells us how the stress energy tensor reacts to conformal transformations. It is not an entirely trivial task to explicitly work out the transformation of $T(z)$, but the result can be cast in the formula

$$T(z)\mathrm{d}z^2 = T'(z')\mathrm{d}z'^2 + \frac{c}{12}\{z', z\}\mathrm{d}z^2 ,$$

where the so-called Schwarzian derivative of the map $z \mapsto z' = f(z)$ is defined as

$$\{z', z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

The conformal anomaly mentioned above can now be computed easily by making use of the just given transformation law of T for $f(z) = -\mathrm{i}\frac{L}{2\pi}\log(z)$.

[1] The reader should take care that $L_0 + \bar{L}_0$, considered on the z-plane, generates dilatations. Only in the u-strip does it generate time translations.

We may now further confine the system to a box of size L, M, with periodic boundary conditions on both sides. Then the partition function of such a system reads

$$Z = Z(L, M) = \mathrm{tr}\ \exp\left\{-2\pi\frac{M}{L}(L_0 + \bar{L}_0 - \frac{c}{12})\right\}. \tag{7.1}$$

A box with periodic boundary conditions has the topology of a torus. The central observation is now that, since we deal with a Euclidean theory, space and time are completely symmetric to each other. It follows that in such a framework a physical sensible partition function should satisfy $Z(L, M) = Z(M, L)$.

More generally, one could consider a periodicity, where a time translation by M is always accompanied by a space translation, generated by $\mathrm{i}(L_0 - \bar{L}_0)$.[2] Let us assume that this addition space translation is by N. Then the partition function would read

$$Z = Z(L, M, N) = \mathrm{tr}\ \exp\left\{-2\pi\frac{M}{L}(L_0 + \bar{L}_0 - \frac{c}{12}) + 2\pi\mathrm{i}\frac{N}{L}(L_0 - \bar{L}_0)\right\}.$$

Introducing complex numbers $\omega_1 = L$, $\omega_2 = N + \mathrm{i}M$, $\tau = \omega_2/\omega_1$, one can rewrite this with $q = \exp(2\pi\mathrm{i}\tau)$ and $\bar{q} = \exp(-2\pi\mathrm{i}\bar{\tau})$ elegantly as

$$Z(\tau, \bar{\tau}) = \mathrm{tr}\left(q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - c/24}\right).$$

7.1 The Moduli space of the torus

As a general rule of thumb, one usually assumes that all states in a theory contribute to loop diagrams. This may be seen as a motivation, why we expect that it is useful to study CFT on the simplest loop diagram, the torus. Essentially, a torus is a cylinder whose ends have been sewn together. Mathematically, it is usually described as the complex plane modulo a lattice. Let the lattice be spanned by two basic lattice vectors, ω_1 and ω_2. Then two points z, z' in the complex plane are identified with each other, if there exist two integers n_1, n_2 such that $z' = z + n_1\omega_1 + n_2\omega_2$. Since the overall size and orientation of the torus shouldn't matter (due to global scaling, translational and rotational invariance of the CFT), we may choose more conveniently one of the base lattice vectors to lie on the real axis with length one, starting at the origin, and the other can without loss of generality be taken to lie in the upper half plane, $\tau \sim \omega_2/\omega_1$, $\Im\mathrm{m}\tau > 0$. In effect, the entire lattice is described by one complex number $\tau \in \mathbb{H}$.

The key observation is now that the lattice, and consequently the torus, does not change at all if we replace τ by $\tau + 1$, since this spans the same lattice. Such a transformation is called unimodular. In the same manner, the lattice does not change if we replace τ by $1/\tau$, where we implicitly have to rescale the lattice, though (the overall since of the torus is irrelevant). Since $\tau \sim \omega_2/\omega_1$, we see that $-1/\tau$ basically interchanges the role of ω_2 and ω_1. The group spanned by these transformations $T : \tau \mapsto \tau + 1$, $S : \tau \mapsto -\frac{1}{\tau}$ is called the modular group $PSL(2, \mathbb{Z})$ and is the set of all 2×2 matrices $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathbb{Z}$ and $\det M = ad - bc = +1$. The action of this group on τ is given by $M(\tau) = \frac{a\tau + b}{c\tau + d}$ which explains why we restrict the sign of the determinant and identify matrices $\pm M$ with each other (this is what the P stands for: $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$).

[2]On the z-plane, $\mathrm{i}(L_0 - \bar{L}_0)$ generates rotations.

Since the torus does not really change under a $PSL(2,\mathbb{Z})$ transformation of its modulus τ, we should expect that a physical sensible theory does not change under such a transformation either, as we have motivated in the preceeding section. Thus we impose as a condition on our (L)CFT that its partition function be modular invariant. In the following, we often use the variables $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ and $\bar{q} = \mathrm{e}^{-2\pi\mathrm{i}\bar{\tau}}$ instead of τ and $\bar{\tau}$. A series expansion in $q, \bar{q}$ is then an expansion around the point $\tau = +\mathrm{i}\infty$, i.e. where the torus is considered in the extreme case where it is more like a cylinder.

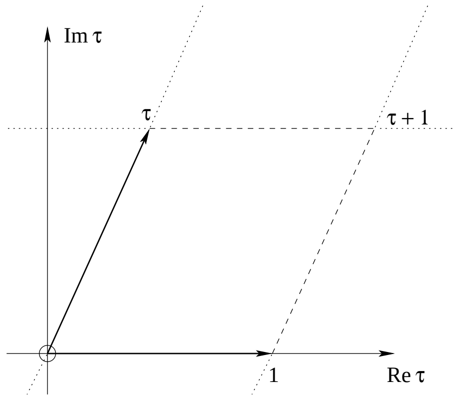

Figure 8: The upper half plane and the modular parameter τ defining a lattice, i.e. torus.

We so far have made elaborate use of the fact that much in conformal field theory can be considered separately for holomorphic and anti-holomorphic fields, or left-chiral and right-chiral fields, respectively. Although one of the not so nice features of LCFT is that correlation functions do not any longer factorize into holomorphic and anti-holomorphic parts, we still can consider most entities in factorized form, as long as we do not impose the physical constraint that observables should be single-valued. This is particularly true for the representation theory of the CFT under consideration. We call a CFT rational, if it has only finitely many highest-weight representations. Then, as Cardy observed a long time ago, the partition function of such a rational CFT can be written as a sesqui-linear form over the characters of these representations. Thus, denoting the finite set of representations by $\mathcal{R}$, the partition function takes the form

$$Z(\tau,\bar{\tau}) = \sum_{h,\bar{h}\in\mathcal{R}} N_{h\bar{h}}\chi_h(\tau)\chi^*_{\bar{h}}(\tau)\,, \tag{7.2}$$

where $N_{h\bar{h}}$ is a certain matrix with non-negative integer entries. Here, the character of the highest-weight representation $M_{h,c}$ is defined as usual,

$$\chi_h(\tau) = \mathrm{tr}_{M_{h,c}} q^{L_0 - c/24}\,, \tag{7.3}$$

and analogously for $\chi^*_{\bar{h}}(\tau)$.

Since the partition function is modular invariant, the characters from which it is built must transform covariantly under the modular group. Therefore, in the present setting of a rational theory, i.e. $|\mathcal{R}| < \infty$, they form a finite-dimensional representation of the modular group. As a consequence, the transformations $S : \tau \mapsto -1/\tau$ and $T : \tau \mapsto \tau + 1$ are represented as matrices acting on the characters, that is,

$$\chi_h(-\frac{1}{\tau}) = \sum_{h'\in\mathcal{R}} S_h^{\ h'}\chi_{h'}(\tau)\,, \tag{7.4}$$

$$\chi_h(\tau+1) = \sum_{h'\in\mathcal{R}} T_h^{\ h'}\chi_{h'}(\tau)\,. \tag{7.5}$$

One of the most astonishing deep results in CFT is that the S-matrix fulfills a certain algebraic property, which on first glance seems to be pure magic. Eric Verlinde suggested namely, that the S-matrix also yields the so-called fusion rules, which essentially count the multiplicities of representations appearing on the right hand side of the fusion product of two representations. The latter is, in analytical terms, provided by the OPE, and might be thought of as some kind of tensor product algebraically. To ease notation, let us arbitrarily enumerate the weights $h \in \mathcal{R}$ as h_i, $i = 0, \ldots, |\mathcal{R}| - 1$ with the convention that h_0 refers to the vacuum representation. Then, the seminal so-called Verlinde formula reads

$$[h_i] * [h_j] = \sum_k N_{ij}{}^k [h_k] \quad \text{with} \quad N_{ij}{}^k = \sum_r \frac{S_i{}^r S_j{}^r (S^{-1})_r{}^k}{S_0{}^r} \,. \tag{7.6}$$

Although, the entries of the S-matrix may be very complicated algebraic numbers (made out of $\exp(2\pi\mathrm{i}\rho)$ expressions with ρ rational numbers), the $N_{ij}^{\ \ k}$ are always non-negative integers.

This brief tour through modular invariance just scratched at the surface of this concept. We would like to note that the fact that the characters of a rational CFT form a finite dimensional representation of the modular group $PSL(2,\mathbb{Z})$ is one of the motivations for the deep interest mathematicians take in CFT. It also provides one of the most restrictive structures underpinning a rational CFT. Therefore, much effort is poured into using modular invariance to classify all possible rational CFTs. This classification is an important task, since only with this knowledge are we able to make full use of this great toolkit of theoretical physics, which CFT constitutes.

8. Conclusion

These notes by no means provide a comprehensive introduction to the vast theme of conformal field theory. Many topics of great importance have been skipped completely, or mentioned only in a half-sentence. These notes pretty much consist of the material presented in the actual lectures, which were mainly designed to fill in the gaps left by – and to provide the bare necessities to follow – the main series of the *String Theory Crash Course* held fall 2000 at Hannover university. Therefore, the selection of covered material was made along the lines of this course. The main series of lectures on string theory itself by Olaf Lechtenfeld and the introduction to supersymmetry by Jan Plefka are also be availbale as written notes, see [6]. The nature of the course, to provide a preliminary survey of string theory in a very short time, is reflected in the incompleteness of these notes. The bibliography might help the reader to find some more comprehensive introductions to the subject. Again, also the bibliography does not attempt to be thorough in any sense, but is intended to list easily accessible reviews or books on conformal field theory and string theory.

References

[1] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, *Nucl. Phys.* **B 241** (1984) 333.

[2] P. Ginsparg, in Les Houches 1988 Summer School, E. Brézin and J. Zinn-Justin, eds. (1989) Elsevier, Amsterdam, [http://xxx.lanl.gov/hypertex/hyperlh88.tar.gz];
J. Cardy, in Les Houches 1988 Summer School, E. Brézin and J. Zinn-Justin, eds. (1989) Elsevier, Amsterdam;
C. Gomez, M. Ruiz-Altaba, Rivista Del Nuovo Cimento **16** (1993) 1–124.

[3] J. Fuchs, *Lectures on conformal field theory and Kac-Moody algebras*, to appear in Lecture Notes in Physics, Springer, [`hep-th/9702194`];
A.N. Schellekens *Conformal Field Theory*, Saalburg Summer School lecture notes (1995), [http://www.itp.uni-hannover.de/~lechtenf/Events/Lectures/schellekens.ps];
M. Gaberdiel, Rept. Prog. Phys. **63** (2000) 607-667, [`hep-th/9910156`];
S.V. Ketov, *Conformal Field Theory* (1995) World Scientific;
Ph. Di Francesco, P. Mathieu, D. Sénéchal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics (1997) Springer;
A.B. Zamolodchikov, Al.B. Zamolodchikov, *Conformal Field Theory and Critical Phenomena in Two-Dimensional Systems*, Soviet Scientific Reviews/Sec. A/Phys. Reviews (1989) Harwood Academic Publishers;
M. Schottenloher, *Eine mathematische Einführung in die konforme Feldtheorie*, Vorlesung, [http://www.mathematik.uni-muenchen.de/~gkadmin/pr/gk_mp_9510_21.ps.gz];
C. Schweigert, J. Fuchs, J. Walcher, *Conformal field theory, boundary conditions and applications to string theory*, [`hep-th/0011109`].

[4] C. Itzykson, H. Saleur,J.-B. Zuber, *Conformal Invariance and Applications to Statistical Mechanics* (1988) World Scientific.

[5] M. Green, J. Schwarz, E. Witten, *String Theory*, vols. 1,2 (1986) Cambridge University Press;
M. Kaku, *String Theory* (1988) Springer;
D. Lüst, S. Theisen, *Lectures on String Theory*, Lecture Notes in Physics (1989) Springer;
L. Alvarez-Gaumé, Helv. Phys. Acta **61** (1991) 359-526;
R. Dijkgraaf, *Les Houches Lectures on Fields, Strings and Duality*, [`hep-th/9703136`].

[6] O. Lechtenfeld, *String theory*, [http://www.itp.uni-hannover/~lechtenf/???];
J .Plefka, *Supersymmetry*, [http://www.???].

[7] R. Dijkgraaf, E. Verlinde, H. Verlinde, *Comm. Math. Phys.* **115** (1988) 649.

[8] V.S. Dotsenko, V.A. Fateev, *Nucl. Phys.* **B 249** [FS12] (1984) 312; *Nucl. Phys.* **B 251** [FS13] (1985) 691; *Phys. Lett.* **B 154** (1985) 291.

[9] B.L. Feigin, D.B. Fuks, *Funkt. Anal. Appl.* **16** (1982) 114; *Funkt. Anal. Appl.* **17** (1983) 241;

[10] D. Friedan, E. Martinec, S. Shenker, *Nucl. Phys.* **B 271** (1986) 93.

[11] E. Verlinde *Nucl. Phys.* **B 300** [FS22] (1988) 360.