On a New Universal Class of Phase Transitions and Quantum Hall Effect

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Abstract

We study the possible phase transitions between (2+1)-dimensional abelian Chern-Simons theories. We show that they may be described by non-unitary rational conformal field theories with $c_{eff} = 1$. As an example we choose the fractional quantum Hall effect and derive all its main features from the discrete fractal structure of the moduli space of these non-unitary transition conformal field theories and some large scale principles. Rationality of these theories is intimately related to the modular group yielding severe conditions on the possible phase transitions. This gives a natural explanation for both, the values and the widths, of the observed fractional Hall plateaux.

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1 Introduction

The quantum Hall effect (QHE) certainly is one of the most exciting phenomena in condensed matter physics, since theory more or less fails to describe it in terms of two- or three-dimensional continuous second order phase transitions of the state of its electrons. The same is true for other famous phenomena such as high temperature superconductivity. Thus, there are condensed matter systems in reality, whose physics can be described more or less completely in two dimensions, which show a kind of phase transitions, but which cannot be explained by the well known universality classes of two-dimensional statistical systems.

The aim of this paper is to introduce a new class of two-dimensional phase transitions. As expected in two dimensions, they obey conformal invariance at the critical point, such that the theories at the critical point can be described by certain non-unitary rational conformal field theories (RCFTs). This is a remarkable fact by itself, since up to now there are not many "useful" applications of non-unitary theories known.

Actually, our new class of phase transition describes the interpolation *between* phases of a two-dimensional system which can be described by rational models of current algebras. In this paper we confine ourselfs to the case of abelian current algebras, i.e. to the well known rational models with $c = 1$. As it turns out, the possible transitions between two such models are determined by severe restricitions which are deeply connected with the modular group.

As an example, we choose the quantum Hall effect and show how the transitions between the Hall plateaux can be described within the above mentioned new class. Our transitions connect two different quantum Hall states by connecting the corresponding chiral conformal theories living on their edges. Many of the features of this class of phase transitions are very general and automatically link together many viewpoints common in the field of two-dimensionally confined systems such as the QHE or high-T_c superconductivity. We mention only a few keywords: Anyons, Chern-Simons theory, duality, modular group, etc.

We are able to derive all the essential macroscopic data of the QHE only from a careful study of the nature of the new class of transitions. The fact that the transitions are described by RCFTs imposes severe restricitons to which states a given quantum Hall state may change, which are intimately related to the modular group. Most remarkebly, we find a natural explanation not only for the fact that the Hall conductance is quantized, but also that it remains constant in some range of variation of the magnetic field.

Allthough we mainly concentrate on the QHE, it is clear from the general structure of our class of transitions that it should equally well addapt to other phenomena with similar phase diagrams, such as high-T_c superconductivity. More generally, we expect that these transitions appear, whenever a system may be described by Chern-Simons theory, since the latter is equivalent to a chiral con-

formal theory on the boundary of the system in space describing a certain phase of it.

As pointed out above, we confine ourselfs in this paper to the study of transitions belonging to abelian Chern-Simons theories. The paper is organized as follows:

In section two we review briefly the microscopic description of the QHE and the macroscopic large scale behaviour which is completely governed by the Chern-Simons action. We develop a graphical description of the possible QHE states in terms of an $1/N$ expansion of Chern-Simons-QED Feynman graphs, form which we easily can read off the interesting observables.

In the third section we introduce some tools from conformal field theory and give a short survey on the relation of QHE wave functions to correlators of certain $c = 1$ conformal field theories.

The fourth section proposes our scheme of phase transitions between different QHE states, i.e. between different Chern-Simons theories. We argue that certain non-unitary rational conformal theories may interpolate between different chiral $c = 1$ conformal field theories. We use the concept of fusion well known in rational conformal theory to describe the attachment of flux quanta to particles according to Jain's picture of the QHE. By this procedure we move from one sector of such a non-unitary theory to another such that chiral projection leads to different $c = 1$ theories.

Section five discusses the structure of possible phase transition of the proposed kind due to an action of the modular group on the parameters of the discrete fractal moduli space of our non-unitary theories. Together with the large scale principles of the first section this enables us to predict the observable fractional values as well as the widths of their plateaux. They correspond to certain attractor regimes in the moduli space.

In the last section we summarize our results and give some more details on the attractor band structure. We also point out possible generalizations and further applications of our scheme to other two-dimensional phenomena in condensed matter physics, such as high-T_c superconductivity.

2   From Quantum Hall Effect to Chern-Simons-Theory

The experimental discoveries of the integer quantum Hall effect (IQHE) [31] 1980 and of the fractional quantum Hall effect (FQHE) [44] 1982 are one of the most interesting physical phenomena in solid state physics in recent years [41, 43]. The transversal conductance of a two-dimensional electron gas in a high magnetic field at low temperature exhibts quantized plateau values of the form $\sigma_{xy} = \frac{e^2}{h}\nu$, where the filling factor ν is an integer or fractional number. In many respects, both

the integer as well as the fractional effect share very similar underlying physical characteristics and concepts, for instance the two-dimensionality of the system, the quantization of the Hall conductance with simultaneous vanishing of the longitudinal resistance, and the interplay between disorder and the magnetic field giving rise to the existence of extended states. In other respects, they encompass entirely different physical principles and ideas. In particular, while the IQHE is essentially thought of as a noninteracting electron phenomenon [33], the FQHE is believed to arise from a condensation of the two-dimensional electrons into a new incompressible state of matter as a result of interelectron interaction [34], the so called quantum droplet.

This condensation phenomenon could be extended to a whole hierarchic structure of quasi-particles and -holes, which is based on the fundamental states with $\nu = 1/(2p + 1)$ [22, 24]. However, the repeated condensation of quasi-paritcles seems somehow unphysical. Another phenomenological theory of J.K. Jain considers composite particles built from electrons and attached flux quanta of the magnetic field. In this model IQHE and FQHE appear in a unified way. Recent experimental results are in good agreement with this theory [12, 50, 25].

In most of the works on the FQHE, the ansatz of R.B. Laughlin for the wave functions to the fundamental fractions $\nu = \frac{1}{3}, \frac{1}{5}, \frac{1}{7} \ldots$ plays an important rôle [34]. This ansatz cannot yet prooven rigorously, but has an extremely high overlap with numerical exact solutions. The ground state takes the simple form

$$\psi_p = \prod_{i<j}(z_i - z_j)^{2p+1} \exp\left(-\frac{1}{2}\sum_i |z_i|^2\right), \tag{2.1}$$

where p should be an integer due to the Pauli principle. In J.K. Jain's picture, where the electrons are bound to $2p$ flux quanta, the wave functions are obtained from the ones of the IQHE, ϕ_n, with $\nu = n$, by

$$\psi_\nu = D^{2p}\phi_n \quad \text{with} \quad D = \prod_{i<j}(z_i - z_j). \tag{2.2}$$

Mean field arguments yield the filling factor $\nu = n/(2pn \pm 1)$. The Laughlin wave functions are obtained for $n = 1$. The assumption, that an even number of flux quanta is attached, results from the requirement that the statistics of the composite particles remains fermionic.

The incompressibility of these quantum fluids is explained by an finite energy gap above the ground state. This incompressibility also results in an infinite symmetry which describes the area preserving nonsingular deformations of the quantum droplet and commutes with the hamiltonian [8, 16, 28]. The quantization of this symmetry is well known in physics as the nonsingular part of a $\mathcal{W}_{1+\infty}$ and arises e.g. in string theories or two-dimensional gravity [40, 2, 53]. These deformations are directly related to edge excitations which should live on the one-dimensional boundaries and were studied by a number of authors

[23, 47, 6, 17, 19, 7]. The dynamics of these edge states is mainly based on the relation of Chern-Simons gauge theories and conformal field theory [51, 52].

2.1 Microscopical Description

Let us consider a two-dimensional electron in a uniform transversal magnetic field B. The Schrödinger-Equation then takes the form

$$H\psi = \frac{1}{2m}\left(p - \frac{e}{c}A^2\right)^2 \psi = E\psi\,, \tag{2.3}$$

where the momentum $p = -i\hbar\nabla$ and the gauge potential A are defined in the plane. Let us choose the symmetric gauge $\mathbf{A} = \frac{B}{2}(-y, x)$ and introduce complex variables: $z = x + iy$, $\bar{z} = x - iy$ and $\partial = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$. Defining all lengths in units of the magnetic length,

$$l = \left(\frac{2\hbar c}{eB}\right)^{\frac{1}{2}}, \tag{2.4}$$

and the energies in units of the Landau level spacing,

$$\omega_c = \frac{eB}{mc}\,, \tag{2.5}$$

the Hamiltonian can be reexpressed as:

$$H = 2\hbar\omega_c l^2\left(-\partial\bar{\partial} + \frac{1}{2l^2}(\bar{z}\bar{\partial} - z\partial) + \frac{1}{4l^4}z\bar{z}\right). \tag{2.6}$$

Letting $\hbar = m = l = 1$ the hamiltonian and the angular momentum J can be written in terms of a pair of independent harmonic oscillators:

$$
\begin{aligned}
H &= a^\dagger a + aa^\dagger\,, & (2.7)\\
J &= b^\dagger b - a^\dagger a\,, & (2.8)
\end{aligned}
$$

where these operators are

$$
a = \frac{z}{2} + \bar{\partial}\,, \qquad\qquad a^\dagger = \frac{\bar{z}}{2} - \partial\,, \tag{2.9}
$$

$$
b = \frac{\bar{z}}{2} + \partial\,, \qquad\qquad b^\dagger = \frac{z}{2} - \bar{\partial}\,, \tag{2.10}
$$

and satisfy the canonical commutation relations

$$[a, a^\dagger] = 1\,, \qquad\qquad [b, b^\dagger] = 1\,, \tag{2.11}$$

with all other commutators vanishing. The vacuum is defined by the condition $a\psi_{0,0} = b\psi_{0,0} = 0$ which yields

$$\psi_{0,0} = \frac{1}{\sqrt{\pi}}\exp(\frac{1}{2}|z|^2)\,. \tag{2.12}$$

4

The solutions of the Schrödimger-Equation are divided into infinitely degenerated Landau levels (due to rotational invariance around the axis of the magnetic field) with energies $2n + 1$ and eigen functions

$$\psi_{n,l} = \frac{(b^\dagger)^l (a^\dagger)^n}{\sqrt{l! n!}} \psi_{0,0} \,. \tag{2.13}$$

Restricting the model to finite size of area A reduces the degeneracy, since higher angular momentums yield wave functions with larger support. The degeneracy of each Landau level is then given as $N_A = \Phi_{mag}/\Phi_0$, where $\Phi_{mag} = BA$ is the magnetic Flux through the plane and $\Phi_0 = (h/e)$ is the elementary Flux quantum.

Suppose now, there are N such electrons. If there is no interaction between them, then only the magnetic field B controls the number of states and thus the density of electrons per state, acting as an external pressure. The many-particle problem splits in N copies of the single-particle problem with identical operators $a_i = a, b_i = b$, where the label i refers to the i-th electron. Since the electron density per state is the correct quantum measure of the electron density, i.e. the filling fraction ν, the latter is given as $\nu = N/N_A$ and is forced to be an integer due to certain gauge conditions. In fact, ν can be viewed as the Chern-character of an U(1) line bundle over the parameter torus of the magnetic fluxes, and thus is an integer valued topological invariant [38, 32, 1, 39]. It also may be related to an element in the cyclic cohomology of a C^* algebra [5].

Let us now introduce a Chern-Simons type interaction of flux quanta that in Jain's picture are thought of being attached to the electrons. To that issue we redefine the operators $b_i, a_i^\dagger$,

$$
\begin{aligned}
b_i &= \partial_i + \frac{\bar{z}_i}{2} - 2p \sum_{i \neq j} \frac{1}{z_i - z_j} \,, \\
a_i^\dagger &= -\partial_i + \frac{\bar{z}_i}{2} + 2p \sum_{i \neq j} \frac{1}{z_i - z_j} \,.
\end{aligned}
\tag{2.14}
$$

Of course, we now get additional terms of the form $2p\pi\delta(z_i - z_j)$ in the commutation relations,

$$[a_i, a_i^\dagger] = 1 - 2p\pi \sum_{i \neq j} \delta(z_i - z_j) \,, \tag{2.15}$$

$$[a_i, a_j^\dagger] = 2p\pi\delta(z_i - z_j) \quad \text{for} \quad i \neq j \,. \tag{2.16}$$

But these terms may be neglected as long as we require fermionic statistics, i.e. vanishing of the wave functions, if two paricle coordinates approach each other. The resulting theory describes the fractional quantum Hall effect with filling $\nu = 1/(2p + 1)$, where the macroscopic observables are obtained by addition as in the case without interaction, $H = \sum_{i=1}^{N}(a_i^\dagger a_i + a_i a_i^\dagger)$ and the Laughlin wave function (2.1) is eigen function of H to the lowest Landau level.

As has been widely pointed out in the literature (see e.g. [8, 28, 16]), the incompressibility of the Laughlin quantum droplet is equivalent with the invariance of the theory under non singular area preserving diffeomorphisms. With our definitions from above and $n, m \geq -1$ we can define operators

$$\mathcal{L}_{m,n} = \sum_{i=1}^{N} (b_i^\dagger)^{m+1} (b_i)^{n+1} \, , \tag{2.17}$$

which all commute with the Hamiltonian[i]. They generate the algebra $\mathcal{W}_{1+\infty}^+ = \mathcal{W}^+(1, 2, 3, 4, \ldots)$ of non-singular area preserving diffeomorphisms with commutation relations

$$[\mathcal{L}_{n,m}, \mathcal{L}_{k,l}] = \sum_{s=0}^{\min(m,k)} \frac{(m+1)!(k+1)!}{(m-s)!(j-s)!(s+1)!} \mathcal{L}_{n+k-s,m+l-s} - (m \leftrightarrow l, n \leftrightarrow k) \, . \tag{2.18}$$

Moreover, the Laughlin wave functions ψ_p from (2.1) are lowest-weight states, i.e. with $W_n^{(s)} \sim \mathcal{L}_{n+s-2,s-2}$ for $s \geq 1$, $n \geq -s+1$ as the fourier modes of the generators of the algebra with spin s we have $W_n^{(s)} \psi_p = 0$ for $-s < n \leq -1$.

Let us make a few remarks: The redefined operators b_i and $a_i^\dagger$ for $p > 0$ do not longer have $b_i^\dagger$ and a_i as their Hermitian adjoints. This can be corrected by introducing an inner product

$$\langle \Psi_1 \mid \Psi_2 \rangle = \int \Psi_1^\dagger \mu \Psi_2 \, , \tag{2.19}$$

where the non trivial singular measure μ is given as:

$$\mu(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N) = \prod_{i<j} \mid z_i - z_j \mid^{-4p} \, . \tag{2.20}$$

We may observe that the Laughlin wave functions are also eigen functions to the free Hamiltonian of the original $a_i, a_i^\dagger$ operators with the same eigen value. Thus, the Chern-Simons interaction does not destroy the Landau level structure.

The configuration space for distinguishable particles is given by

$$C_N = \{ (z_1, \ldots, z_N) \in \mathbb{C}^N; \quad z_i \neq z_j \quad \text{for} \quad i \neq j \} \, . \tag{2.21}$$

The $(a_i, a_i^\dagger)$ can be considered as covariant derivatives on a $U(1) \otimes \ldots \otimes U(1)$ bundle over C_N as in the paper of E. Verlinde on the non-abelian Aharanov-Bohm effect [46]. Thus, the curvature is given by (2.20) which describes a constant magnetic field plus $2p$ flux quanta added to each electron. This is exactly the FQHE interpretation of J.K. Jain by the composite fermion theory mentioned above. These flux quanta can be described in an abelian Chern-Simons theory by localized Wilson loops. Considering the N-point function of these flux quanta

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[i]Note that there even do not appear any terms with δ-functions.

6

localized at the positions z_i of the electrons one sees that it is proportional to the measure μ (2.20) using that those Wilson loop operators can be expressed by vertex operators [3]. This explains the former observation on the relation between vertex operator correlators and the Laughlin wave function [20, 42, 10, 37].

This picture is in good agreement with an argument of A. Lopez and E. Fradkin [35] that adding an even number of flux quanta to each electron leaves all expectation values invariant. One can see this, if one calculates expectation values via path integrals, because then only closed paths contribute to the partition function which is $tr \exp(-\beta H)$. Closed paths are given by exchanging electrons or moving them around each other. The phase associated to a path loop has two contributions, the statistics of the particles and the Aharanov-Bohm phase due to the enclosed flux. The former is fermionic, since adding an even number of flux quanta to the electrons does not change it, the latter is trivial, since each whole flux quantum produces a phase of unity. Calculating the expectation values of the Laughlin wave function with the inner product (2.19),

$$\int \psi_p^\dagger \, \mu \, \psi_p dz^N \, , \tag{2.22}$$

it is easy to see that this expression is independent of p, thus, adding flux quanta indeed does not change the expectation value.

Thus, in our formulation of the FQHE we consider a Hamiltonian without explicit interelectron interaction as in the IQHE, but describing the interaction with the help of a nontrivial measure coming from the N-point correlation function of the flux quanta in an abelian Chern-Simons theory.

Let us emphasize again that this picture of the FQHE is not only a complicated view of the IQHE. Since adding of two flux quanta does change the effective magnetic field and thus, the size of the wave functions, not all expectation values remain unchanged. Moreover, E. Verlinde [46] has shown that a free Hamiltonian (without magnetic field) acquires a non trivial S-matrix when a substitution of the kind

$$\partial_i \to \partial_i + \alpha \sum_{i \neq j} \frac{1}{z_i - z_j} \, , \qquad \bar{\partial}_i \to \bar{\partial}_i \tag{2.23}$$

is applied in the covariant derivatives. On the other hand, a Hamiltonian with Chern-Simons interaction can be rewritten as a free Hamiltonian of non interacting particles with anyonic statistics via a singular gauge transformation eliminating the Chern-Simons gauge field. The particles still are subject to a non trivial S-matrix. Both these aspect will be very important in the following.

We conclude our short introduction to the microscopic aspects of the QHE by mentioning that one can easily generalize the Chern-Simons interaction terms to the case of different independent quantum fluids (i.e. sets of eventually interacting Landau levels or different layers), see e.g. [47, 42, 48, 27, 19, 16]. Such systems

7

lead to Laughlin type wave functions of the form

$$\psi_K(\{z_i^I\}) \sim \prod_{\substack{I \\ i<j}} (z_i^I - z_j^I)^{K_{II}} \prod_{\substack{I<J \\ i\leq j}} (z_i^I - z_j^J)^{K_{IJ}} \exp(-\frac{1}{4l^2} \sum_{I,i} \mid z_i^I \mid^2)\,, \qquad (2.24)$$

where now the electrons are distributed to n different subbands[ii] which are labeled by $I, J = 1, \ldots, n$, each subband I containing N_I electrons. The Pauli principle and the requirement of single valuedness of the wave function restrict the $n \times n$ matrix K_{IJ} to be symmetric, positive, integer valued with odd integers on the main diagonal. The filling fraction of such states is given by

$$\nu = \sum_{IJ} (K^{-1})_{IJ}\,. \qquad (2.25)$$

The filling fractions of J.K. Jain's model are obtained from the simple $n \times n$ matrices $K = \mathbb{1}_n + 2pC_n$, where C_n is the $n \times n$ matrix with all entries equal to 1. The resulting Lagrangian describes a system of n independent currents which, however, are coupled together by just the global Chern-Simons coupling via $2p$ flux quanta.

The fractional fillings can also be understood in terms of topological invariants, namely the first Chern-number of a vector bundle divided by its rank, where the latter is equal to the degeneracy of the ground state [38]. For a recent treatment which relates the observed fractions to the class of stable vector bundles see [45].

2.2 Macroscopic Description

Let as again consider a system of non interacting electrons in a strong transverse magnetic field B, confined to a two-dimensional domain (thus, we live in $(2+1)$-dimensional space time). The current J^μ can be written as the curl of a vector potential α, i.e.

$$J^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu \alpha_\lambda\,. \qquad (2.26)$$

Since J^μ is invariant under gauge transformations of α, α_μ is a gauge field. By gauge invariance, the simplest possible local term in the Lagrangian density is just the Chern-Simons term, i.e.

$$\mathcal{L} = \frac{\eta}{4\pi} \epsilon^{\mu\nu\lambda} \alpha_\mu \partial_\nu \alpha_\lambda + \ldots\,, \qquad (2.27)$$

where η is a dimensionless coefficient. As has been discussed by J. Fröhlich and A. Zee [19], there could be other terms including the Maxwell term $\frac{1}{g^2} f_{\mu\nu}^2$

[ii]Subbands can be realized as different layers, different Landau levels or additional quantum numbers for the first Landau level.

and other short-range dynamical interactions. But these additional terms will be invisible in the scaling limit, if $\eta \neq 0$. Actually, as argued by them, every $(2+1)$-dimensional gauge theory at zero temperature with a strictly positive energy gap will be completly governed at very long distances by the topological Dirac-Aharanov-Bohm type interactions between charged sources carrying magnetic vorticity. These topological interactions are described by a Chern-Simons Lagrangian as given by (2.27).

In the simplest case of one filled Landau level, it turns out that $\eta = \nu = 1$. Of course, we could take different currents J_I and we could introduce electron-electron interactions. The universality of the long distance behaviour, however, forces that the only effect of electron-electron interactions is to modify the coefficient η, as long as the Landau level structure and the positive energy gap are preserved. Thus the effective long distance Lagrangian is just given as

$$\mathcal{L} = \frac{1}{4\pi} \sum_{I,J} K_{IJ} \epsilon^{\mu\nu\lambda} \alpha_{I,\mu} \partial_\nu \alpha_{J,\lambda} + \dots , \qquad (2.28)$$

with the same matrix K_{IJ} as introduced at the end of the last section. This way of describing the QHE with abelian Chern-Simons theory has been extensively studied [17, 19, 35].

There is a second principle, which further restricts the influence of the microscopic behaviour to the macroscopic observables: As has been pointed out in the last section, we can always write the N-particle Hamiltonian in an form which exhibits $O(N)$-symmetry, eventually moving non trivial effects to some anyonic behaviour of the particles. But $O(N)$-invariant Hamiltonians admit an $1/N$-expansion of the correlation functions. Since $N \gg 1$, only the leading terms will survive. As we will see later, this can explain the independence of observables, such as the transversal conductivity within a plateau, from the variation of the external parameters, such as the external magnetic field. In fact, a small variation leads to microscopical influences which, however, do only contribute to higher terms of the $1/N$-expansion.

To see this in more detail, let us view *abelian* Chern-Simons theory as an ordinary quantum electrodynamics (QED) with massive bosons. Thus, we can introduce Feynman graphs: A current of n non interacting Landau levels (or sublevels) is depicted by a fermionic line $\underrightarrow{\quad n \quad}$ whose direction corresponds to the moving direction of the electrons. The Chern-Simons interaction of $2p$ attached flux quanta is depicted by a bosonic line (the "photon") $\sim\!\!\sim^{1/2p}\!\!\sim$. This syntactically makes sense, since the charge carriers are fermions and the attachment of an even number of flux quanta does not change the statistics of the charge carriers, hence is of bosonic character.

Let us assume that we consider the Feynman graphs of first order in the $1/N$-expansion, for example a graph with two free vertices corresponding to a two-point correlation function. But what does this correlation function measure?

In fact, the macroscopic current-current correlation of the N electrons. But this is directly related to the conductivity, see e.g. [17]. Moreover, if the contributing graphs of the $1/N$-expansion deal with macroscopic values as the currents, they must also satisfy the ordinary laws of classical, macroscopic electrodynamics, such as the Kirchhoff rules. This can be translated into a particularly nice graphical description. The $1/N$-two-point-graphs may be viewed as networks of certain conductivities driven by certain currents generated by magnetic flux through closed loops of the graph. We may measure the conductivity between the two free vertices. A fermionic loop of a current of n Landau levels has conductivity n, the boson has conductivity $1/2p$. Note, that conductivity and conductance are the same in our situation, hence the networks are topological in the sense that the overall conductivity does not depend on the positions of the vertices. This also mirrors the fact that Chern-Simons theory is topological.

The case $p = 0$ corresponds to a shortcut, since a conductor in series, which has infinite conductivity, does not change the overall conductivity. The bosonic propagator reduces to a point interaction with bosons of infinite mass. This is exactly, what we should expect, since $1/N$-expansion is meaningless in the IQHE because there is no interaction between the particles. The macroscopic observables are just given by summing up the N identical contributions of the single-particle theory. But the graph of first order of the single-particle theory is nothing else than a point-interaction. In general, the elementary first order graph is

$$(2.29)$$

Since we normally measure two-point correlations, we have to close two of the external lines to a loop. Keeping one line fixed as the incoming line, there are up to symmetry two possibilities to connect two lines. The first is

$$(2.30)$$

which gives the transversal conductivity $\sigma_{xy} = n/(2pn + 1)$ to first order. If we change the direction of the magnetic field which drives the current in the loop denoted n, we get $\sigma_{xy} = n/(2pn - 1)$ by formally replacing $n \mapsto -n$.

The longitudinal conductivity of a Hall sample near zero temperature is essentially zero. There may by corrections coming from the Chern-Simons Lagrangian,

but the only possible contributions are self-energy terms. We get the self-energy to first order by the second possibility of connecting two lines,

$$(2.31)$$

This graph gives a vanishing contribution, as is observed in experiments, since no non-zero conductivity, which is driven by a current from a loop enclosing magnetic flux enters the graph.

It is easy to see that there are 15 different possibilities to get second order contributions to two-point correlators. But since we have to keep in mind the directions of the currents, we see that many terms cancel, because they contribute with opposite signs. There again only two graphs survive, which are not cuttable, i.e. not separable into two disconnected subgraphs by cutting one fermionic line. One of them gives the second order transversal conductivity, the other is a self-energy contribution corrected by a first-order Hall current. The two graphs are

$$(2.32)$$

and

$$(2.33)$$

Remarkebly, the longitudinal conductivity does not vanish to second order. One may think of this correction that the longitudinal current to second order somehow "sees" the Hall current of first order. In fact, there are mesurements of Hall conductivities, which cannot be obtained in first order and indeed have non-vanishing longitudinal resistance. Presumably, they are driven by a lower number of electrons such that second order effects become visible. A possible explanation of these Hall conductivities is given by J. Fröhlich et. al. [18] by a classification of quantum Hall fluids.

A futher remark is necessary here. Whether a Lagrangian of form (2.28) has higher order terms in the $1/N$-expansion, depends on the form of the matrix K.

11

Viewing K as a network incidence matrix, it is easy to derive the corresponding graph which will correctly yield ν as given in (2.25). This also determines the highest order (of loops) which will contribute to the theory, since the Chern-Simons theory for one global current essentially is a one-loop theory.

Every symmetric, positive, integer valued matrix K with odd integer entries on the main diagonal can be built up by a repeated procedure of globally attaching even numbers of flux quanta and extending the matrix by adding new currents (eventually using negative numbers). Therefore, the graph determining the filling fraction is just a further extended version of (2.32). But note, that the Hall conductivity will be given just by the first order contribution in most of the cases.

Our discussion may be generalized to n-point-graphs, but one has to ask in which way one could implement a physical measurement of these expectation values.

We have now the following macroscopic picture: The universal behaviour of the large scale physics and its topological nature explain the fact that the Hall conductivity is quanitzed. The fact, that the macroscopic observables are given by the $N \to \infty$ limit of the microscopic description, may explain that the quantization is stable against small variations of the external magnetic field. This will become more clear in the following.

3   From Quantum Hall Effect to Conformal Field Theory

It is by now a well known fact that $(2 + 1)$-dimensional Chern-Simons theory is equivalent to $(1 + 1)$-dimensional chiral rational conformal field theory living on the boundary of the space domain [51, 52]. Much work has been done to evaluate this connection in the case of the QHE [37, 20, 10]. In particular, the Laughlin wave functions could be expressed as N-point correlation functions of certain vertex operators of rational $c = 1$ Gaussian models. This also explained in a nice way the occurence of non abelian statistics and anyons.

Usually, the conformal field theory (CFT) lives on the cylinder made out of the the edge of the quantum droplet times a time axis. But if one wants to relate Laughlin wave functions to chiral conformal blocks of the CFT, one has to consider an appropriate analytical continuation back into the plane (a Wick-rotation), which we will implicitly assume in the following. The following statements can be found in the literature:
• The CFTs for the so called abelian QHE states, which correspond to their abelian Chern-Simons theories, have $c = 1$;
• The attached flux quanta can be described by vertex operators, which correspond to the localized Wilson loops of the Chern-Simons theory;

• The wave functions are then given by the correlation functions of the vertex operators;

• There is a principle of chirality at least for the FQHE at fillings $\nu = 1/(2p+1)$, i.e. the wave functions are essentially given by the chiral conformal blocks;

3.1 Preliminaries in CFT

To be more specific, let us consider vertex operators of a free field construction of a CFT with central charge $1 - 24\alpha_0^2$. The fourier modes of the current $j = \partial\phi$ of a scalar free field $\phi(z)$ obey the U(1)-Kac-Moody algebra

$$[j_m, j_n] = n\delta_{m+n,0},\tag{3.1}$$

which is known to describe the chiral edge waves, i.e. the energy gapless excitations of the QHE states. The irreducible lowest-weight representations are realized as Fock spaces $\mathcal{F}_{\alpha,\alpha_0}$ over the lowest-weight states $\mid \alpha, \alpha_0 \rangle$ with

$$j_n \mid \alpha, \alpha_0 \rangle = 0 \;\; \forall n < 0, \qquad j_0 \mid \alpha, \alpha_0 \rangle = \sqrt{2}\alpha \mid \alpha, \alpha_0 \rangle.\tag{3.2}$$

These Fock spaces $\mathcal{F}_{\alpha,\alpha_0}$ carry the structure of Virasoro modules, if the Virasoro field is defined by

$$L(z) = \mathcal{N}(j,j)(z) + \sqrt{2}\alpha_0 \partial_z j(z),\tag{3.3}$$

where $\mathcal{N}$ means normal ordering. The Virasoro algebra has then the central charge $c = 1 - 24\alpha_0^2$. The lowest-weight states of the $\widehat{U(1)}$ algebra become Virasoro lowest-weight states,

$$L_n \mid \alpha, \alpha_0 \rangle = 0 \;\; \forall n < 0, \qquad L_0 \mid \alpha, \alpha_0 \rangle = h(\alpha) \mid \alpha, \alpha_0 \rangle,\tag{3.4}$$

where the conformal weight is given by $h(\alpha) = \alpha^2 - 2\alpha\alpha_0$. Finally, the vertex operators map Fock spaces into each other, $\psi_\alpha : \mathcal{F}_{\beta,\alpha_0} \mapsto \mathcal{F}_{\alpha+\beta,\alpha_0}$. Their explicit form is

$$\psi_\alpha = \exp\left(-\sum_{n>0} \sqrt{2}\alpha j_n \frac{z^n}{n}\right) \exp\left(-\sum_{n<0} \sqrt{2}\alpha j_n \frac{z^n}{n}\right) c(\alpha) z^{-\sqrt{2}\alpha\alpha_0},\tag{3.5}$$

where $c(\alpha)$ commutes with all $j_n, n \neq 0$ and maps lowest-weight states into lowest-weight states. Products of vertex operators are only defined for radial ordered coordinates, i.e. $\psi_\alpha(z_1)\psi_\beta(z_2)$ is only defined for $\mid z_1 \mid > \mid z_2 \mid$. The other half is obtained by analytic continuation, where the non trivial statistics of vertex operators shows up, $\psi_\alpha(z_1)\psi_\beta(z_2) = \exp(2\pi i\alpha\beta)\psi_\beta(z_2)\psi_\alpha(z_1)$.

From this construction we can in particular obtain the $c = 1$ Gaussian models, i.e. the U(1)-theory of mappings of the unit circle onto a circle of radius R. We choose our free field $\phi(z)$ to be compactified on a circle with radius R. The partition function is then

$$Z(R) = \mid \eta(\tau) \mid^{-2} \sum_{(p,\bar{p})\in\Gamma_R} q^{\frac{1}{2}p^2}\bar{q}^{\frac{1}{2}\bar{p}^2},\tag{3.6}$$

where τ is the modular parameter of the torus, $q = \exp(2\pi i\tau)$, and $\eta(\tau) = q^{-1/24}\prod_{n>0}(1-q^n)$ is the Dedekind function. The summation of the "momenta" is over the lattice

$$\Gamma_R = \left\{(p,\bar{p}) = \left(\frac{n}{R} + \tfrac{1}{2}mR, \frac{n}{R} - \tfrac{1}{2}mR\right)\Big| n,m \in \mathbb{Z}\right\},\qquad(3.7)$$

which is self-dual, if we adopt a Lorentzian metric. The self-duality assures that $Z(R)$ is modular invariant. The normalized vertex operators are given by

$$V_{nm}^+(z,\bar{z}) = \sqrt{2}\cos\left[p\phi(z) + \bar{p}\bar{\phi}(\bar{z})\right],\, V_{nm}^-(z,\bar{z}) = \sqrt{2}\sin\left[p\phi(z) + \bar{p}\bar{\phi}(\bar{z})\right],\qquad(3.8)$$

where the relation of $(p,\bar{p})$ and (n,m) is defined by the lattice Γ_R. The combinations $V_{nm}^+ + iV_{nm}^-$ create states with momentum $\pm\frac{1}{2}(p+\bar{p})$ and winding number $\pm(p-\bar{p})$.

The models described above yield RCFTs, whenever $2R^2 = p/q$, $p,q \in \mathbb{N}$. In these cases, $Z(R)$ can be written as a finite bilinear form of the characters of the underlying RCFT. The latter are of the form $\chi_\lambda = \Theta_{\lambda,k}/\eta$, where the elliptic functions are $\Theta_{\lambda,k}(\tau) = \sum_{n\in\mathbb{Z}} q^{(2kn+\lambda)^2/4k}$. The partition function for $2R^2 = p/q$ can then be written as

$$Z(R) = \frac{1}{\eta(\tau)\eta(\bar{\tau})}\sum_{\lambda\,\mathrm{mod}\,2pq}\Theta_{\lambda,pq}(\tau)\Theta_{\lambda',pq}(\bar{\tau}),\qquad(3.9)$$

where with $\lambda = nq + mp \bmod 2pq$ we have $\lambda' = nq - mp \bmod 2pq$. In the following we will use a slightly different notation $Z[2R^2] \equiv Z(R)$ for convenience. The duality of the Gaussian partition function takes a particular simple form in this notation, $Z[x] = Z[1/x]$ for every $x \in \mathbb{R}_+$.

3.2 Wave Functions as CFT Correlators

Let us consider a generic correlation function of the chiral vertex operators (3.5) on the plane (on a genus zero Riemannian manifold). We have the well known result

$$\langle\Omega_{-\alpha_0}^*, \psi_{\alpha_1}(z_1)\dots\psi_{\alpha_N}(z_N)\Omega_0\rangle = \prod_{i<j}(z_i - z_j)^{2\alpha_i\alpha_j},\qquad(3.10)$$

if $|z_1| > \dots > |z_N|$ and $\sum_{i=1}^N \alpha_i = \alpha_0$, where α_0 denotes the background charge and Ω_α the ground state to the superselection sector of charge α.

Since this is purely holomorphic, we cannot reproduce the non-holomorphic parts $\exp(-1/2\sum_i |z_i|^2)$ of the Laughlin wave functions (2.1). Either we include this term explicitly in the integral measure μ, or we insert a term $\exp(-i\alpha\int d^2z'\, \bar{\rho}\phi(z'))$ into the correlator (3.10), where ϕ is again the free field and $\bar{\rho}$ is an avaraged density $(\pi\alpha^2)^{-1}$. If one integrates this term over a disk of area $2\pi\alpha^2 N$, then the real part correctly yields the desired exponential term for N electrons,

14

while the imaginary part contributes a singular phase. The latter can be eliminated by an also singular gauge transformation corresponding to the uniform external magnetic field [37]. In the following we will often neglect the exponential term and absorb the external magnetic field in $\Omega^*_{-\alpha_0}(N)$, since the integral also modifies the background charge.

Let us consider the chiral $c = 1$ RCFT with compactification radius $R^2 = 2p+1$. It can be shown [37, 20, 10] that the vertex operators ψ_α with $\alpha = \sqrt{2p+1}$ exactly reproduce the the Laughlin wave functions,

$$\langle \Omega^*_{-\alpha_0}(N), \prod_{i=1}^N \psi_{\sqrt{2p+1}}(z_i)\Omega_0 \rangle = \prod_{i<j}(z_i - z_j)^{2p+1} \exp\left(-\tfrac{1}{2}\sum_i \mid z_i \mid^2\right), \qquad (3.11)$$

while the other fundamental vertex operators with charges $\alpha = \lambda/\sqrt{2p+1}$, $\lambda = 1, \ldots, 2p+1$, produce excitations with anyonic statistics $\theta = \pi\alpha^2$. From (3.7) we learn that the chiral vertex operators have electric charge $\alpha/R = \lambda/R^2$ and magnetic vorticity $\alpha R = \lambda$. Thus, our vertex operators $\psi_{\sqrt{2p+1}}(z)$ have charge 1 and vorticity $2p + 1$ which is, what we would expect from composite fermions with $2p$ attached flux quanta.

There is a broad discussion in the literature on the properties of the CFT picture of the QHE. A particular nice point is, that one can easily obtain the wave functions for arbitrary (periodic) boundary conditions and arbitrary genus Riemannian manifolds. In fact, the real physical system has much of a torus, since one has to close the circuits in order to measure currents or voltages. If one thinks of the longitudinal current generated by a magnetic field and measures the Hall voltage by an induced magnetic flux, one gets the so called magnetic torus via gauge invariance modulo flux quanta. If the torus has the modular parameter τ with (complex) lengths L_x, L_y, then the generic N-point correlator is

$$\langle \Omega^*_{-\alpha_0}(N), \psi_{\alpha_1}(z_1)\ldots\psi_{\alpha_N}(z_N)\Omega_0 \rangle_l^{g=1} =$$

$$\prod_{i<j}\left(\frac{\Theta[^{1/2}_{1/2}](\frac{z_i-z_j}{L_x}|\tau)}{\partial_z \Theta[^{1/2}_{1/2}](0|\tau)}\right)^{2\alpha_i\alpha_j} \Theta\begin{bmatrix} l/\alpha_0^2 \\ 0 \end{bmatrix}\left(\frac{Z\alpha_0^2}{L_x}\middle| \tau\alpha_0^2\right), \qquad (3.12)$$

where $Z = \sum_{i=1}^n \alpha_i z_i/\alpha_0$ is the center of charge coordinate. Here we have introduced the Θ-functions with characteristic,

$$\Theta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) = \sum_{n\in\mathbb{Z}} e^{2i\pi\tau\frac{(n+a)^2}{2}+2i\pi(n+a)(z+b)}.$$

In the case of the Laughlin wave functions we get the well known result

$$\langle \Omega^*_{-\alpha_0}(N), \prod_{i=1}^N \psi_{\sqrt{2p+1}}(z_i)\Omega_0 \rangle_l^{g=1} =$$

$$\prod_{i<j}\left(\frac{\Theta[^{1/2}_{1/2}](\frac{z_i-z_j}{L_x}|\tau)}{\Theta'[^{1/2}_{1/2}](0|\tau)}\right)^{2p+1} \Theta\begin{bmatrix} l/(2p+1) \\ 0 \end{bmatrix}\left(\frac{(2p+1)Z}{L_x}\middle| (2p+1)\tau\right), \qquad(3.13)$$

15

which now is $(2p+1)$-fold degenerated. This degeneracy stems from the possibility of an additional quantum number carried by the ground state with background charge on the torus and follows from the properties of the representation of the braid group on the torus. We denote the functions (3.13) by $\psi_{p,l}(z_1, \ldots, z_n)$. They form a $(2p+1)$-dimensional space closed under the action of the magnetic translations S_a and T_b. For the elementary translations by Steps $a = L_x/(2p+1)$ and $b = L_y/(2p+1)$ one has

$$ S_a \psi_{p,l} = e^{\pi i \frac{l}{2p+1}} \psi_{p,l}, \qquad T_b \psi_{p,l} = \psi_{p,l+1}. $$

Moreover, $\psi_{p,l}$ transforms covariantly under the exchange of L_x and L_y, i.e. $\tau \to -1/\tau$, since

$$ \Theta \begin{bmatrix} l/(2p+1) \\ 0 \end{bmatrix} \left(\frac{z\sqrt{2p+1}}{\tau} \middle| -\frac{(2p+1)}{\tau} \right) = $$
$$ e^{i\pi z^2/\tau} \sqrt{\frac{\tau}{i(2p+1)}} \sum_{l'=1}^{2p+1} e^{-2\pi i \frac{ll'}{2p+1}} \Theta \begin{bmatrix} l'/(2p+1) \\ 0 \end{bmatrix} \left(z\sqrt{2p+1} \middle| -(2p+1)\tau \right). $$

Altough here we have discussed only the case of the Laughlin wave functions, there exist generalizations of this approach to the Laughlin type wave functions (2.24) using rational non integer compactification radii.

4 From Conformal Field Theory to Phase Transitions

Since the seminal work of A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov in 1984 [4], we have a deep theoretical understanding of the universality classes of second order phase transitions of two-dimensional statistical systems. Such a phase transition is related to a CFT, since scaling invariance at the critical point implies full conformal invariance of the statistical field theory in all known cases.

But nature is much richer, and there are two-dimensional systems with a completely different phase transition behaviour – such as the QHE. Here the *phases itself* are described by chiral $c = 1$ RCFTs. Therefore, the transition between two phases must map *different* RCFTs into each other. Moreover, the QHE phases have a very high symmetry due to the incompressibility of the states. While phase transitions usually also show up a very high symmetry, one cannot expect to keep the full symmetry of non-singular area preserving diffeomorphisms, since the size of the system must change (see section two). On the other hand, we might well expect to have conformal invariance at the transition point, since we again have scaling invariance.

4.1 The Modular Group in the Game

The phase diagrams of the QHE and similarly that of high-T_c superconductivity have been studied in much detail [29, 36, 25]. The mainpoint is the assumption of an infinite discrete symmetry group acting on the parameter space. This is nothing strange and, in fact, an old idea which lead to the discovery of S-duality by J. Cardy and E. Rabinovici [9].

The most prominent infinite discrete group is the modular group $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ which is the free span of $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with the relations $S^2 = (ST)^3 = \mathbb{1}$. It operates on the upper half complex plane $\mathbb{H}$ by $S : \tau \mapsto -1/\tau$ and $T : \tau \mapsto \tau + 1$ which we may extend to include the real line. This group, or certain subgroups as the main congruence subgroups $\Gamma(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid a \equiv d \equiv 1 \bmod N, b \equiv c \equiv 0 \bmod N\}$, presumably govern the phase diagram structure of many models in both, condensed matter physics as well as string theories. Common to all these models is the existence of an infinite number of *different* states into which the model may condense in dependency of the parameters.

As we have seen above, the quantization of the magnetic flux motivates the following discrete operations, which map QHE states into each other:

$$
\begin{aligned}
\nu &\mapsto \nu/(2\nu + 1) & \text{attaching two } \uparrow -\text{flux quanta}, \\
\nu &\mapsto \nu/(2\nu - 1) & \text{attaching two } \downarrow -\text{flux quanta}, \\
\nu &\mapsto \nu + 1 & \text{adding a further Landau level}, \\
\nu &\mapsto 1 - \nu & \text{particleholeduality}.
\end{aligned}
\tag{4.1}
$$

The first three transformations generate the subgroup $\Gamma_T(2)$ of the modular group $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$, which is spanned by $ST^{-2}S$ and T. Every real filling factor $\nu \in \mathbb{R}$ can be arbitrarily well approximated by an infinite continued fraction expansion generated by words $\ldots U_{p_3}^{n_3} U_{p_2}^{n_2} U_{p_1}^{n_1} \in \Gamma_T(2)$, where $U_p^n = (ST^{-2}S)^p T^n = \begin{pmatrix} 1 & n \\ 2p & 2pn+1 \end{pmatrix}$ and $n_i, p_i \neq 0 \ \forall i > 0$. Rational fillings are given by finite continued fractions

$$
\begin{aligned}
\nu &= [n_1, 2p_1, n_2, 2p_2, n3, 2p_3, \ldots, n_k, 2p_k, n_{k+1}] \\
&= n_{k+1} + \cfrac{1}{2p_k + \cfrac{1}{\ddots + \cfrac{1}{n_2 + \cfrac{1}{2p_1 + \cfrac{1}{n_1}}}}},
\end{aligned}
\tag{4.2}
$$

which can directly be interpreted in our graphical expansion of section two as the conductivity of a network of successive loops[iii]. As an example see (2.32) as the second order graph with $\nu = [n, 2p, m, 2q]$.

[iii]This is always the leading graph to a given loop order.

Given a finite word $U = \prod_{i=1}^{k} U_{p_i}^{n_i} \in \Gamma_T(2)$ we define its length as $\ell(U) = \sum_{i=1}^{k}(n_i + 2p_i)$ and its order as $\mathcal{O}(U) = k$, if $p_k \neq 0$, otherwise $\mathcal{O}(U) = k - 1$. This means that we consider fillings which are equivalent modulo 1 as of the same order. Since $U_q^0 U_p^n = U_{p+q}^n$ and $U_q^m U_0^n = U_q^{n+m}$, we assume that all words are given in the form of minimal order and length.

The difference between this phase diagram structure and the one e.g. of the oblique confinement phases in $\mathbb{Z}_p$ lattice gauge theory is that different phases correspond to in general different boundary CFTs.

Therefore, we are looking for a class of RCFTs which might be able to furnish the proposed phase transitions between different QHE states. Since both states, which we want to connect, are described by chiral RCFTs with central charge $c = 1$ but different boundary conditions given by their compactification radii R_1 and R_2, the operator content of their bulk CFTs will be different. Moreover, since they both have the same central charge, they cannot be connected by a renormalization group flow via unitary theories which always decreases the central charge. Nonetheless, they might be connected by *non-unitary* CFTs as long as the *effective* central charge remains constant. In non-unitary theories we have to distinguish between the vacuum with its su(1,1)-invariance, and the state of lowest energy, i.e. lowest L_0 eigen value $h_{min} < 0$. The effective central charge is then defined as $c_{eff} = c - 24h_{min} \geq 0$. Since we are looking for rational theories, it is clear that they must posess an extended chiral symmetry algebra[iv].

The RCFTs with $c_{eff} = 1$ have been completely classified [21, 11, 30, 14, 15]. Besides the well known theories with $c = 1$, which are the Gaussian models mentioned above[v], there exist several classes of non-unitary RCFTs:

The "bosonic" ones are the simplest, they have central charge $c = 1 - 24pq$, maximal extended chiral symmetry algebra $\mathcal{W}(2, 3pq)$, and partition function

$$Z_{bos}[p/q, p'/q'] = \tfrac{1}{2}(Z[p/q] + Z[p'/q']) \,, \tag{4.3}$$

where we must have $p'q' - pq = 1$. Note, that this means that there exists a RCFT with partition function (4.3) if and only if the two parameters correspond to an element of the modular group, since we may rewrite the condition as

$$\det \begin{pmatrix} p' & p \\ q & q' \end{pmatrix} = 1 \,. \tag{4.4}$$

We have duality in both arguments, i.e. $Z_{bos}[p/q, p'/q'] = Z_{bos}[q/p, p'/q'] = Z_{bos}[q/p, q'/p'] = Z_{bos}[p/q, q'/p']$. It will soon become clear that this class of

---

[iv]Let k denote the number of bosonic, l the number of fermionic generators spanning the chiral symmetry algebra $\mathcal{W}$ of a RCFT. Then we strictly have $0 < c_{eff} < k + l/2$ [13].

[v]There are also the $\mathbb{Z}_2$-orbifolds with partition function $(Z[2R^2] + 2Z[4] - Z[1])/2$ and the three exceptional solutions $(2Z[9] + Z[4] - Z[1])/2$, $(Z[16] + Z[9] + Z[4] - Z[1])/2$ and $(Z[25] + Z[9] + Z[4] - Z[1])/2$.

RCFTs generates the phase transitions for a $\Gamma(1)$-phase diagram, i.e. where the infinite discrete group acting on the parameter space is the full modular group.

The second class are the "fermionic" theories with central charge $c = 1 - 12pq$, maximal extended chiral symmetry algebra $\mathcal{W}(2, \frac{3}{2}pq)$, and partition function

$$
\begin{aligned}
Z_{ferm}[p/q, p'/q'] &= \tfrac{1}{2}(Z[2p/q] + Z[2p'/q'] + Z[p/2q] + Z[p'/2q']) \quad (4.5)\\
&= Z_{bos}[2p/q, 2p'/q'] + Z_{bos}[p/2q, p'/2q'],
\end{aligned}
$$

where we must now have $p'q' - pq = 2$. Thus, the fermionic theories will generate a phase diagram structure governed by $\Gamma(2)$ which is generated by T^2 and ST^2S. Our condition takes the form

$$
\det \begin{pmatrix} p' & p \\ q & q' \end{pmatrix} = 2. \quad (4.6)
$$

It is worthwile, to emphasize a little pecularity: Matrices with determinant two do not belong to $\Gamma(2)$. A nice property of $\Gamma(2)$ acting on rational numbers is to preserve parity in both, numerator and denominator, separately. Our condition together with the fact that p, q must both be odd to yield a fermionic theory selects the equivalence class of completely odd rational numbers. Choosing an arbitrary matrix of $\Gamma(2)$ we can select the three possible equivalence classes by multiplying it with one of the matrices $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ to get $(0/1) = 0$, $(1/0) = \infty$ or $(1/1) = 1$ respectively. Note that these three matrices all have determinant two.

These both sets of RCFTs have $\mathbb{Z}_2$-orbifolds and also $N = 1$ supersymmetric extensions which will not be important to us in the scope of this paper. Nonetheless, they may well play a rôle in other condensed matter systems such as high-T_c superconductivity where an appearence of $N = 1$ supersymmetry has been conjectured.

4.2 Transitions between chiral RCFTs

In this section we will make use of a detailed knowledge of the representation theory of the non-unitary theories mentioned above. The interested reader is refered to [14, 15]. Since we want to describe aspects of the QHE, we concentrate in the following mainly on the fermionic theories defined in (4.5) and (4.6).

The partition function (4.5) is nothing else than the sum of two partition functions of fermionic $c = 1$ RCFTs, $Z_{ferm}[p/q, p'/q'] = (Z_{bos}[2p/q, p/2q] + Z_{bos}[2p'/q, p'/2q'])/2$, here expressed in terms of (4.3). But what does it mean to sum partition functions? The direct sum of (the underlying algebras of) different CFTs yields the tensor product of their Hilbert spaces und thus the product of the partition functions. The only meaning of the sum of partition functions can be that one CFT lives on two disjunct boundaries. In fact, from the form of the partition function we learn that the theory consists of two sectors each one

belonging to the specific periodicity conditions of one of the two compactification radii, expressed in the statistics parameters $\theta_1/\pi = p'/q'$ and $\theta_2/\pi = p/q$. But a closer look to the representation theory of the maximal extended chiral symmetry algebra $\mathcal{W}(2, \frac{3}{2}p'q')$ shows that the decomposition of the Hilbert space into a direct sum of irreducible lowest-weight representations does not respect this sector structure.

To be more specific, the characters of the vacuum representation and of the representation on the lowest-weight state $|\,h_{min}\rangle$ involve modular forms from both sectors of the partition function decomposed according (3.9),

$$\chi_{vac}(\tau) = \sum_{n \in \mathbb{Z}_+} \chi_{h_{n,n}}^{Vir}(\tau) = \frac{1}{2\eta(\tau)} \left(\Theta_{0,p'q'/2}(\tau) - \Theta_{0,pq/2}(\tau) \right), \qquad (4.7)$$

and for $\chi_{h_{min}}(\tau)$ one has to replace the difference by the sum. Here the conformal weights are denoted as usual by $h_{r,s} = \frac{1}{4}((r\alpha_- + s\alpha_+)^2 - (\alpha_- + \alpha_+)^2)$, where $\alpha_\pm = \alpha_0 \pm \sqrt{1 + \alpha_0^2}$ and in our case $\alpha_0 = \sqrt{p'q'/2}$.

The fact that some characters involve modular forms of different moduli has highly non trivial consequences. Firstly, this means that the theory somehow twists the different boundary conditions, such that it cannot be completely decomposed into two disjunct parts with "homogenous" boundary conditions. Therefore, the S-matrix, which describes the modular transformations of the characters under $\tau \mapsto -1/\tau$, does not have block structure. Secondly, the fusion rules of the RCFT, which can be obtained from the the S-matrix via the Verlinde formula, have the property that fusing two $\mathcal{W}$-conformal families from one sector can yield $\mathcal{W}$-conformal families of the other sector on the right hand side. Exactly this property will enable us to shift from one QHE state to another.

Before we demonstrate how this works, we have to make one essential remark: QHE states are described by *chiral* RCFTs, phase transitions by some tensor product of a left- and right-chiral CFT to one, whose vertex operators are all local and whose correlators are all well defined single valued functions. If we look at the chiral part of one of our non-unitary RCFTs, we find that it is isomorphic to a direct sum of two chiral $c = 1$ RCFTs (if we carefully work with effective central charge and effective lowest-weights). Thus, we can recover the QHE states by appropriate *chiral projections* of our non-unitary RCFT.

We start with the best understood states, the Laughlin states to filling factors $\nu = 1/(2p+1)$. We rewrite the Laughlin wave functions as correlators of local chiral vertex operators of a fermionic non-unitary theory showing explicitly how we get the chiral QHE state as chiral projection. Let us denote the full vertex operators by $V_{(k,l|m,n)}(z, \bar{z}) \equiv \psi_{\alpha_{k,l}}(z) \otimes \psi_{\alpha_{m,n}}(\bar{z})$. We consider the non-unitary theory with central charge $c = 1 - 24\frac{2p+1}{2}$ and chiral symmetry algebra $\mathcal{W}(2, \frac{3}{2}(2p+1))$. The diagonal partition function is characterised by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 2p+3 & 2p+1 \end{smallmatrix} \right)$. Looking at the irreducible representation to $|\,h_{min}\rangle$ we find a vertex operator $V_{(0,0|1,1)}$ of conformal weight $(h, \bar{h}) = (-(2p+1)/2, 0)$. Its N-point correlator on the

sphere is

$$\langle \Omega^*_{-\sqrt{(2p+1)/2}}(N), \prod_{i=1}^{N} V_{(0,0|1,1)}(z_i, \bar{z}_i)\Omega_0\rangle = \prod_{i<j}(z_i - z_j)^{2p+1}\exp\left(-\frac{1}{2}\sum_i |z_i|^2\right),$$
(4.8)

hence identical with (3.11). Actually, as a rigorous fact, the chiral sectors of our non-unitary theories are indistinguishable from a disjunct union of two chiral $c = 1$ CFTs.

Let us now assume that the external magnetic field is slowly increased. Our present QHE state is realized by a uniform distribution of magnetic flux and electrons such that each electron has $2p$ flux quanta attached to it. Increasing the magnetic field inserts additional flux quanta which at the beginning are not bound to an electron. We may now imagine that there will be an amount of additional flux quanta such that a reconfiguration of the system becomes possible in which again all flux quanta are somehow bound to electrons. Certainly, there is at least the possibility to add $2N$ flux quanta such that each electron could carry $2(p+1)$ at all. Note, that all possible QHE phases described above are given by a certain partitioning of uniform composite fermion subbands, i.e. electrons having all the same number of attached flux quanta within the same subband, such that *all* available flux quanta are eaten up that way.

The reason for this is that the wave functions (2.24) will be single valued only, if there are no "free" flux quanta around. This explains, why the filling factors are given by continued fractions, since the latter exactly implement all electron density partitionings of the required kind.

A closer look to the spectrum of our non-unitary theory in question shows that there is a good candidate vertex operator for describing a single flux quantum. While the composite fermions are described by $V_{(0,0|1,1)}(z, \bar{z})$, the flux quantum is realized by the operator $V_{(\frac{2p}{2p+1},\frac{2p}{2p+1}|\frac{2p}{2p+1},\frac{2p}{2p+1})}(w, \bar{w})$, whose conformal dimension is $(h, \bar{h}) = (\frac{4p+1}{4p+2}, \frac{4p+1}{4p+2})$. If we insert M such flux quanta into the correlator (4.8), we obtain

$$\langle \Omega^*_{-\sqrt{(2p+1)/2}}(N, M), \prod_{j=1}^{M} V_{(\frac{2p}{2p+1},\frac{2p}{2p+1}|\frac{2p}{2p+1},\frac{2p}{2p+1})}(w_j, \bar{w}_j) \prod_{i=1}^{N} V_{(0,0|1,1)}(z_i, \bar{z}_i)\Omega_0\rangle =$$

$$\prod_{j<j'}|w_i - w_{j'}|^{1/(2p+1)} \prod_{i,j}(z_i - w_j) \prod_{i<i'}(z_i - z_{i'})^{2p+1}e^{-\frac{1}{2}\sum_i |z_i|^2 - \frac{1}{2(2p+1)}\sum_j |w_i|^2}$$
(4.9)

Indeed, the flux quanta have the fractional statistics parameter $\theta/\pi = 1/(2p+1)$, and the fractional charge $-e/(2p + 1)$. Thus, they behave as anyons [49, 37]. In this way, we reproduce the basic excitations of the Laughlin wave functions. Of cource, the anti-holomorphic part $\prod_{j<j'}(\bar{w}_j - \bar{w}_{j'})^{1/2(2p+1)}$ drops out in the chiral projection but cannot be avoided due to mathematical consistency.

The main idea now is the following: If we read "attaching of flux quanta" literally, we must let approach the coordinates of the flux quanta to the ones of the

particles, $w_i \rightarrow z_i$. For simplicity let us first consider the case $M = N$. Then we can attach one flux quantum to each particle. We now insert the operator product expansion (OPE) of $V_{(0,0|1,1)}(z, \bar{z}) V_{(\frac{2p}{2p+1}, \frac{2p}{2p+1} | \frac{2p}{2p+1}, \frac{2p}{2p+1})}(w, \bar{w})$ which is valid for $| z - w | \ll 1$. The OPE has the general form

$$V_{(\alpha|\beta)}(z, \bar{z}) V_{(\gamma|\delta)}(w, \bar{w}) =$$
$$\sum_{\zeta, \eta} (z - w)^{h(\zeta) - h(\alpha) - h(\gamma)} (\bar{z} - \bar{w})^{\bar{h}(\eta) - \bar{h}(\beta) - \bar{h}(\delta)} C_{\alpha\gamma}^{\zeta} C_{\beta\delta}^{\eta} \Phi_{(\zeta|\eta)}(w, \bar{w}), \quad (4.10)$$

where $\Phi_{(\zeta|\eta)}$ denotes a generic field $f(\partial\phi, \partial^2\phi \ldots) V_{(\zeta|\eta)}$. The fusion rules of our RCFT tell us which $\mathcal{W}$-conformal families will contribute to the right hand side of the OPE. Since we want to take the limit $w \rightarrow z$, we may restrict ourselfs to the term of leading order. Thus, with "attaching flux quanta" we mean the fusion product, i.e. the projection of the OPE to its leading order in the limit $w \rightarrow z$.

If we do this in (4.9) with $M = N$, we see that the right-chiral part of $V_{(0,0|1,1)}(z, \bar{z})$ acts as identity. Thus, inserting the OPE we obtain the left-chiral part tensorized with $\psi_{\alpha \frac{2p}{2p+1}, \frac{2p}{2p+1}}(\bar{z})$. Obviously, this has *no* purely chiral projection! But let us repeat this procedure with a second set of N flux quanta, hence attaching *two* flux quanta. Studying the fusion rules of our RCFT [14], we find the surprising result that the leading right-chiral part again is the identity $\psi_{\alpha_{1,1}}(\bar{z})$, while in the left-chiral part the identity does not appear and the leading vertex operator $\psi_{\alpha \frac{2p+1}{2p+3}, -\frac{2p+1}{2p+3}}(z)$ belongs to the second sector of our RCFT! Thus, by chiral projection, we moved from one $c = 1$ CFT to another.

This shows, that attaching an *even* number of flux quanta, described by fusion product, changes the periodicity conditions and the statistics of the state from $1/(2p+1)$ to $1/(2p+3)$. In symbolical notation of fusion we have for the left-chiral part

$$\left([\phi_{0,0}] \star \left[\phi_{\frac{2p}{2p+1}, \frac{2p}{2p+1}} \right] \right) \star \left[\phi_{\frac{2p}{2p+1}, \frac{2p}{2p+1}} \right] = \left[\phi_{\frac{2p+1}{2p+3}, -\frac{2p+1}{2p+3}} \right] + \ldots \quad (4.11)$$

to leading order. Up to this moment, we arrived at a $c = 1$ CFT with compactification radius $R^2 = 1/(2p + 3)$, but with some excited state. Nonetheless, the description of the phase transition is not yet complete, since we still have to change the size of the system, such that the total magnetic flux density remains constant [26, 27]. This dissipation of the system (since we must decrease the electron density) will cost energy to compensate the external pressure of the magnetic field, and therefore, cool down the QHE state. Once we have changed the statistics by the fusion product, the system cannot cool down to the old ground state, it is forced to find a stable (hence chiral) state in the $c = 1$ CFT with compactification $R^2 = 1/(2p + 3)$. This state is the Laughlin wave function with $\nu = 1/(2p + 3)$. The amount of energy will be proportional to the ratio of the filling factors, $\nu_{new}/\nu_{old} = \nu^*/\nu = (2p + 1)/(2p + 3)$. This is exactly the excitation energy of our leading term, since $h(\alpha_{\frac{2p+1}{2p+3}, -\frac{2p+1}{2p+3}}) = -(2p + 1)/(2p + 3)$.

Now we could start the game again since, after renormalization of the size of the system, our starting point will be the non-unitary RCFT with $c = 1 - 12(2p + 3)$ and chiral symmetry algebra $\mathcal{W}(2, \frac{3}{2}(2p + 3))$. We will see, that the Laughlin states are characterized by the property that the filling factor is equal to the statistics of the basic anyonic excitations, $\nu = \theta/\pi$. More generally, we can form sequences of the following kind. Let $\mathcal{C}[p/q]$ denote a chiral CFT with compactification $R^2 = p/q$, and $\mathcal{W}[p/q, r/s]$ a non-unitary fermionic RCFT with $c_{eff} = 1$. Further let π denote (left-) chiral projection. Then we have sequences

$$\mathcal{C}[p_1/q_1] \xrightarrow{\pi^{-1}} \mathcal{W}[p_1/q_1, p_2/q_2] \xrightarrow{\pi} \mathcal{C}[p_2/q_2] \xrightarrow{\pi^{-1}} \mathcal{W}[p_2/q_2, p_3/q_3] \xrightarrow{\pi} \dots, \quad (4.12)$$

where according to general arguments on the renormalization group flow of CFTs we must have a decreasing central charge in direction of the flow, i.e. $c_1 = 1 - 12p_1q_1 > c_2 = 1 - 12p_2q_2 > \dots$. This in turn means that the product of p_iq_i has to increase along the sequence.

In this way we can obtain other filling factors with odd numerators and denominaters. Let us start with the filling $\nu = p/q'$, given by a certain continued fraction. Let us choose a $c = 1$ theory with suitable compactification radius, i.e. $R^2 = p'/q'$, which at the same time selects the statistics parameter of anyonic excitations of our QHE state to be $\theta/\pi = p'/q'$. Now, if it is possible to find an positive integer q, such that (4.6) can be fulfilled, we found a possible QHE state with filling factor $\nu^* = p'/q$ given by a $c = 1$ CFT with $R^2 = p/q$. Note, that the condition (4.6) is non trivial and restricts the possible phase transitions between QHE states.

For every admissible filling $\nu = p/q'$ where is a infinite set of fillings $\nu_i^* = p_i'/q_i$, such that (4.6) is fulfilled. For example, $\nu = p/q'$ has $\begin{pmatrix} p & p' \\ q' & p'q'+2 \end{pmatrix}$ as admissible matrices allowing transitions to $\nu^* = p'/(p'q' + 2)$ for $p' > 0$ odd. On the other hand, for every statistics parameter $\theta/\pi = p/q$ where is only a finite (but never empty) set of statistical parameters $\theta^*/\pi = p'/q'$ such that condition (4.6) can be satisfied. Therefore, a given statistics parameter can yield only a finite set of filling factors and QHE states.

If $p, p' \neq 1$, the partition function (4.5) is no longer diagonal. This affects the fusion rules by an automorphism [14] and will in general change our leading order in (4.11). Moreover, the non trivial factorization $p'q'$ or pq guarantees that there are chiral vertex operators of (half-) integer conformal weight above the ground state vertex operator $\psi_{\alpha_{0,0}}$ such that we can have new chiral projections with single valued correlators, i.e. wave functions of the more general form (2.24).

5 From Phase Transitions to Fractals

Let us return for a moment to our class of non-unitary RCFTs with $c_{eff} = 1$ and condition (4.6) for the fermionic case. Our partition function depends on two

parameters. We have studied the moduli space of these theories in much detail, see [15] for the bosonic case. The strange fact that we are not allowed to combine two arbitrary rational $c = 1$ CFTs, i.e. combine two Gaussian partition functions $Z[x], Z[y]$ with x, y rational, in order to get a new *rational* theory, is intimately related to the modular group. As has been explained above, we can obtain all admissible partition functions by the orbit of Γ (or a subgroup as $\Gamma(2)$ for the fermionic case) on a suitable matrix, in our case $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

The set of all admissible pairs $(x, y) \in (\mathbb{R}_+)^2$ yields a multi fractal of measure zero (actually a Cantor set) which nonetheless lies dense in the plane[vi]. figure 1 shows a crude approximation of it, where we mapped all pairs into the fundamental region $\{(x, y) \in (\mathbb{R}_+)^2 \mid 0 \le x \le y \le 1\}$ via duality of the partition function in each of its arguments. Due to purely esthetic reasons we mirrored everything at the diagonal.

5.1 Particle-Hole Duality

At this point the interested reader may wonder, where the filling factors with even numerators appear. Up to now, we implemented the following operations on filling factors:

$$\begin{aligned} \nu &\mapsto \nu/(2\nu + 1) & \text{attaching two } \uparrow -\text{flux quanta,} \\ \nu &\mapsto \nu + 2 & \text{adding two further Landau levels,} \end{aligned} \tag{5.1}$$

which are generated by the main congruence subgroup $\Gamma(2)$ of the modular group Γ, spanned by $ST^{-2}S$ and T^2. This subgroup preserves the parity of numerator and denominator separately. Our condition on the admissible partition functions chooses the odd-odd parity, assured by multiplying a matrix $A \in \Gamma(2)$ with $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

Firstly, if we change the direction of the external magnetic field, we easily obtain a mapping between fractions of the form

$$\nu \mapsto \nu/(2\nu - 1) \quad \text{attaching two } \downarrow -\text{flux quanta,} \tag{5.2}$$

where we just replace $ST^{-2}S$ by its inverse ST^2S. This does not change the parity. But it changes the sign of the determinant (4.6). Thus, we have to exchange the columns of the matrix which gives us a canceling sign. We see that in this way our "flow" between QHE states according (4.12) keeps its direction which never decreases denominators, i.e. which never leads to "less anyonic" statistics.

But there is another mapping of QHE states, the so called particle-hole duality which transforms filling fractions as

$$\nu \mapsto 1 - \nu \quad \text{particle} - \text{hole duality.} \tag{5.3}$$

[vi]The proof of these facts can be found in [15] for the bosonic case. It is easy to see that things remain true in the fermionic case, since $\Gamma(2) \subset \Gamma$ is an infinite discrete subgroup of index two.

This transformation is not contained in $\Gamma(2)$, even not in the whole modular group. But it has a natural explanation within our scheme. Our partition function $Z_{ferm}[p/q, p'/q']$ posesses duality in each of its arguments. This means that the compactification and statistics p/q have the same spectrum as q/p, and therefore, can generate the same QHE states. The analogous holds for p'/q'. Duality corresponds to exchanging the elements of the main- or off-diagonal of the matrix $A = \begin{pmatrix} p & p' \\ q' & q \end{pmatrix}$, which we denote by $D_/$ or $D_\backslash$ respective. Obviously we have $D_\backslash D_/ = D_/ D_\backslash = D_\times$. We see that $D_\times$ just exchanges the order of the compactification radii, since $p/q < p'/q'$ implies $q'/p' < q/p$. This also changes the direction of our "flow" between QHE states and it might happen that a QHE state is mapped to one with less anyonic statistics. But this is impossible from the sector structure of our theories, since the fusion product always leads to vertex operators of more anyonic statistics. Moreover, as a consequence, the renormalization group flow between the non-unitary RCFTs with $c_{eff} = 1$ would not decrease the central charge. The same happens, if we apply the transformation $TS \notin \Gamma(2)$ to A which maps ν to $1 - 1/\nu$.

Combining both, we obtain what we want. In fact, the pair $(\nu, \nu^*) = (p/q', p'/q)$ is mapped to $(1 - \nu^*, 1 - \nu) = (1 - p'/q, 1 - p/q')$, thus yielding the particle-hole duality. The funny point is now that this operation changes the parity of the numerators! In this way we obtain rational numbers with even numerator but still odd denominator.

We pause here to remark that in the frame of QHE the word hole does not mean the same as in general semiconductor theory. Here a hole is realized by quasiparticles consisting of an appropriate number of flux quanta. It is well known [37, 34, 43, 41] that an inserted single flux quantum behaves as a quasihole of charge $q = -e/(2p + 1)$ and statistics $\theta/\pi = 1/(2p + 1)$, see (4.9). So, $2p + 1$ such quasiholes make up a "real" fermionic hole of charge $q = -e$ and statistics $\theta/\pi = 1$. But there is a slight difference. While real electrons have antiperiodic boundary conditions, since they *have* the freedom to change their spin relatively to the external magnetic field (although it is unlikely), flux quanta have not. This we can incorporate by giving a hole fermionic, but periodic boundary conditions. This is by no way unnatural. We only change from the Neveu-Schwarz sector of a fermionic CFT to its Ramond sector, and naturally we will get statistics with even numerators.

Therefore, depending on whether we live in Neveu-Schwarz or in Ramond sector, our RCFT in question will describe a transition between two electron QHE states or between two hole QHE states. The transitions by itself preserve parity.

In this discussion we used the fact that we are not allowed to use duality of the partition function without redefining the filling fraction such that it remains invariant. Otherwise we could not avoid to get transitions in the wrong direction. This has two consequences. Firstly, there will be no trouble with even denominators after using particle-hole duality, which gives us even numerator fillings.

Secondly, we do not obtain the pure IQHE states by this procedure, since we cannot use duality on the Laughlin states. But we do not expect to obtain IQHE states (except $\nu = 1 = 1/(2p-1)$ with $p = 1$ and its descendents), since they lack a real Chern-Simons interaction and therefore, are not described by a Lagrangian of the form (2.28).

5.2 Attractors and Fractals

Finally, we would like to explain the plateaux within our scheme. Let us consider a sequence of type (4.12) and its induced sequence of matrices $\begin{pmatrix} p_2 & p_1 \\ q_1 & q_2 \end{pmatrix} \longrightarrow$ $\begin{pmatrix} p_3 & p_2 \\ q_2 & q_3 \end{pmatrix} \longrightarrow \begin{pmatrix} p_4 & p_3 \\ q_3 & q_4 \end{pmatrix} \longrightarrow \dots$. As is well known, the corresponding sequence of filling factors will converge to a real number, if we assure that the length $\ell(A_i)$ does not decrease. But this is exactly achieved by the direction of the renormalization group flow. Let us consider a sequence, where even the order $\mathcal{O}(A_i)$ is strictly increasing. Then the sequence of pairs $(\nu_i = p_{i+1}/q_i, \nu_i^* = p_i/q_{i+1})$ converges to $(\nu_\infty = \nu_\infty^*)$ where ν_∞ lies somehow in the intervall $[\nu_1, \nu_1^*]$. Actually, starting with a matrix A_1, all matrices $B = A \cdot A_1$ such that $\mathcal{O}(B) > \mathcal{O}(A_1)$ correspond to filling fractions inside this intervall. Matrices with this property are denoted by $B \succ A_1$. It has been shown in [15] that the set of all such matrices, i.e. of all pairs of compactifications $(p_i/q_i, p_{i+1}/q_{i+1})$ plotted as in figure 1 (but without identfying points via duality), is confined to a band in the following way: Let w^2 be the product of the compactifications, $w^2 = (p_i/q_i)(p_{i+1}/q_{i+1}) = \nu_i \nu_i^*$. Then the orbit of all matrices $B = A \cdot A_1 \succ A_1$ is confined to a hyperbolic band of width $\varepsilon = 2w\Delta\nu + (\Delta\nu)^2$, where $\Delta\nu = |\nu_1 - \nu_1^*|$, defined by the equation $x/y = \alpha$ with $\alpha \in [w^2 - \varepsilon, w^2 + \varepsilon]$.

Let us choose ν_1 and A_1 of order $\mathcal{O}(A_1) = 1$. Then the orbit of all matrices $B \succ A_1$ will intersect the diagonal in figure 1 at a point x near w. The diagonal corresponds to unitary fermionic $c = 1$ CFTs, since $Z_{ferm}[x, x] = Z[2x] + Z[x/2]$. This unitary theory is a never reached fixpoint for a certain sequence of matrices $B \succ A_1$ with $\nu_\infty = x$. Moreover, if we can truncate our $1/N$-expansion to first order, all filling fractions corresponding to matrices $B \succ A_1$ will have the Hall conductivity $\sigma_{xy} = \nu_1$, since ν_1 is the first order continued fraction expansion for all $\nu(B)$, $B \succ A_1$. We call the set of all points $(\nu(B), \nu^*(B))$ to $B \succ A_1$ the attractor band of $(\nu(A_1), \nu^*(A_1))$.

Physically this means the following: Let us start from a QHE state with filling fraction $\nu = [n_1, 2p_1, n_2, 2p_2, \dots, 2p_k]$ and let us first assume that the Hall conductivity is given by a first order effect, $\sigma_{xy} = [n_1, 2p_1, n_2]$. If we increase the external magnetic field very slowly, the filling fraction may change a bit. If the change of the magnetic field is small enough, ν will just change by a small amount coming from minor corrections in the highest orders of its continued fraction expansion, eventually the order itself may change a bit. This corresponds to a reconfiguration of the system and the Lagrangian (2.28) such that again all

flux quanta are bounded to electrons. Since the Hall conductivity is a first order effect, it does not change. Thus, we are moving inside an attractor band of the kind described above.

If now the increment of the external magnetic field is strong enough such that ν changes to a number expressible in a continued fraction expansion, whose *first* orders are different, the Hall conductivity changes. We heve then moved from one first order band to another. How can this happen, if our flow of theories never decreases the order? Firstly, as we have seen for the Laughlin states, there are transitions between fractions of the *same* order, and in this case the first order changed, such that the Hall conductivity changed too. Secondly, if we approach a rational number with a short continued fraction by a sequence of continued fractions of larger length, the system will certainly choose the much simpler inner configuration as soon as it can, since this decreases the number of states. Consideration of a sequence of longer and longer continued fractions assumes at the same time, that the magnetic field is increased slower and slower. But there is a limit, since we either insert at least one flux quantum, or nothing changes. Thus, the general situation will be that the increment of the magnetic field will affect lower orders too, since it is done within a finite time.

figure 2 shows a crude approximation of the attractor bands in a double logarithmic plot (where we inverted the hyperbolas to origin lines, $x = 2R_1^2, y = 1/2R_2^2$, which become parallel due to the logarithmic scales). We calculated the bands only with matrices $A \cdot A_1$ with $\ell(A) \leq 10$, since otherwise the plot would be overcrowded and neighbouring bands would be undistinguishable. One sees, that the bands can have small overlaps which are due to possible transitions which change the first order in the continued fraction. In this figure we only show the bands to the experimentally observed Hall conductivities of first order. There are other observed Hall conductivities, e.g. the both hole QHE states $\frac{4}{11} = [1, 2, 1, 2]$ and $\frac{4}{13} = [-2, 2, -2, 4]$, which are of higher order and have non vanishing longitudinal conductivity. They presumably belong to Hall samples, where the number N of electrons (or holes) is small enough to allow second order effects to contribute.

The attractor bands are defined relatively to a start *pair* of fillings (ν_1, ν_1^*). For a given ν_1 we choose ν_1^* such that the statistics parameters have the smallest possible denominators. Usually, this implies the largest possible $\Delta\nu$ resulting in some overlap of the bands.

6   From Summary to Discussion

The aim of this paper was to introduce a new class of phase transitions in two dimensions and, treated as an example, to explain the main features of the QHE with this kind of transitions. Since this class is defined by general considerations on RCFT, it can be applied to similar phenomena of condensed matter physics

which are essential two-dimensional. The main assumption is a phase diagram whose topological structure is completely determined by an infinite discrete group such as the modular group. In our case the group in question is $\Gamma(2)$.

In a forthcoming work we will study high-T_c superconductivity which is supposed to have a very similar phase diagram as QHE. Due to a possible decoupling of the electron charges from spin we expect that the phase transitions may be described by $N = 1$ supersymmetric extensions of the RCFTs used in this work.

The logic of this paper works with two strategies.

Firstly, following a work of J. Fröhlich and A. Zee [19] showing that attaching flux quanta is an in first order globel Chern-Simons interaction which couples to the overall current of the electrons, we develop a graphical description of Chern-Simons interactions in $1/N$-expansion, which is similar to the Feynman graphs. These graphs give us a simple way to read off the filling factor ν. The macroscopic conductivity observables are obtained by truncation to the maximal macroscopic contributiong order, i.e. for $N \gg 1$ only the first order. We argued that the graph may be viewed as a classical conductor network, since the macroscopic observable currents are subject to classical electrodynamics. Then the topological nature of Chern-Simons-QED shows up in the way that there is no difference between conductance and conductivity. The fact that there exists a first order, i.e. that there is a Chern-Simons term in the Lagrangian which will dominate the large scale physics, is related to the existence of impurities at which the flux quanta are localized.

It depends crucialy on the experimental situation, whether and how many fractional QHE states can be observed between the IQHE states. Typically, the FQHE can only be observed at a tenth of the IQHE temperature range, it needs a stonger magnetic field and a carefully choosen range of the impurity density. But a further improvment of these presumptions seems not to lead to a more refined plateau structure. Nearly all measured FQHE conductivities are of first order, i.e. of the form $\sigma_{xy} = \frac{n}{2pn\pm1} + m$, $n \in \mathbb{N}$, $m,p \in \mathbb{Z}_+$. This supports the theoretical prediction that the Chern-Simons interaction is macroscopically of first order only.

Therefore, we interpret our graphs on one hand classicaly, and then only to first order, and on the other hand as Feynman graphs of a Cherm-Simons-QED. Since the graphs also encode a continued fraction expansion of the filling fraction, we see that higher orders contribute smaller corrections to the filling factor.

Secondly, using the equivalence of $(2 + 1)$-dimensional Chern-Simons theory with chiral $(1+1)$-dimensional CFT, we construct transitions between QHE states by non-unitary RCFTs with $c_{eff} = 1$, which connect two different chiral $c = 1$ CFTs. The structure of the moduli space of these theories incorporates the subgroup $\Gamma(2)$ of the modular group which is in agreement with theoretical results on the phase diagramm of the QHE. Moreover, it predicts certain fixpoints and attractor bands which are related to the observed plateau widths.

Every matrix A defines a hyperbolic band in the moduli space which contains

all points to matrices $B \succ A$. The width of the band is parametrised by $\Delta\nu = |$ $\nu(A) - \nu^*(A) |$, the average opening width of the hyperbolic band by $\nu(A) \cdot \nu^*(A)$. The pairs of filling factors $(\nu(B), \nu^*(B))$ for all matrices $B \succ A$ yield hyperbolic bands contained in the one of A. Mapping all matrices $B = \begin{pmatrix} p' & p \\ q & q' \end{pmatrix} \succ A$ by

$$Q : B = \begin{pmatrix} p' & p \\ q & q' \end{pmatrix} \mapsto \left(\frac{p'}{q'}, \frac{p}{q} \right) \tag{6.1}$$

into the plane fills a hyperbolic band of width ε,

$$\varepsilon \sim 2\bar{\nu}\Delta\nu + (\Delta\nu)^2 \sim 2\sqrt{\nu \cdot \nu^*} \mid \nu - \nu^* \mid + (\nu - \nu^*)^2 . \tag{6.2}$$

Note that in figure 1 the parts of the hyperbolic bands with $x > 1$ or $y > 1$ are folded inside the plot and would appear there as origin lines. In figure 3 we show just one single attractor band, plotted in the way of figure 1.

From our $1/N$ argumentation we conclude that (nearly) all observed Hall conductivities $\sigma_{xy} = [n, 2p, m]$ are given by first order effects. The corresponding matrices $A \in \Gamma(2)$ with $(\nu = \sigma_{xy}, \nu^*)$ with $\Delta\nu$ maximal generate as germs of orbits $B = A' \cdot A$, $A' \in \Gamma(2)$, hyperbolic bands of maximal width. These matrices A correspond in general to transitions which change the first order of the continued fraction expansions, since ν is supposed to be of first order $\Delta\nu$ is choosen maximally. Thus, they define a range of variations of the filling factor due to variation of the external magnetic field, where the Hall conductivity remains constant.

Let the Hall sample be in an arbitrary generic state with an smeared out filling factor $\bar{\nu} \pm \delta\nu$, since there are fluctuations of thermal or quantum mechanical kind, or the external magnetic field fluctuates. The system may no rearrange itself to a state of simpler form, e.g. a state corresponding to a first order graph Lagrangian (2.28), which lies inside the attractor band to the state $\bar{\nu} \pm \delta\nu$. In any case, the system will "cool down" to the simplest possible state within the allowed ν-range. The attractor bands then define equivalence classes of QHE states, between which the system can change without affecting the macroscopic observables. Since the filling factor $\bar{\nu}$ is to first order a linear function of the external magnetic field (similar to the *classical* Hall conductivity), the attractor band widths directly correspond to plateau widths of the quantum Hall conductivity. We have the surprising result that a macroscopic observable is quantized while more quantum like parameters as the filling factor vary more or less smoothly. The smoothness of ν is related to the maximal possible order of the continued fraction. The latter in turn depends directly on N which is supposed to be very large.

Therefore, the plateau widths should be proportional to ε according to (6.2). The experimental data do indeed support this. The plateau width is the larger the smaller the numbers n, p, m in the first order expansion $\sigma_{xy} = [n, 2p, m]$ are. Very small bands correspond to very small plateaux which will naturally be difficult to observe, since they are stable only in a very small range of the magnetic field. If the fluctuations $\delta\nu$ are stronger then the width of the plateau, the system will

switch to a neighbouring plateau of larger width. This explains the dominating of small numbers.

Finally, one can show that the attractor bands of the already experimentally observed Hall conductivities cover the parameter plane almost completely. The bands to the Laughlin states with $\sigma_{xy} = 1/(2p-1)$ have width $4/(4p^2-1)^{3/2} + 4/(4p^2-1)^2$ centered around $\bar{\nu} = (4p^2-1)^{-1/2}$ and thus, do not overlap. Between them there is space for bands with $n > 1$. In fact, we can cover the region $\mathcal{M} = \{(x,y) \in \mathbb{R}^2 \mid x,y > 0,\, xy < 1\}$ with small numbers n, p. This region is shifted by one along the diagonal under the operation of $T : m \mapsto m+1$. Therefore, including particle-hole duality and changing of the direction of the magnetic field, we can cover almost the whole region $\mathcal{M}$ with numbers $p \leq 4, \mid n \mid \leq 8, p \mid n \mid < 8$, which correspond to the experimentally observed Hall conductivities.

Figure 2 shows exactly this in a crude approximation. There seems to be some space left between some bands. This is the case, if bands lie near prominent "forbidden" zones corresponding to even denominator fillings, which are difficult to approximate with matrices of small length (we plotted only up to $\ell \leq 10$), see [15] for details. In fact, even denominator fillings are not observed in ordinary Hall samples. In figure 2 we see such gaps for $\nu = 1/2$ and $\nu = 1/4$. The region around $\nu = 1$ seems to be rather empty, since approximation of the "unitary" line $x = y$ is difficult with low order expansions. In addition figure 2 shows with straight lines the position of that Hall conductivities which should have the next stable plateaux, i.e. which should show up in more precise experiments most likely. From top left to bottom right they belong to the values $\nu \in \{\frac{8}{15}, \frac{7}{15}, \frac{4}{15}, \frac{3}{13}, \frac{2}{11}, \frac{2}{13}, \frac{1}{9}\}$. One sees that they further approch the "forbidden" zones $\nu = \frac{1}{2}$ and $\nu = \frac{1}{4}$.

If second order effects may contribute, the width of the attractor bands is smaller, since $\Delta\nu$ has to be defined appropriately such that the germ matrix corresponds to the maximal change of ν in the second order. Then we naturally will have new gaps where Hall plateaux to second order effects may fit in. It would be worthwile to study under which experimental conditions the Hall sample supports second order QHE states, i.e. under which circumstances the total number of electrons is low enough.

With the model of the FQHE proposed in this work, we have given an unified view of several aspects, Chern-Simons theory, conformal field theory, phase transitions, linked together by the modular group and its action on a new class of non-unitary rational conformal field theories. We used the FQHE as an example for a new class of phase transitions which are related to these non-unitary RCFTs. Within this frame we are able to explain the main features of the FQHE including the plateau widths and the selection of observed plateaux.

The modular group once more showed up in theoretical physics, connecting so different fields as arithmetic and fractal geometry, rational conformal field theory, phase transitions in two dimensions, and – most fascinating – experimentally observable real systems as the QHE.

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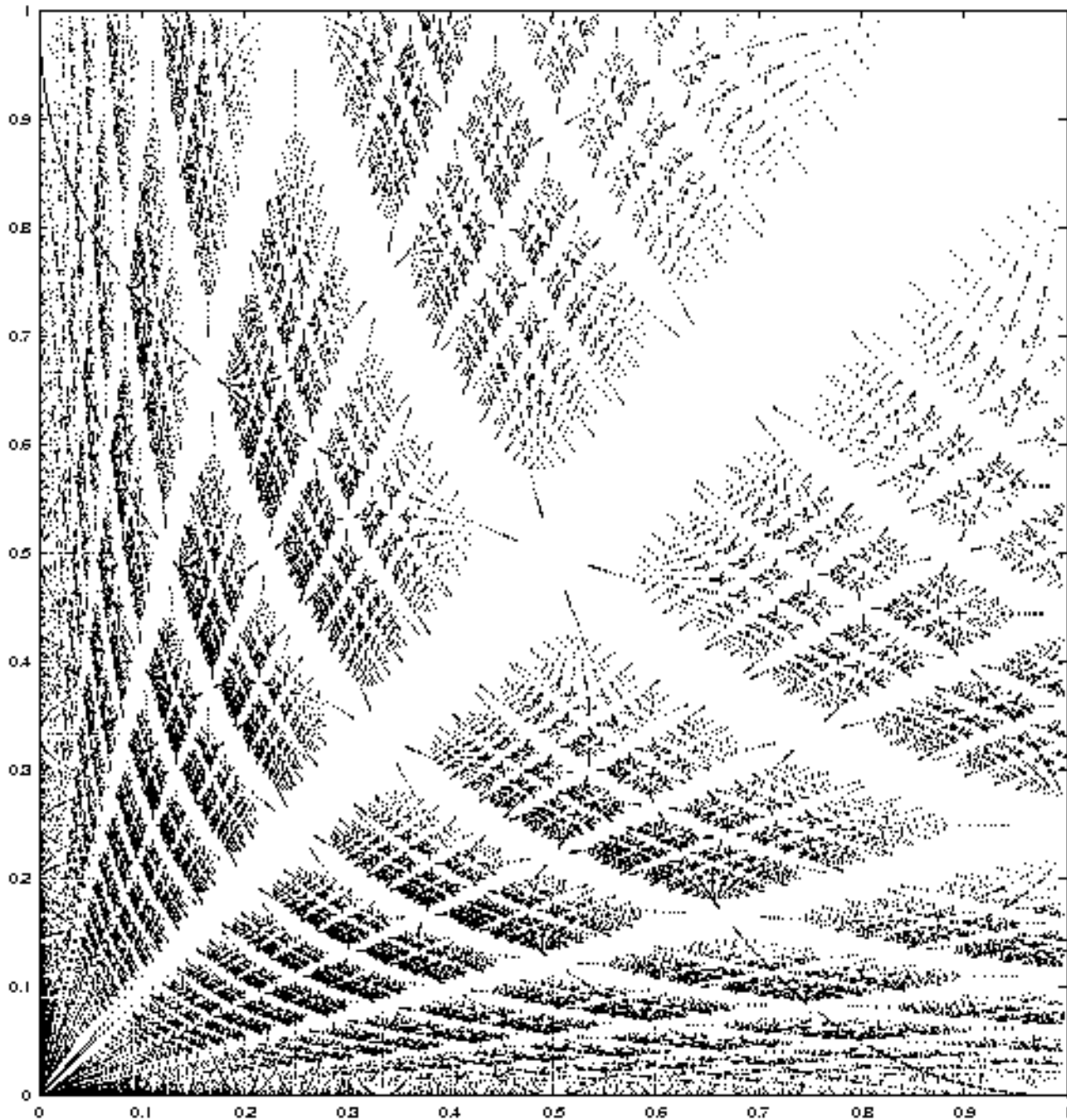[53] E. Witten *Ground Ring of Two-Dimensional String Theory*, Nucl. Phys. **B373** (1992) 187-213

FIGURE 1. The moduli space of the fermionic $c_{eff} = 1$ theories with partition function $Z_{ferm}[p/q, p', q']$. Plotted are all admissible pairs ($x = p/q, y = p'/q'$) of completely odd rational numbers which fulfill $p'q' - pq = 2$. The plot is restricted to $(x, y) \in [0, 1] \times [0, 1]$ since points outside this region are identified via duality of the partition function. The plot is generated from matrices $A \cdot \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, $A \in \Gamma(2)$, with length $\ell(A) \leq 10$.
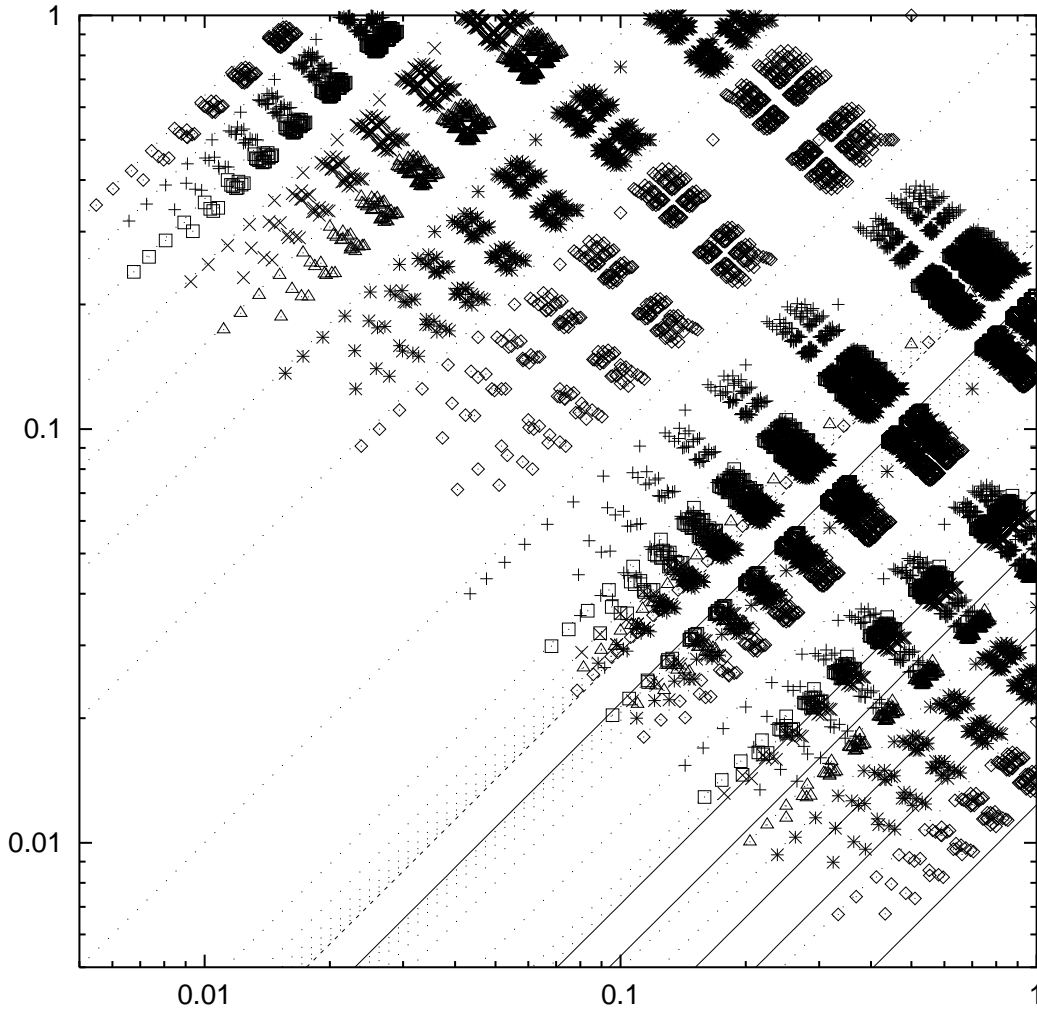
FIGURE 2. Attractor bands of all experimentally observed Hall plateaux of first order. We generated parallel lines by logarithmic scales in order to improve clarity of the plot. Plotted are all points (ν, ν^*) to matrices $A' \cdot A$ with $\ell(A') \leq 10$. Here the start matrix A is choosen as described in the text. From top left to bottom order the different fractions are $\nu \in \{8, 7, 6, 5, 4, 3, 2, 1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \frac{6}{11}, \frac{7}{13}, \frac{6}{13}, \frac{5}{11}, \frac{4}{9}, \frac{3}{7}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7}, \frac{3}{11}, \frac{2}{9}, \frac{1}{5}, \frac{1}{7}\}$, distinguished by the plot symbols.
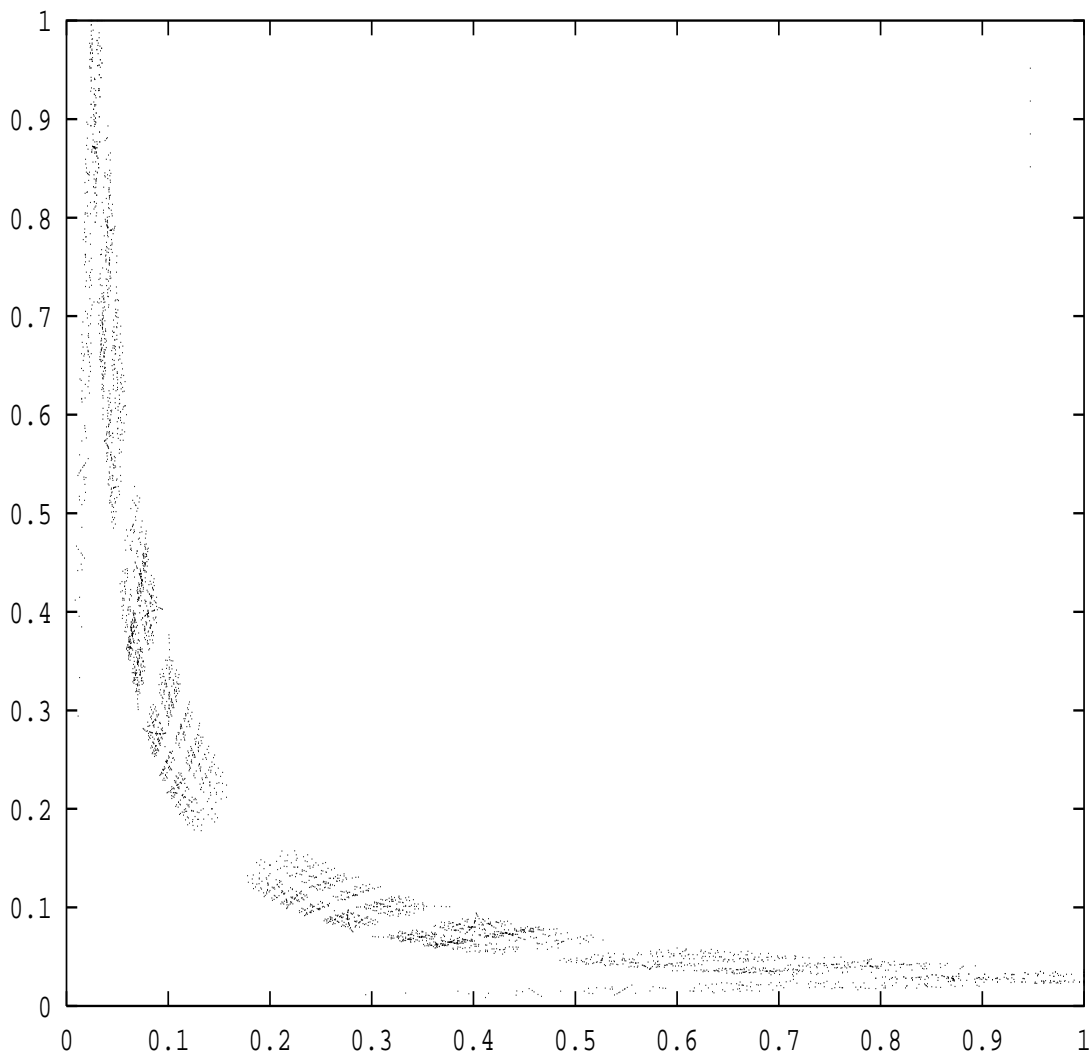
FIGURE 3. Attractor band to the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 5 & 7 \end{smallmatrix}\right)$ which corresponds to all QHE state transitions whose new filling factors lie in $[1/7, 1/5]$. Since nearly all such QHE states will have the Hall conductivity $\sigma_{xy} = 1/5$ (or eventually $1/7$) to first order, this defines a region of variation of the magnetic field, where the Hall conductivity remains constant. The mean slope of the broad lines is ν/ν^* and ν^*/ν respective. The dirty dust inside is an artefact of the used algorithm.

38