The Haldane-Rezayi Quantum Hall State and Conformal Field Theory

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Abstract

We propose field theories for the bulk and edge of a quantum Hall state in the universality class of the Haldane-Rezayi wavefunction. The bulk theory is associated with the $c = -2$ conformal field theory. The topological properties of the state, such as the quasiparticle braiding statistics and ground state degeneracy on a torus, may be deduced from this conformal field theory. The 10-fold degeneracy on a torus is explained by the existence of a logarithmic operator in the $c = -2$ theory; this operator corresponds to a novel bulk excitation in the quantum Hall state. We argue that the edge theory is the $c = 1$ chiral Dirac fermion, which is related in a simple way to the $c = -2$ theory of the bulk. This theory is reformulated as a truncated version of a

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doublet of Dirac fermions in which the $SU(2)$ symmetry – which corresponds to the spin-rotational symmetry of the quantum Hall system – is manifest and non-local. We make predictions for the current-voltage characteristics for transport through point contacts.

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I. INTRODUCTION

In 1987, Willet, *et al.* [1] discovered a fractional quantum Hall plateau with conductance $\sigma_{xy} = \frac{5}{2} \frac{e^2}{h}$. Shortly thereafter, Haldane and Rezayi [2] suggested

$$\Psi_{\mathrm{HR}} \;=\; \mathcal{A} \left(\frac{u_1 v_2 - v_1 u_2}{(z_1 - z_2)^2} \; \frac{u_3 v_4 - v_3 u_4}{(z_3 - z_4)^2} \; \cdots \right) \prod_{i>j}(z_i - z_j)^2 \; e^{-\frac{1}{4\ell_0^2}\sum |z_i|^2} \tag{1}$$

($\mathcal{A}$ means antisymmetrization over all exchanges of electrons, u_i, v_i are respectively up and down spin states of the ith electron, and ℓ_0 is the magnetic length) as a variational *ansatz* for the incompressible state of electrons observed in this experiment.[1] Despite the passage of nearly 10 years, this proposal has been neither confirmed nor ruled out by experiment, in large part because the theoretical understanding of this state is still primitive. In this paper, we try to address this deficiency by proposing effective field theories of the bulk and edge of a system of electrons with ground state given by (1).

Our faith in Laughlin's wavefunctions for the principal odd-denominator states stems not only from their large overlap with the exact ground state of small systems. Rather, their success lies in the fact that they exhibit properties – 'topological ordering' [3,4] – which are plausibly far more robust than the specifics of any trial wavefunction. The 'topological ordering', which refers to the fractional statistics of quasiparticles [5,6] and off-diagonal long-range-order of certain non-local order parameters [4], could be demonstrated for Laughlin's wavefunctions because the plasma analogy facilitates calculations with these wavefunctions. The 'topological ordering' is summarized by the (abelian) Chern-Simons effective field theories of the quantum Hall effect (see also [7–9]). The fractional statistics is the linchpin of the theory and its most startling prediction, and, hopefully, will be confirmed some day soon. The Chern-Simons effective field theory led, in turn, to a conformal field theoretic

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[1](1) is a wavefunction for electrons at $\nu = 1/2$. It is assumed that at $\nu = 5/2$ the lowest Landau levels of both spins are filled and the appropriate analog of (1) in the second Landau level describes the additional $\nu = 1/2$.

description of the edge excitations [10]. Detailed predictions based on the edge theory have recently been spectacularly confirmed [11–14]. Unfortunately, there is no plasma analogy for the Haldane-Rezayi wavefunction, nor for a number of other proposed wavefunctions such as the Pfaffian state. Consequently, the correct Chern-Simons theory of the quasiparticle statistics and the conformal field theory of the edge excitations have remained elusive.

A way of skirting this obstacle was suggested by Moore and Read [15]. They observed that the Laughlin state and a number of other quantum Hall effect wavefunctions, including the Haldane-Rezayi state, could be interpreted as the conformal blocks of certain conformal field theories. This observation gains great power in light of the equivalence, discovered by Witten [16], between the states of a Chern-Simons theory and conformal blocks in an associated conformal field theory. It is often the case that a quantum Hall state can be reproduced by the conformal blocks of a conformal field theory. Since this conformal field theory is equivalent to a Chern-Simons theory, it is very tempting to close the circle, following Moore and Read, and conjecture that this Chern-Simons theory is the desired effective theory of the bulk, which would be obtained by a direct calculation using brute force or some generalization of the plasma analogy. This conjecture is true for states with abelian statistics, such as the Laughlin states and their hierarchical descendents. A general argument in favor of this premise was given in [17], where it was used to deduce the $SO(2n)$ non-abelian statistics of quasiholes in the Pfaffian state once the correspondence between this state and the conformal blocks of the $c = \frac{1}{2} + 1$ conformal field theory was demonstrated in some detail. If correct, this conjecture implies that the conformal blocks are the preferred basis of the quantum Hall wavefunctions since they make the quasiparticle statistics transparent.

It has been observed [15,18,19] that the Haldane-Rezayi state is a conformal block in the $c = -2$ conformal field theory. Here, we derive some consequences of this fact. Among these is the 10-fold degeneracy of the ground state on the torus. The ground state degeneracy on a torus is not merely mathematical trivia. It is equal to the number of 'topologically dictinct' quasihole excitations – ie. that have inequivalent braiding properties (so, for instance, the combination of any excitation with a bosonic excitation does not produce a topologically

distinct excitation) – which there are in the system. As we will explain, the 10-fold degeneracy is a surprise, and is due to the existence of an excitation which cannot be found in other proposed even-denominator quantum Hall states, such as the Pfaffian and $(3, 3, 1)$ states. The 10-fold degeneracy is due, in the $c = -2$ conformal field theory, to the existence of a logarithmic operator [27]. We elucidate the structure of the $c = -2$ theory, with particular emphasis on the calculation of conformal blocks and on the logarithmic operator. The former allow us to obtain the non-abelian statistics of the quasiholes.

In principle, it should be possible to use the Chern-Simons theory of the bulk to deduce the conformal field theory of the edge excitations. However, a more direct approach is possible. As can be explicitly shown for the Laughlin states [20] (see, also, the second ref. in [10]), the states of the edge conformal field theory can be enumerated by direct construction of the corresponding lowest Landau level wavefunctions which are the exact zero-energy eigenstates of certain model Hamiltonians. Under mild asumptions about the confining potential at the edge of the system, which gives these excitations non-zero energy, the energy spectrum can be obtained as well. Milovanovic and Read [19] generalized this construction to the Pfaffian, $(3, 3, 1)$, and Haldane-Rezayi states. In the case of the Pfaffian and $(3, 3, 1)$ states, their construction led immediately to the correct edge theory. We propose that the edge theory of the Haldane-Rezayi state is a theory of a chiral Dirac fermion, with $c = 1$. This theory possesses a global $SU(2)$ symmetry which becomes manifest when recast as a truncated version of a $c = 2$ theory. The $SU(2)$ symmetry – which is just the spin-rotational symmetry, an approximate symmetry in GaAs with its small g factor and effective mass – is unusual in that the local spin-densities do not have local commutation relations (see, also, [40]). This indicates the impossibility of localizing spin at the edge which, we argue, is supported by an analysis of the explicit wavefunctions. The relationship between the $c = -2$ and $c = 1$ theories [28,34,40] – they have nearly the same states, spectra, and, therefore, partition functions – is very natural in this context since these theories describe the bulk and edge of the Haldane-Rezayi state.

Section II is a review of the relevant facts and standard lore regarding the Haldane-Rezayi

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state. In section III we recapitulate, in order to make our exposition as self-contained as possible, the conformal field theory approach to the bulk wavefunctions in the quantum Hall effect. In section IV, we discuss the $c = -2$ conformal field theory and, in section V, we apply it to study quasiparticles in the Haldane-Rezayi state. Section VI is devoted to the relationship between the $c = -2$ and $c = 1$ conformal field theories. The edge theory of the Haldane-Rezayi state is discussed in section VII and the physical consequences following from the results of sections V and VII are discussed in section VIII. Parts of this paper are rather technical. Readers who are uninterested in the subtleties and finer points of the $c = -2$ and $c = 1$ conformal field theories but are interested in the observable consequences which follow from them may wish to skip or merely skim sections III, IV, and VI.

II. THE HALDANE-REZAYI STATE

In this paper, we will be studying the zero-energy eigenstates of the Hamiltonian [2]

$$H = V_1 \sum_{i>j} \delta'(z_i - z_j), \tag{2}$$

where $V_1 > 0$. While this is almost certainly not the Hamiltonian governing any experiment, it has the advantage of tractability, and the properties which interest us are universal and should be stable against perturbations. We have assumed that the Zeeman energy vanishes so (2) is invariant under $SU(2)$ spin rotations. In GaAs, the Zeeman energy is $\frac{1}{60}$ of the cyclotron energy, so $SU(2)$ will be a reasonable approximate symmetry.

This Hamiltonian shares with other simple, short-range lowest Landau level Hamiltonians the property that not only the ground state, but also states with quasiholes and edge excitations, have zero energy. This should not be troubling since the incompressibility of the quantum Hall state depends upon the existence of a discontinuity in the chemical potential. If quasiparticles have finite energy but quasiholes do not, there will be such a discontinuity. For a more realistic interaction, both quasiholes and quasiparticles have finite energy, but the discontinuity persists. Since they are annihilated by the Hamiltonian, the

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multi-quasihole states are easier to enumerate, so in what follows, we will discuss them exclusively; the properties of quasiparticles – though difficult to study directly – are trivially related to those of quasiholes. The vanishing energy of the edge excitations should not be a surprise, either. These have finite energy only if there is a confining potential at the edge of the system, as there will be in any real 2D electron gas. As we discuss further below, we will simply assume that, in the presence of a confining potential, the energy of an edge excitation is proportional to its angular momentum. For these reasons, we will refer to the state with no quasiholes and no edge excitations – the maximally compressed state – as 'the ground state' and refer to the other zero-energy states as 'quasihole states' and 'edge states', respectively, despite the fact that, strictly speaking, all of these states are ground states of (2). The multi-quasihole states, which are bulk excitations, can be distinguished from the edge states by the property that the former must be homogeneous in the z_i's since only such wavefunctions can be extended to the sphere (where there is no edge and, hence, no edge excitations). The inhomogeneous zero-energy wavefunctions are edge states.

As we mentioned above, the ground state of (2) is the Haldane-Rezayi state,

$$\Psi_{\text{HR}} \;=\; \text{Pf}\left(\frac{u_i v_j - v_i u_j}{(z_i - z_j)^2}\right)\;\prod_{i>j}(z_i - z_j)^2\; e^{-\frac{1}{4\ell_0^2}\sum |z_i|^2} \tag{3}$$

where Pf, the Pfaffian, is the square root of the determinant of an anti-symmmetric matrix, or, equivalently, the antisymmetrized product over pairs introduced in (1). It resembles its cousins, the 'Pfaffian' and $(3,3,1)$ states:

$$\Psi_{\text{Pf}} \;=\; \text{Pf}\left(\frac{u_i u_j}{z_i - z_j}\right)\;\prod_{i>j}(z_i - z_j)^2\; e^{-\frac{1}{4\ell_0^2}\sum |z_i|^2} \tag{4}$$

$$\Psi_{(3,3,1)} \;=\; \text{Pf}\left(\frac{u_i v_j + v_i u_j}{z_i - z_j}\right)\;\prod_{i>j}(z_i - z_j)^2\; e^{-\frac{1}{4\ell_0^2}\sum |z_i|^2} \tag{5}$$

The Pfaffian factors are reminiscent of the real space form of the BCS pairing wavefunction. The Haldane-Rezayi state can be interpreted as a quantum Hall state of spin-singlet d-wave pairs while the Pfaffian and $(3,3,1)$ states can be interpreted as spin-triplet p-wave paired states with $S_z = 1$ and $S_z = 0$, respectively. These states are discussed in [21,15,22,23,17,24].

The quasiholes in the Haldane-Rezayi state are, like the vortices in a superconductor, half flux quantum excitations. A wavefunction for a state with two such quasiholes at η_1 and η_2 can be written down by modifying the factor inside the Pfaffian in (3):

$$\Psi = \text{Pf}\left(\frac{(z_i - \eta_1)(z_j - \eta_2) + i \leftrightarrow j}{(z_i - z_j)^2}\ (u_i v_j - v_i u_j)\right)\ \prod_{i>j}(z_i - z_j)^2 \qquad (6)$$

Here, and henceforth, we will be sloppy and omit the Gaussian factor in the wavefunction so as to avoid excessive clutter. The half flux quantum quasiholes have charge $\frac{1}{4}$. As in the case of the Pfaffian and $(3,3,1)$ states, there is not a unique state of $2n$ quasiholes at $\eta_1, \eta_2, \ldots, \eta_{2n}$. Rather, there is a degenerate set of states. This degeneracy is the *sine qua non* for non-Abelian statistics. Consider the four-quasihole case. Define the three polynomials

$$P_\sigma(z_i, z_j) = \left(z_i - \eta_{\sigma(1)}\right)\left(z_i - \eta_{\sigma(2)}\right)\left(z_j - \eta_{\sigma(3)}\right)\left(z_j - \eta_{\sigma(4)}\right)\ +\ i \leftrightarrow j \qquad (7)$$

where σ is a permutation of $\{1, 2, 3, 4\}$. The following four-quasihole states

$$\Psi = \text{Pf}\left(\frac{P_\sigma(z_i, z_j)}{(z_i - z_j)^2}\ (u_i v_j - v_i u_j)\right)\ \prod_{i>j}(z_i - z_j)^2 \qquad (8)$$

are annihilated by (2). These wavefunctions are not all linearly independent. Linear relations, found in [17], reduce the set (8) to a basis set of 2 linearly independent states. There are also states which are not spin-singlets. When there are $2n$ quasiholes, there are 2^{2n-3} linearly independent states; the following particularly elegant basis was found in [24]:

$$\Psi = \mathcal{A}\left(z_1^{p_1}\chi_1 \ldots z_{k-1}^{p_{k-1}}\chi_{k-1} \frac{(u_k v_{k+1} - v_k u_{k+1})\ P_\sigma^{2n}(z_k, z_{k+1})}{(z_k - z_{k+1})^2}\right. \qquad (9)$$
$$\left. \times\ \frac{(u_{k+2} v_{k+3} - v_{k+2} u_{k+3})\ P_\sigma^{2n}(z_{k+2}, z_{k+3})}{(z_{k+2} - z_{k+3})^2}\ \ldots\right) \prod_{i>j}(z_i - z_j)^2$$

where χ_i is the spin wavefunction of the ith electron, $k \leq n$, $p_j \leq n - 2$ and σ is some permutation which is fixed once and for all. The most general multi-quasihole excitation can also have charge $\frac{1}{2}$ 'Laughlin quasiholes' at $\lambda_1, \lambda_2, \ldots, \lambda_l$,

$$\Psi = \mathcal{A}\left(z_1^{p_1}\chi_1 \ldots z_{k-1}^{p_{k-1}}\chi_{k-1} \frac{(u_k v_{k+1} - v_k u_{k+1})\ P_\sigma^{2n}(z_k, z_{k+1})}{(z_k - z_{k+1})^2}\right. \qquad (10)$$
$$\left. \times\ \frac{(u_{k+2} v_{k+3} - v_{k+2} u_{k+3})\ P_\sigma^{2n}(z_{k+2}, z_{k+3})}{(z_{k+2} - z_{k+3})^2}\ \ldots\right) \prod_{i,\alpha}(z_i - \lambda_\alpha)\ \prod_{i>j}(z_i - z_j)^2$$

Although the charge $\frac{1}{2}$ 'Laughlin quasiholes' can be made by bringing together two charge $\frac{1}{4}$ quasiholes, we distinguish them because they do not affect the Pfaffian, or 'pairing,' part of the wavefunction.

It is instructive to consider the Haldane-Rezayi state on a torus. The Hamiltonian (2) no longer has a unique ground state. Its degenerate ground states are:

$$\Psi_{\text{HR}}^{a,b} \;=\; \text{Pf}\left(\frac{(u_i v_j - v_i u_j)\;\vartheta_a(z_i - z_j)\vartheta_b(z_i - z_j)}{\vartheta_1^2(z_i - z_j)}\right)\;\prod_{i>j}\vartheta_1^2\,(z_i - z_j)\;\prod_{k=1}^{2}\vartheta_1\left(\sum_i z_i - \zeta_k\right)\;(11)$$

where ζ_k, $k = 1, 2$ are arbitrary complex numbers. Here $a, b = 2, 3, 4$, but there is a linear relationship between $\vartheta_2^2, \vartheta_3^2, \vartheta_4^2$, so there is a 5-fold degeneracy arising from this choice[2]. There is an additional factor of two from the choice of the ζ_k's, so the total ground state degeneracy is 10 (see, also, [25]). The degeneracy on a torus is not only an important way of distinguishing quantum Hall states found in numerical studies, but also has a simple physical significance. The different degenerate ground states are obtained from each other by creating a quasihole-quasiparticle pair, taking one around a non-trivial cycle of the torus, and annihilating them. There are as many degenerate ground states as there are distinct, non-trivial ways of doing this. In other words, the ground state degeneracy on a torus is equal to the number of *distinct* bulk excitations that the quantum Hall state admits, where *distinct* refers to the braiding properties of the excitations. We will return to this issue in the next section.

As in the case of the bulk excitations, the edge excitations may be naturally divided into a direct product of those which do not affect the pairing part of the wavefunction with those which only affect the Pfaffian factor. The former are generated by multiplying the ground state by symmetric polynomials. They form a $1 + 1$-dimensional chiral bosonic theory with $c = 1$. In a Laughlin state at $\nu = \frac{1}{m}$, these would be sufficient to span the space of edge excitations. In the Haldane-Rezayi state, however, we also have the wavefunctions which

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[2]For the definition of the standard elliptic ϑ-functions and their use in constructing the wave functions on a torus see, for example, ref. [24].

modify the pairing part of the wavefunction [19]. These are closely related to the form (9) of the multi-quasihole wavefunctions

$$\Psi = \mathcal{A}\left(z_1^{p_1}\chi_1 \ldots z_{k-1}^{p_{k-1}}\chi_{k-1}\frac{u_k v_{k+1} - v_k u_{k+1}}{(z_k - z_{k+1})^2}\frac{u_{k+2}v_{k+3} - v_{k+2}u_{k+3}}{(z_{k+2} - z_{k+3})^2} \ldots\right)\prod_{i>j}(z_i - z_j)^2 \tag{12}$$

In these states, $k - 1$ of the electrons are unpaired. There is no restriction on the p_j's, so the unpaired electrons increase the angular momentum (the total number of powers of z) above that of the ground state by $p_1 + 1, p_2 + 1, \ldots, p_{k-1} + 1$, with positive p_i, no more than two of which may be equal (because of the requirements of analyticity and antisymmetry). These electrons have spins $\chi_1, \ldots, \chi_{k-1}$. Dividing the p_i's into those associated with up-spin electrons, n_i's, and those associated with down-spin electrons, m_i's, this sector of the edge theory is composed of states

$$|n_1, n_2, \ldots, n_\alpha; m_1, m_2, \ldots, m_\beta\rangle \tag{13}$$

with $n_i \neq n_j$, $m_i \neq m_j$ if $i \neq j$. These states correspond to wavefunctions

$$\Psi = \mathcal{A}\left(z_1^{n_1}u_1 \ldots z_\alpha^{n_\alpha}u_\alpha \ z_{\alpha+1}^{m_1}v_{\alpha+1} \ldots z_{\alpha+\beta}^{m_\beta}v_{\alpha+\beta}\right. \tag{14}$$

$$\left.\frac{u_{\alpha+\beta}v_{\alpha+\beta+1} - v_{\alpha+\beta}u_{\alpha+\beta+1}}{(z_{\alpha+\beta} - z_{\alpha+\beta+1})^2}\frac{u_{\alpha+\beta+2}v_{\alpha+\beta+3} - v_{\alpha+\beta+2}u_{\alpha+\beta+3}}{(z_{\alpha+\beta+2} - z_{\alpha+\beta+3})^2} \ldots\right)\prod_{i>j}(z_i - z_j)^2$$

and have angular momenta (which, in the $1 + 1$-dimensional edge theory, are just the momenta along the edge)

$$\sum_i (n_i + 1) \ + \ \sum_i (m_i + 1) \tag{15}$$

and

$$S_z = \frac{1}{2}(\alpha - \beta) \tag{16}$$

This is simply a Fock space of spin-$\frac{1}{2}$ fermions. The fermions are neutral, since the number

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of filled fermionic levels can be changed without changing the electron number.[3] If we assume that the energy of a state is proportional to its angular momentum relative to that of the ground state (in general, it will be some function of the angular momentum, which we expand in powers of the angular momentum; the higher powers will be irrelevant, in a renormalization group sense), then the low-energy effective field theory of the edge must be a theory of spin-$\frac{1}{2}$ neutral (and, hence, real) fermions. Since the spin-$\frac{1}{2}$ representation of $SU(2)$ is not a real representation, it is difficult to see what this theory should be. We return to this puzzle in section VII.

The states (15) are the vacuum sector of the edge theory. There are also sectors in which charge has been added to the edge. The wavefunctions for these sectors have fractionally-charged quasiholes in the interior which results in fractional charges being added to the edge. When an integer charge is added to the edge, the vacuum sector is recovered again. When a charge corresponding to half-integral flux is added to the edge, a quasihole of the type (6) is present in the bulk. The edge excitations are still of the form (15), but the angular momenta associated with them are now half-integral, $n_i + \frac{1}{2}$ and $m_i + \frac{1}{2}$. This is a 'twisted sector' [19].

III. CFT FOR THE BULK OF A QUANTUM HALL STATE

The signature of a quantum Hall state is the braiding statistics of the localized excitations, the quasiholes and quasiparticles. Following [6], we would calculate them using the Berry's phase technique, according to which the states $|i\rangle$ transform as

$$|i\rangle \rightarrow \mathrm{P}exp\left(i \oint \gamma_{ij}\right)|j\rangle \tag{17}$$

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[3]This is not quite true. The fermion number is congruent to the electron number modulo 2; see [19] for a dicussion of this point. However, this does not affect the conclusion that the fermionic excitations are neutral since pairs of fermions can be created without changing the charge.

(Pexp is the path-ordered exponential integral) when the αth quasihole, with position η_α, is taken in a loop enclosing others, where

$$\gamma_{ij} = \langle i | \frac{\partial}{\partial \eta_\alpha} | j \rangle \tag{18}$$

The fractional statistics of quasiholes in the Laughlin states were established in this way [6]. However, the matrix elements (18) cannot be directly evaluated for more complicated states such as the Haldane-Rezayi state.

To calculate the braiding statistics of quasiholes in the Haldane-Rezayi state, we will take the approach suggested by Moore and Read [15], which is, essentially, to guess the Chern-Simons effective field theory of this state. To motivate this guess, we look for a conformal field theory which has conformal blocks which are equal to the quantum Hall wavefunctions. As a warm-up, let's see how this works in the case of the Laughlin state at $\nu = \frac{1}{m}$ where the Berry phase calculation can be used as a check for the correctness of this procedure.

The Hamiltonian

$$H = \sum_{k=0}^{m-1} V_k \sum_{i>j} \delta^{(k)}(z_i - z_j) \tag{19}$$

annihilates the Laughlin ground and multi-quasihole (at positions $\eta_1, \ldots, \eta_n$) states.

$$\Psi_{1/m} = \prod_{i>j} (z_i - z_j)^m \tag{20}$$

$$\Psi_{1/m}^{\mathrm{qh}} = \prod_k (z_k - \eta_1) \, \ldots \prod_k (z_k - \eta_n) \prod_{i>j} (z_i - z_j)^m \tag{21}$$

The ground state is equal to the following conformal block in the $c = 1$ theory of a chiral boson, ϕ, with compactification radius $R = \sqrt{m}$:

$$\Psi_{1/m} = \left\langle e^{i\sqrt{m}\phi(z_1)} \, e^{i\sqrt{m}\phi(z_2)} \, \ldots \, e^{i\sqrt{m}\phi(z_N)} \, e^{-i \int d^2z \, \sqrt{m}\rho_0\phi(z)} \right\rangle \tag{22}$$

in which electrons are represented by the operator $e^{i\sqrt{m}\phi}$. The last factor in the correlation function corresponds to a neutralizing background (ρ_0 is the electron density); without it, this correlation function would vanish. Multi-quasihole wavefunctions are obtained by inserting $e^{i\phi/\sqrt{m}}$ in this correlation function:

$$\left\langle e^{\frac{i}{\sqrt{m}}\phi(\eta_1)} \ldots e^{\frac{i}{\sqrt{m}}\phi(\eta_n)} \; e^{i\sqrt{m}\phi(z_1)} \, e^{i\sqrt{m}\phi(z_2)} \ldots e^{i\sqrt{m}\phi(z_N)} \, e^{-i\int d^2z \sqrt{m}\rho_0\phi(z)} \right\rangle \tag{23}$$

$$= \prod_{\alpha>\beta}(\eta_\alpha - \eta_\beta)^{1/m} \prod_{k,\alpha}(z_k - \eta_\alpha) \prod_{i>j}(z_i - z_j)^m \; .$$

As may be seen directly from (20) or (22), the electrons are fermions, as they must be. It may, furthermore, be seen by inspection from (21) or (23) that a phase of $e^{2\pi i} = 1$ is aquired when an electron is taken around a quasiparticle. However, the advantage of the conformal block construction is evident when we turn to the phase aquired when one quasiparticle is taken around another. According to (23), this phase is $e^{2\pi i/m}$. Whereas the η's are merely parameters in an electron wavefunction, so that their braiding properties must be determined from the Berry's phase, the conformal blocks put the η's and z's on an equal footing. *The key conjecture is that the braiding properties of both can be seen by inspection of the conformal blocks* [15,26,17]. These conformal blocks are isomorphic to the states of an abelian Chern-Simons theory which describes these braiding properties

$$\mathcal{L} = ma_\mu \epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda + a_\mu j^\mu \tag{24}$$

where j^μ is the quasihole current and an electron is simply an aggregate of m quasiparticles.

In the $c = 1$ theory, the electron is represented by $e^{i\sqrt{m}\phi}$; the quasihole, by $e^{i\phi/\sqrt{m}}$. The primary fields of the algebra generated by the Virasoro algebra together with the electron operator, i.e. the rational torus, are $1, e^{i\phi/\sqrt{m}}, e^{2i\phi/\sqrt{m}}, \ldots, e^{(m-1)i\phi/\sqrt{m}}$. These operators create excitations consisting of $0, 1, \ldots, m - 1$ quasiholes. The primary fields correspond to the topologically inequivalent, non-trivial excitations at $\nu = 1/m$, since electrons have trivial braiding properties with all excitations. An excitation consisting of $k + m$ quasiholes is equivalent to one comprised of k quasiholes because the additional m quasiholes have no effect on braiding, or, in conformal field theory language, because the former is a descendent of the latter in the rational torus algebra. Similarly, a quasiparticle is equivalent to $m - 1$ quasiholes. The m different inequivalent conformal blocks of the vacuum – which correspond to the degenerate quantum Hall ground states – on the torus are constructed via the Verlinde algebra by creating a pair of conjugate fields, taking one around a loop, and annihilating.

In the case of the Pfaffian state, a correspondence can be made between the ground state and multi-quasihole states and the conformal blocks of the $c = \frac{1}{2} + 1$ conformal field theory. The braiding matrices, which are embedded in a spinor representation of $SO(2n)$, can be obtained from the latter. However, a direct check cannot be done using the plasma analogy to compute the Berry's phase matrix elements; the more indirect arguments of [17] must be used to justify the guess based on conformal field theory. The $c = 1$ part of the theory must be present in any quantum Hall state; it simply 'keeps track' of the charge. In the wavefunction, it yields the Jastrow factors, which determine the filling fraction. In the edge theory, the $c = 1$ sector of the theory describes the surface density waves of an incompressible quantum Hall droplet. The $c = \frac{1}{2}$ part of the theory is responsible for the Pfaffian factor in the wavefunction, and, hence, for the non-Abelian statistics. It also describes the neutral fermionic excitations at the edge of the Pfaffian state.

If we wish to follow the approach outlined in this section to study the Haldane-Rezayi state, we must, first, find a conformal field theory which reproduces this state. As usual, there must be a $c = 1$ sector, which is the theory of a chiral boson, with compactification radius $\sqrt{2}$ as would be expected as $\nu = \frac{1}{2}$. According to [15,18,19], the pairing part of the Haldane-Rezayi ground state is given by a correlation function in the $c = -2$ conformal field theory. We will discuss this at length in the following section, but, for now, we state, without justification, that there are dimension 1 fermionic fields, $\partial\theta_\alpha$, in the $c = -2$ theory and $\langle \partial\theta_\alpha \partial\theta_\beta \rangle = -\epsilon_{\alpha\beta}/z^2$. The electron can be represented as $\Psi^{\text{el}} = \partial\theta_1 e^{i\sqrt{2}\phi} u + \partial\theta_2 e^{i\sqrt{2}\phi} v$ since

$$\langle \Psi^{\text{el}} \Psi^{\text{el}} \ldots \Psi^{\text{el}} \rangle = \langle \partial\theta \ldots \partial\theta \rangle \, \langle e^{i\sqrt{2}\phi} \ldots e^{i\sqrt{2}\phi} \rangle = \text{Pf}\left(\frac{u_i v_j - v_i u_j}{(z_i - z_j)^2} \right) \prod_{i>j} (z_i - z_j)^2 \qquad (25)$$

In the next section, we discuss the $c = -2$ conformal field theory, with an eye towards calculating its conformal blocks. In the following sections, we use these conformal blocks to discuss the bulk excitations of the Haldane-Rezayi state.

IV. CORRELATION FUNCTIONS OF THE $C = -2$ THEORY

The $c = -2$ theory has been extensively studied (refs. [33], [27], [30], [34], [35]). Here we want to give a self-contained account which includes all of the developments relevant to our discussion of the Haldane-Rezayi state. Some of what we present here has not, to our knowledge, been published before.

The $c = -2$ theory can be represented as a pair of ghost fields, or anticommuting fields $\theta, \bar{\theta}$ with the action (ref. [27])

$$S = \int \partial \theta \bar{\partial} \bar{\theta} \tag{26}$$

This action has an $SU(2)$ (actually even an $SL(2, C)$) symmetry which becomes evident if we introduce the 'spin-up' and 'spin-down' fields $\theta_1 \equiv \theta$ and $\theta_2 \equiv \bar{\theta}$ in terms of which the action is

$$S \propto \int \epsilon_{\alpha\beta} \partial \theta_\alpha \bar{\partial} \theta_\beta \tag{27}$$

where ϵ is the antisymmetric tensor. Acting on θ by SU(2) matrices does not change the action. The SU(2) algebra is generated by the SU(2) triplet of generators

$$W_{\alpha\beta} \propto \partial \theta_\alpha \partial^2 \theta_\beta + \partial \theta_\beta \partial^2 \theta_\alpha \tag{28}$$

of dimension 3, which form a $\mathcal{W}$-algebra rather than a Kac-Moody algebra (ref. [33])[4].

The fields θ are complex. Nevertheless writing down the full action

$$S \propto i \int \epsilon_{\alpha\beta} \partial \theta_\alpha \bar{\partial} \theta_\beta - i \int \epsilon_{\alpha\beta} \partial \theta_\alpha^\dagger \bar{\partial} \theta_\beta^\dagger \tag{29}$$

shows that $\theta^\dagger$ decouple from θ and we can consider them independently. If, on the other hand, we include them, the central charge for the theory (29) is $c = -4$. We emphasize that

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[4] As has been noted in a number of publications, the dimension 1 fields $\theta\partial\theta$ have logarithms in their correlations functions and therefore do not form a Kac-Moody algebra.

$\bar{\theta}$ is not a complex conjugate of θ, but is just another field. Alternatively, we could take θ, $\bar{\theta}$ to be real fields with an $SL(2, R)$ symmetry.

To quantize the theory (27) we have to compute the fermionic functional integral

$$\int \mathcal{D}\theta \mathcal{D}\bar{\theta} \exp(-S) \tag{30}$$

We note that computed formally this fermionic path integral is equal to zero due to the "zero modes" or constant parts of the fields θ which do not enter the action (27). To make it nonzero we have to insert the fields θ into the correlation functions (compare with ref. [32]), as in

$$\int \mathcal{D}\theta \mathcal{D}\bar{\theta} \, \bar{\theta}(z)\theta(z) \exp(-S) = 1 \tag{31}$$

Therefore, the vacuum $|0\rangle$ of this theory is somewhat unusual. Its norm is equal to zero,

$$\langle 0|0 \rangle = 0 \tag{32}$$

while the explicit insertion of the fields θ produces nonzero results

$$\left\langle \bar{\theta}(z)\theta(w) \right\rangle = 1 \tag{33}$$

Furthermore, if we want to compute correlation functions of the fields $\partial\theta$ we also need to insert the zero modes explicitly,

$$\left\langle \partial\theta(z)\partial\bar{\theta}(w) \right\rangle = 0, \quad \text{but} \tag{34}$$
$$\left\langle \partial\theta(z)\partial\bar{\theta}(w)\bar{\theta}(0)\theta(0) \right\rangle = -\frac{1}{(z-w)^2}$$

The second correlation function is computed by analogy to the free bosonic field.

From the point of view of conformal field theory, the strange behavior of (32), (33), and (34) can be explained in terms of the logarithmic operators which naturally appear at $c = -2$. As was discussed in [27], the theory $c = -2$ must necessarily possess an operator $\tilde{I}$ of dimension 0, in addition to the unit operator I, such that

$$[L_0, \tilde{I}] = I \tag{35}$$

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(where L_0 is the Hamiltonian). Moreover, it can be proved by general arguments of conformal field theory, such as conformal invariance and the operator product expansion, that the property (35) necessarily leads to the correlation functions (refs. [27] and [31])

$$\langle\, II \,\rangle = 0, \tag{36}$$

$$\langle\, I(z)\tilde{I}(w) \,\rangle = 1,$$

$$\langle\, \tilde{I}(z)\tilde{I}(w) \,\rangle = -2\log(z-w)$$

These relations force us to conclude that the operator $\tilde{I}$ must be identified with the normal ordered product of θ and $\bar{\theta}$ [39],

$$\tilde{I} \equiv - : \theta\bar{\theta} := -\frac{1}{2}\epsilon_{\alpha\beta}\theta_\alpha\theta_\beta \tag{37}$$

The stress energy tensor of the theory (27) is given by

$$T =: \partial\theta\partial\bar{\theta} : \tag{38}$$

and it is easy to see that its expansion with $\tilde{I}$ is indeed given by

$$T(z)\tilde{I}(w) = \frac{1}{(z-w)^2} + \frac{\partial\tilde{I}}{z-w} + \dots \tag{39}$$

The mode expansion of the fields θ has to be written in the form

$$\theta(z) = \sum_{n\neq 0} \theta_n z^{-n} + \theta_0 \log(z) + \xi \tag{40}$$

where ξ are the crucial zero modes (they disappear in the expansion for $\partial\theta$). Here $n \in Z$ in the untwisted sector (ie. with periodic boundary conditions) and $n \in Z + \frac{1}{2}$ in the twisted sector (anti-periodic boundary conditions).

To be consistent with the earlier results (34) and (39) we have to impose the following anticommutation relations (compare with ref. [30])

$$\left\{\theta_n, \bar{\theta}_m\right\} = \frac{1}{n}\delta_{n+m,0}, \ n \neq 0 \tag{41}$$

$$\left\{\theta_0, \bar{\theta}_0\right\} = 0$$

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$$\{\theta_m, \theta_n\} = \{\bar{\theta}_n, \bar{\theta}_m\} = 0$$

$$\{\xi, \bar{\xi}\} = 0$$

$$\{\xi, \bar{\theta}_0\} = 1$$

$$\{\theta_0, \bar{\xi}\} = -1$$

The last two relations are absolutely crucial in keeping (39) intact. The mode expansion θ_n should not be confused with the notations θ_1 and θ_2 introduced earlier. To avoid confusion we will primarily use the $\theta, \bar{\theta}$ notation.

Note that the modes ξ become the creation operators for logarithmic states. Indeed,

$$\theta_n|0\rangle = 0 \text{ for } n \geq 0 \tag{42}$$

and

$$\tilde{I}|0\rangle = \bar{\xi}\xi|\rangle \tag{43}$$

The mode expansion (40) together with (41) and

$$\langle 0|0 \rangle = 0, \ \left\langle \bar{\xi}\xi \right\rangle = 1 \tag{44}$$

can be used to compute any correlation function in the theory.

For instance, we can reproduce the correlation functions of (36)

$$\left\langle I(z)\tilde{I}(w) \right\rangle = \left\langle \bar{\theta}(w)\theta(w) \right\rangle = \left\langle \bar{\xi}\xi \right\rangle = 1 \tag{45}$$

while

$$\left\langle \tilde{I}(z)\tilde{I}(w) \right\rangle = \left\langle : \bar{\theta}(z)\theta(z) :: \bar{\theta}(w)\theta(w) : \right\rangle = \tag{46}$$

$$\left\langle \bar{\xi}\theta(z)\bar{\theta}(w)\xi \right\rangle + \left\langle \bar{\theta}(z)\xi\bar{\xi}\theta(w) \right\rangle = -2\log(z - w)$$

The last line of (46) can be computed either directly in terms of modes or by comparison with (34).

As has been discussed at length in the literature, the fields W introduced in (28) form a $\mathcal{W}$-algebra and in fact all the states of the $c = -2$ theory can be classified according

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to various representations of that algebra. A clear review can be found in ref. [28]. Six representations are listed in that paper. They can easily be represented in terms of the fields of our theory. We have the unit operator I, the logarithmic operator $\tilde{I} = - : \theta\bar{\theta} :$, the SU(2) doublet of dimension 1 fields $\partial\theta$ and $\partial\bar{\theta}$, the twist field μ of dimension $-1/8$, a doublet of twist fields $\sigma_\alpha \equiv (\theta_\alpha)_{-\frac{1}{2}}\mu$ of dimension[5] $3/8$, and finally a structure of fields θ, $\partial\theta$ and $\theta\partial\theta$ connected with each other by the action of the Virasoro generators L_n.

With all the preliminaries completed we can proceed to construct the correlation functions of the fields θ. The correlation function

$$\left\langle \partial\theta(z_1)\partial\bar{\theta}(w_1)\dots\partial\theta(z_n)\partial\bar{\theta}(w_n)\tilde{I} \right\rangle = \sum_\sigma \operatorname{sign}\sigma \prod_{i=1}^{n} \frac{1}{(z_i - w_{\sigma(i)})^2}, \tag{47}$$

where $\sigma(i)$ is the permutation of the numbers 1, 2, ..., n, reproduces the Haldane-Rezayi wave function. Note the explicit insertion of the logarithmic operator $\tilde{I} =: \bar{\theta}\theta :$ to make (47) nonzero. For convenience, we express the correlation functions in this section in 'z-w' notation in which the θ's are at the points z_i and the $\bar{\theta}$'s are at the w_i's. In the next section, we revert to the 'u-v' notation which is better adapted for a discussion of wavefunctions.

The correlation functions in the twisted sector can be found by splitting the logarithmic operator into two twist fields μ according to the general formula (ref. [27])

$$\mu(z)\mu(w) \approx I\log(z - w) + \tilde{I} \tag{48}$$

and is equal to

$$\left\langle \partial\theta(z_1)\partial\bar{\theta}(w_1)\dots\partial\theta(z_n)\partial\bar{\theta}(w_n)\mu(\eta_1)\mu(\eta_2) \right\rangle = \tag{49}$$

$$(\eta_1 - \eta_2)^{\frac{1}{4}} \sum_\sigma \operatorname{sign}\sigma \prod_{i=1}^{n} \frac{(z_i - \eta_1)(w_{\sigma(i)} - \eta_2) + (z_i - \eta_2)(w_{\sigma(i)} - \eta_1)}{(z_i - w_{\sigma(i)})^2\sqrt{(z_i - \eta_1)(z_i - \eta_2)(w_{\sigma(i)} - \eta_1)(w_{\sigma(i)} - \eta_2)}}$$

Note that we do not need the logarithmic operator any more. It has been split into two twist fields. Alternatively, we can say that in the twisted sector the summation in (40) is over half integer numbers and the zero modes no longer enter the expansion for the fields θ.

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[5]$\theta_{-\frac{1}{2}}$ is the mode expansion (40) for θ where $n \in Z + \frac{1}{2}$ to reproduce the twisted sector. The zero modes are naturally absent in that sector.

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Correlation functions of the type (49) are, as we will see below, useful for constructing the bulk excitations. However, the twist fields are not the only way of doing it. We could also split the logarithmic operator according to the operator product expansion

$$\tilde{I}(z)\tilde{I}(w) = -2\log(z-w)\tilde{I} + \ldots \tag{50}$$

which follows from (36). The following correlation function

$$\left\langle \partial\theta(z_1)\partial\bar{\theta}(w_1)\ldots\partial\theta(z_n)\partial\bar{\theta}(w_n)\tilde{I}(u_1)\tilde{I}(u_2) \right\rangle \tag{51}$$

will be needed in the next section. It can be computed by either solving the differential equations of conformal field theory, or by the straightforward mode expansion (40) and (41). Either method results in

$$\left\langle \partial\theta(z_1)\partial\bar{\theta}(w_1)\ldots\partial\theta(z_n)\partial\bar{\theta}(w_n)\tilde{I}(u_1)\tilde{I}(u_2) \right\rangle = \tag{52}$$
$$-2\log(u_1 - u_2)\sum_\sigma \text{sign}\sigma \prod_{i=1}^{n} \frac{1}{(z_i - w_{\sigma(i)})^2} -$$
$$\sum_\sigma \text{sign}\sigma \sum_{k=1}^{n} \left\{ \prod_{i\neq k} \left(\frac{1}{(z_i - w_{\sigma(i)})^2} \right) \frac{(u_1 - u_2)^2}{(u_1 - z_k)(u_1 - w_{\sigma(k)})(u_2 - z_k)(u_2 - w_{\sigma(k)})} \right\}$$

We see that it splits into two terms. One is the product of the Haldane-Rezayi wave function (47) and the logarithm. The other is a nontrivial expression. In fact, it is easy to get rid of the trivial part by taking one of the logarithmic operators to infinity. In doing so we have to remember the transformation law for the logarithmic fields which follows from (35),

$$\tilde{I}(f(z)) = \tilde{I}(z) + \log\left(\frac{\partial f}{\partial z}\right) \tag{53}$$

According to the standard procedure, taking the position of the field $\tilde{I}(z)$ to infinity corresponds to taking the position of the field $\tilde{I}(1/z) = \tilde{I}(z) - 2\log(z)$ to the origin. Therefore the trivial part of (52) disappears.

V. TOPOLOGICAL PROPERTIES OF BULK EXCITATIONS IN THE

HALDANE-REZAYI STATE.

Armed with the preceding results, we can discuss the bulk excitations of the Haldane-Rezayi state. The discussion will be more complicated than but otherwise directly analogous to the discussion of the Laughlin state in section III and of the Pfaffian state in [17].

The primary fields of the $c = -2 + 1$ theory are: $1, e^{i\phi/\sqrt{2}}$, which create the ground state and the Laughlin quasiparticle; $\partial\theta, \partial\theta e^{i\phi/\sqrt{2}}$, which create neutral fermions in the bulk; $\mu e^{i\phi/2\sqrt{2}}, \mu e^{i\phi/2\sqrt{2}+i\phi/\sqrt{2}}, \sigma_\alpha e^{i\phi/2\sqrt{2}}, \sigma_\alpha e^{i\phi/2\sqrt{2}+i\phi/\sqrt{2}}$, which create half flux quantum quasiholes; and $\tilde{I} e^{i\phi/\sqrt{2}}, \tilde{I} e^{i\phi\sqrt{2}}$. Although there are 10 fields, they are not obtained by simply multiplying the 2 primary fields of the $c = 1$ theory with the 5 of the $c = -2$ theory. The last 3 pairs of fields involve particular combinations of fields from the $c = -2$ and $c = 1$ theories. These are necessary to give wavefunctions which are single-valued in the electron coordinates. Milovanovic and Read have shown that this requirement is equivalent to an orbifold construction [19]. These 10 primary fields correspond to the 10 topologically distinct bulk excitations of the Haldane-Rezayi state. The corresponding wavefunctions are $(p, p_\alpha = 0, 1)$:

$$\Psi_I = \mathrm{Pf}\left(\frac{u_i v_j - v_i u_j}{(z_i - z_j)^2}\right) \prod_i (z_i - \eta)^p \prod_{i>j}(z_i - z_j)^2 \tag{54}$$

$$\Psi_{\partial\theta} = \mathcal{A}\left(\frac{\chi_1}{(\eta - z_1)^2} \frac{u_2 v_3 - v_2 u_3}{(z_2 - z_3)^2} \cdots\right) \prod_i (z_i - \eta)^p \prod_{i>j}(z_i - z_j)^2 \tag{55}$$

$$\Psi_\mu = (\eta_1 - \eta_2)^{3/8}\mathrm{Pf}\left(\frac{(u_i v_j - v_i u_j)\left((z_i - \eta_1)(z_j - \eta_2) + i \leftrightarrow j\right)}{(z_i - z_j)^2}\right) \prod_{i,\alpha}(z_i - \eta_\alpha)^{p_\alpha} \prod_{i>j}(z_i - z_j)^2 \tag{56}$$

$$\Psi_\sigma = (\eta_1 - \eta_2)^{19/8}\mathcal{A}\bigg(\frac{(u_1 v_2 + v_1 u_2)(z_1 - z_2)}{(\eta_1 - z_1)(\eta_2 - z_1)(\eta_1 - z_2)(\eta_2 - z_2)} \times \tag{57}$$
$$\frac{(u_3 v_4 - v_3 u_4)\left((z_3 - \eta_1)(z_4 - \eta_2) + 3 \leftrightarrow 4\right)}{(z_3 - z_4)^2} \cdots\bigg)\prod_{i,\alpha}(z_i - \eta_\alpha)^{p_\alpha} \prod_{i>j}(z_i - z_j)^2$$

$$\Psi_{\tilde{I}} = \mathcal{A}\left(\frac{(u_1 v_2 - v_1 u_2)(\eta_1 - \eta_2)^2}{(z_1 - \eta_1)(z_1 - \eta_2)(z_2 - \eta_1)(z_2 - \eta_2)}\frac{u_3 v_4 - v_3 u_4}{(z_3 - z_4)^2} \dots \right) \times \qquad (58)$$
$$\prod_{i,\alpha}(z_i - \eta_\alpha)^{p_\alpha+1} \prod_{i>j}(z_i - z_j)^2$$

The states (55), (57) are not legitimate lowest Landau level wavefunctions. However, they have the correct braiding properties, and should be thought of as shorthand for the correct wavefunctions which can be constructed along the lines of the neutral fermion wavefunctions given in [22]. If this is done for the state (57), it will vanish, but when there are more quasiholes, there are non-trivial conformal blocks with σ's which are different from those with μ's (see below). In (56)-(58), we have created pairs of excitations, as must be done on the sphere. In the plane, single excitation wavefunctions can be obtained by taking $\eta_2 \to \infty$.[6] The excitations given by (54)-(57) have analogs in other paired states. However, (58), which raises the ground state degeneracy on the torus to 10, rather than 6 or 8 as it is in the Pfaffian and $(3,3,1)$ states, is new. There is an analogous wavefunction in the $(3,3,1)$ state which, as in the Haldane-Rezayi state, can result from bringing together (i.e. fusing) two half-flux quantum quasiholes. However, it has trivial braiding properties, and therefore does not contribute to the ground state degeneracy of the $(3,3,1)$ state on the torus. In the Haldane-Rezayi state, however, the $\tilde{I}$ excitation (58) braids non-trivially with the half-flux quantum quasiholes.

Non-Abelian statistics first raises its head when there are four quasiholes. Unfortunately, we cannot explicitly calculate the corresponding conformal blocks, which would require calculating $\langle \mu\mu\mu\mu\partial\theta \dots \partial\theta \rangle$ and similar conformal blocks with more twist fields. Since $\mu\mu \sim$

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[6]As a general rule, conformal field theory imposes requirements such as charge neutrality, flux quantization, etc. which must be satisfied by wavefunctions on the sphere. These can be relaxed on the plane by taking some of the quasiparticles to infinity. The conformal blocks must also be spin-singlets, which means that they must be invariant under rotations of the σ_α's and the $\partial\theta_\alpha$'s. Taken as electron wavefunctions, however, they need not be singlets under rotations of the electron spins alone.

$I + \tilde{I}$, there are 2^{n-1} conformal blocks with $2n$ μ's (and any number of $\partial\theta$'s). To count the other $2n$ quasihole states, we have to count all conformal blocks with $2n$ fields which can be either μ or σ_α. Recall that σ_α is obtained by fusing μ and $\partial\theta_\alpha$. Hence, if we call a half flux quantum quasihole operator s^a, which could be either $s^0 = \mu$ or $s^{\pm\frac{1}{2}} = \sigma_\alpha$, and if we further write $I^a, \tilde{I}^a$ to denote members of the conformal families of $I, \tilde{I}$ for $a \in Z$, and of $\partial\theta_\alpha, \partial\theta_\alpha\tilde{I}$ (the spin doublet of conformal weight $h = 1$) for $a \in Z + \frac{1}{2}$, we can collect all fusion rules in the compact form $[s^a] \times [s^b] = [I^{a+b}] + [\tilde{I}^{a+b}]$ and $[s^a] \times [I^b] = [s^a] \times [\tilde{I}^b] = [s^{a+b}]$. Conformal blocks as $\langle s^{a_1}(z_1)s^{a_2}(z_2)\ldots s^{a_{2n}}(z_{2n})\rangle$ have an essentially predetermined form with some straighforward products of powers of $(z_i - z_j)$ and products of certain functions depending an all possible crossing ratios. However, these functions depend only on the conformal weights of the fields in the correlator and the internal channels of the conformal block, not on the spin indices directly.

Let us assume momentarily that we work in a basis where the metric on the internal channels is diagonal, so that we don't have to think about additional indices for the endpoints of internal propagaters. What this means is that we only have to keep track of the spin indices modulo integers. Then, counting conformal blocks is very simple. With $2n$ fields $s^{a_i}(z_i)$ in a correlator we have $(n-1)$ internal chanels of type I^a or $\tilde{I}^a$, the other internal chanels being of type s^a. Since each of the former internal chanels can be either I^a or $\tilde{I}^a$, we have 2^{n-1} possible choices. Further, each such internal channel has $a \equiv 0$ modulo integers, or $a \equiv \frac{1}{2}$ modulo integers (the outer chanels, i.e. fields, appropiately chosen). Due to the overall condition that in total we need spin 0 fixes the $(n-1)$-th internal chanel, if the other (n-2) are chosen. So we get in total $2^{n-1}2^{n-2} = 2^{2n-3}$ possible conformal blocks.

We turn now to the monodromy matrices which are generated when quasiholes are taken around one another. Consider the simplest non-trivial case, with four quasiholes. The two degenerate states can be obtained from the conformal blocks of $\langle \mu\mu\mu\mu\partial\theta\ldots\partial\theta\rangle$ (correlation functions with some of the μ's replaced by σ's have identical conformal blocks in the four-quasihole case, which is the simplest instance of the above reduced degeneracy). Even in

the absence of the explicit forms of these conformal blocks, the monodromy matrices can be obtained from the differential equations which the conformal blocks satisfy (they are the equations for the full elliptic integrals, see [27]). Apart from a trivial phase $e^{-\pi i/4}$ which arises from the chiral boson sector of the theory, the monodromies are:

$$\begin{pmatrix} 1 & 0 \\ -2i & 1 \end{pmatrix} \tag{59}$$

when η_1 is taken around η_2,

$$\begin{pmatrix} 1 & -2i \\ 0 & 1 \end{pmatrix} \tag{60}$$

when η_1 is taken around η_4, and

$$\begin{pmatrix} -3 & 2i \\ 2i & 1 \end{pmatrix} \tag{61}$$

when η_1 is taken around η_3. $\eta_1, \eta_2, \eta_3, \eta_4$ are the positions of the four quasiholes and the matrices refer to the preferred basis of four-quasihole states which is given by the conformal blocks (which, again, we are unable to obtain explicitly). The most salient property of these matrices is that they are not unitary. However, they are unitary with respect to the indefinite metric

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{62}$$

It is worth noting that this metric is precisely the metric on the space of 4-point conformal blocks which is necessary to obtain single-valued correlation functions from the left and right conformal blocks (see [27]). It is natural to conjecture that these $SU(1,1)$ monodromy matrices are the analytic continuation to this indefinite metric of the correct $SU(2)$ monodromy matrices of the Haldane-Rezayi quasiholes, which must be unitary with respect to the definite metric of the Hilbert space of lowest Landau level states. This analytic continuation may be done for (59)-(61) but it is not enlightening. In [17], it was found that the monodromy matrices of the Pfaffian state are given by certain $SO(2n)$ rotations in their spinor representation. It is plausible that the non-abelian statistics of the $2n$ quasihole states of the

Haldane-Rezayi state is given by a reducible representation of some group G (and, hence, of the braid group) because states of different spins will not mix. One possibility is that the states can be grouped into a direct sum of the $SO(2n)$ or $SU(n)$ irreducible representations into which a product of $SO(2n)$ spinor representations may be decomposed.

VI. RELATIONSHIP BETWEEN $C = -2$ AND $C = 1$ THEORY.

A number of papers have established a relationship between the the partition functions of $c = -2$ and $c = 1$, $R = 1$ theories (ie. the $c = 1$ theory of a Dirac fermion or a free boson with compactification radius $R = 1$) [34], [28]. While the proof requires a careful construction of $c = -2$ characters taking into account the presence of the logarithmic operators (ref. [34]), there is a way to roughly understand the relationship in rather simple terms (see, especially, [40]).

One can easily check that for each operator of dimension $h_{c=-2}$ in the $c = -2$ theory there is an operator of dimension $h_{c=1}$ in the $c = 1$, $R = 1$ theory such that

$$h_{c=-2} - \frac{-2}{24} = h_{c=1} - \frac{1}{24} \tag{63}$$

Therefore, in the theory on the cylinder, where the partition function is computed, and where the edge theory lives, their zero-point energies are the same.

Moreover, for each descendant state in the $c = -2$ theory there is a corresponding state in the $c = 1$, $R = 1$ theory. Indeed, take the latter theory as represented by a Dirac fermion

$$S = \int \psi^\dagger \bar\partial \psi \tag{64}$$

The modes $\psi_n^\dagger$ and ψ_{-n}, $n > 0$ can be used to construct descendant states,

$$\psi_{n_1}^\dagger \ldots \psi_{n_\alpha}^\dagger \, \psi_{-m_1} \ldots \psi_{-m_\beta} \, |0\rangle \tag{65}$$

which have the same energies and momenta as the states created by θ_{-n} and $\bar\theta_{-n}$,

$$\theta_{-n_1} \ldots \theta_{-n_\alpha} \, \bar\theta_{-m_1} \ldots \bar\theta_{-m_\beta} \, |0\rangle \tag{66}$$

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The mode expansion of the field ψ (on the plane) is

$$\psi(z) = \sum_n \psi_n z^{-n-\frac{1}{2}} \qquad (67)$$

so the Dirac fermion has half-integral momenta in the untwisted sector and integral momenta in the twisted sector, while the opposite is true for the $c = -2$ theory. Therefore, we map the twisted sector of $c = -2$ into the untwisted one of $c = 1$ and *vice versa*. (See [40] for a more detailed discussion of the mapping at the level of the mode expansions.)

Of course, there is still a question of the zero modes ξ; they do not seem to correspond to anything in $c = 1$. However, the zero modes ξ are rather special fields. They must be present as out- (or in-) states of the $c = -2$ theory to make the correlators of the theory nonzero, and they have to be present only once (since $\xi^2 = 0$).

As far as many of their conformal properties are concerned, the theories of anticommuting bosons and of Dirac fermions are the same on the cylinder. Their respective vacua have the same energy and for each operator of $c = -2$ theory there exists an operator at $c = 1$, $R = 1$. However, the higher modes of the energy-momentum tensor are different in the two theories. Consequently, the Virasoro algebra representations are different; in the $c = 1$ theory, they are unitary while in the $c = -2$ theory they are non-unitary.

To understand the equivalence of the $c = -2$ and $c = 1$ theories better, let us take a closer look at the sectors of the $c = -2$ theory. Ordinarily, each primary field $\phi(z)$ of a conformal field theory and all its descendants generate a highest weight representation of the Virasoro algebra, perhaps with a chiral symmetry algebra (see rf. [37]). Moreover, that representation is irreducible. We achieve its irreducibility by removing all the descendants of the primary field which are themselves primary operators. The Hilbert space of a conformal field theory can then be written as a direct sum over irreducible highest weight representations.

The problem we immediately encounter in the $c = -2$ theory is that its states do not constitute ordinary irreducible highest weight representations of the maximally extended chiral symmetry algebra ($\mathcal{W}$-algebra) or even of the Virasoro algebra itself. We know that we have a state there, $|\tilde{I}\rangle$, such that $L_0|\tilde{I}\rangle = |0\rangle$, $|0\rangle$ being the vaccuum while $|\tilde{I}\rangle = \tilde{I}(0)|0\rangle$.

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$|\tilde{I}\rangle$ and $|0\rangle$ have to be considered together, and together they are usually referred to as a reducible but indecomposable representation (ref. [29]). Indeed, we can certainly reduce it by considering a subset of it, consisting of $|0\rangle$ and its descendants only, without $|\tilde{I}\rangle$. However, we cannot decompose it into a direct product of $|0\rangle$ and $|\tilde{I}\rangle$ as $L_0|\tilde{I}\rangle = |0\rangle$.

According to [28], [29], [36] there are four sectors generated by operators with $h_{c=-2} \in \{-\frac{1}{8}, 0, \frac{3}{8}, 1\}$, denoted $\mathcal{V}_{-1/8}, \mathcal{R}_0, \mathcal{V}_{3/8}, \mathcal{R}_1$ in the following (we use the notation of [28]) and the characters of these representations are

$$\chi_{\mathcal{V}_{-1/8}} = \frac{1}{\eta(\tau)}\Theta_{0,2}(\tau) \tag{68}$$

$$\chi_{\mathcal{V}_{3/8}} = \frac{1}{\eta(\tau)}\Theta_{2,2}(\tau)$$

$$\chi_{\mathcal{R}_0} = \chi_{\mathcal{R}_1} = \frac{2}{\eta(\tau)}\Theta_{1,2}(\tau)$$

where $\eta(\tau) = q^{1/24}\prod_{n>0}(1-q^n)$ is the Dedekind eta function, $\Theta_{\lambda,k} = \sum_{n \in Z} q^{(2kn+\lambda)^2/4k}$ are ordinary Theta functions, and $q = \exp(2\pi i \tau)$ is the modular parameter of the torus.

Note the multiplicity of 2 in the last two characters. It forces an overall multiplicity of 4 in the diagonal partition function to ensure modular invariance[7]

$$Z_{c=-2} = |\chi_{\mathcal{R}_0}|^2 + |\chi_{\mathcal{R}_1}|^2 + |2\chi_{\mathcal{V}_{-1/8}}|^2 + |2\chi_{\mathcal{V}_{3/8}}|^2 = 4Z_{c=1}(R=1) \tag{69}$$

such that equivalence of the partition functions of the $c = -2$ theory and the $c = 1$ theory is really established only up to a factor of 4. Moreover, there is no way to avoid the multiplicities of the $\mathcal{V}_{-1/8}$ and $\mathcal{V}_{3/8}$ representations. The overall multiplicity of 4 stems from

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[7]Moular invariance of the torus partition function of a conformal field theory is an important requirement for consistency. In the context of the theory of the bulk of a quantum Hall state, it is just the statement that the theory, when put on the torus, should be independent of the coordinate system on the torus. It has been proven [38] that conformal invariance of a theory on S^2 implies modular invariance on the torus, if L_0 is diagonal. This should extend to the case of logarithmic conformal field theory by the limiting procedure described in [34].

the zero modes $\xi, \bar{\xi}$. It turns out that both indecomposable representations are formed out of four subsectors according to the four possible ways to distribute these zero modes. However, the combinatorics of the subsectors falls into just two different types which coincide with the combinatorics of the irreducible subrepresentations of $\mathcal{R}_0$ and $\mathcal{R}_1$, called $\mathcal{V}_0$ and $\mathcal{V}_1$ respectively. Their characters are [34], [28]

$$
\begin{align}
\chi_{\mathcal{V}_0} &= \frac{1}{2\eta(\tau)} \left(\Theta_{1,2}(\tau) + \eta^3(\tau) \right) \tag{70} \\
\chi_{\mathcal{V}_1} &= \frac{1}{2\eta(\tau)} \left(\Theta_{1,2}(\tau) - \eta^3(\tau) \right)
\end{align}
$$

and each of these two sector types appears twice in each of the indecomposable representations. We thus conclude that the partition functions consists of four copies of the $c = 1$ Dirac fermion partition function, one for each possible combination of the $\xi, \bar{\xi}$ zero modes.

Although we don't need to take this multiplicity into account on the $c = 1$ side, because there everything factorizes, this multiplicity is intrinsic on the $c = -2$ side due to the fact that some representations are indecomposable. However, there are some disadvantages with this approach to the $c = -2$ theory: The modular behavior of the characters (68) is ambigous due to the equivalence of $\chi_{\mathcal{R}_0}$ and $\chi_{\mathcal{R}_1}$. Moreover, the S-matrix for the modular transformation $S : \tau \mapsto -1/\tau$ does not reproduce the correct fusion rules via the Verlinde formula.

In [34] it was attempted to overcome these difficulties by using the fact that the $\xi, \bar{\xi}$ zero modes are necessary to make any n-point function non-zero. That means that there is a way to partially factorize the untwisted part of the partition function by splitting each indecomposable representation into its irreducible subrepresentation and the part with the opposite θ-fermion number (the total fermion number including the ξ zero mode is always even in $\mathcal{R}_0$ and odd in $\mathcal{R}_1$). The result is

$$
\tilde{Z}_{c=-2} = |\chi_{\mathcal{V}_{-1/8}}|^2 + |\chi_{\mathcal{V}_{3/8}}|^2 + \left(\chi_{\mathcal{V}_0} \chi_{\mathcal{W}_0}^* + \chi_{\mathcal{V}_1} \chi_{\mathcal{W}_1}^* + c.c. \right) = Z_{c=1}(R = 1) \tag{71}
$$

where $\chi_{\mathcal{W}_0} = \chi_{\mathcal{W}_1} = \Theta_{1,2}(\tau)/\eta(\tau)$. The non-diagonal structure precisely resembles the non-

diagonal structure of the conformal blocks necessary in the $c = -2$ theory to ensure crossing symmetry and single valuedness of the four point function, see [27].

We conclude by mentioning that this partition function is certainly modular invarinat, but the set of characters $\{\chi_{\mathcal{V}_{-1/8}}, \chi_{\mathcal{V}_{3/8}}, \chi_{\mathcal{V}_0}, \chi_{\mathcal{W}_0}, \chi_{\mathcal{V}_1}, \chi_{\mathcal{W}_1}\}$ is not. One of the results of [34] is that by introducing a regularizing term $\pm i\alpha \log(q)\eta^3(\tau)$ into $\chi_{\mathcal{W}_0}, \chi_{\mathcal{W}_1}$, one recovers modular covariance for the characters. However, the physical meaning of a $\log(q)$ term in character functions remains unclear. As long as α is taken non-zero, one has a well-defined S-matrix which can be used to calculate fusion coefficients via the Verlinde formula. As shown in [34], the latter have physical meaning only in the limit $\alpha \to 0$ and coincide then with explicit results.

VII. EDGE THEORY OF THE HALDANE-REZAYI STATE

The preceeding considerations inspire us to hope for the following happy ending to our story: the neutral sector of the low-energy edge theory of the Haldane-Rezayi state is a $c = 1$ Dirac fermion.

How can we show that this assertion is correct? Since a quantum-mechanical theory is defined by its Hilbert space of states, inner product, and algebra of observable operators, we must show that these structures are identical for the $c = 1$ theory and the edge excitations annihilated by the Hamiltonian (2). Clearly, the Hilbert spaces, (13) and (65), are the same.[8] The spectra (assuming that the energy is proportional to the angular momentum, as before) and, hence, the partition functions are, as well. Of course, the same may be said for the $c = -2$ theory (ignoring subtleties associated with the zero modes, $\xi, \bar{\xi}$), as we discussed in

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[8]Almost. The twisted sector of the $c = 1$ Dirac theory has 2 zero modes, while the edge excitations of the Haldane-Rezayi state begin at angular momentum 1, ie. $k = 2\pi/L$. This zero mode must be projected out of the theory, which can be done very naturally in the truncation from a $c = 2$ theory described below.

the previous section. The observables – such as the local energy and spin densities – and the inner product must distinguish the correct edge theory. However, these are difficult to calculate.

In trying to calculate the inner products of the edge excitations (15), we run into a familiar roadblock: in the absence of a plasma analogy, there is no painless way of doing this calculation. This complicates matters when we turn to the algebra of observables, because we are interested in these operators *projected into the low-energy subspace.* If this were simply a matter of projecting into the lowest Landau level, it would be no problem. However, we must project into the zero-energy subspace of the Hamiltonian (2), since this is the subspace which contains the low-energy edge excitations. If we simply act on a edge excitation with an operator such as the lowest Landau level projected density operator, the resulting state will be in the lowest Landau level, but it will no longer be annihilated by the Hamiltonian (2). Hence we must project the resulting state into the space of edge excitations annihilated by (2). This projection cannot be performed without a knowledge of the inner products of states, so we are stuck again.

Ordinarily, this would not worry us too much because the commutator algebra of the resulting projected operators would be more or less canonical and easily guessed. However, in the case of the Haldane-Rezayi state, the $SU(2)$ spin symmetry must be realized in an unusual way because the edge theory contains two real, i.e. Majorana, fermions, say $\psi_1(x)$ and $\psi_2(x)$. Their Lagrangian is invariant under the $O(2)$ rotations $\psi_i{}' = O_{ij}\psi_j$. There is no local[9] $SU(2)$ transformation law which preserves the reality property of the Majorana spinors. The simplest way of having an $SU(2)$ doublet of fermions is to have two complex, i.e. Dirac, spinors, χ_i, which transform as $\chi_i{}' = U_{ij}\chi_j$. However, since a single Dirac spinor is composed of two Majorana spinors, such a theory will have too many states at each energy level. Since the $SU(2)$ symmetry cannot be realized in the standard way, the algebra of the

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[9]i.e. so that $\psi_i{}'(x)$ depends only $\psi_j(x)$ and not on $\psi_j(x')$ for $x' \neq x$

spin-densities in the $c = 1$ theory can be and – as we will see momentarily – is anomalous.

First, however, we must answer a more basic question: if, as we have conjectured, the $c = 1$ Dirac theory is the correct edge theory, where is the $SU(2)$ symmetry? The answer is that the symmetry is hidden and non-local. The Dirac theory has the Hamiltonian

$$H = \sum_k vk \, \psi_k^\dagger \psi_k \tag{72}$$

where ψ is a complex chiral fermion or, equivalently, two real fermions, ψ_1, ψ_2, with $\psi = \psi_1 + i\psi_2$, and v is the (non-universal) velocity of the neutral fermions. The generators of the $SU(2)$ symmetry are:

$$S^z = \sum_k \psi_k^\dagger \psi_k \tag{73}$$
$$S^+ = \sum_{k>0} \psi_k^\dagger \psi_{-k}^\dagger$$
$$S^- = \sum_{k>0} \psi_{-k} \psi_k$$

These generators commute with the Hamiltonian and, thus, generate a global $SU(2)$ symmetry of the theory. These symmetry generators were constructed in [40].

In mapping the neutral sector of the edge theory onto the $c = 1$ Dirac fermion, we associate the up-spin neutral fermions with the particles, created by $\psi_k^\dagger$ with $k > 0$. The down-spin neutral fermions are associated with the anti-particles, created by ψ_{-k} ($k > 0$). The $SU(2)$ symmetry of the theory rotates up-spin fermions into down-spin fermions, i.e. it mixes particles and anti-particles. As a result, the transformation law is not local.

We can reformulate the Dirac theory in such a way that this $SU(2)$ symmetry is more transparent. We map $\psi_k \to \chi_{1k}$ and $\psi_{-k}^\dagger \to \chi_{2k}$ where $k > 0$. The fields χ_α form an $SU(2)$ doublet with Hamiltonian

$$H = \sum_{k>0} vk \, \chi_{ik}^\dagger \chi_{ik} \tag{74}$$

and symmetry generators

$$S^a = \sum_{k>0} \chi_{\alpha k}^\dagger \, \sigma_{\alpha\beta}^a \, \chi_{\beta k} \tag{75}$$

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As a result of the restriction to $k > 0$, they are 'half' of an $SU(2)$ doublet of Dirac fermions. The $k < 0$ part of the theory has been discarded. It then follows that there is no local $SU(2)$ Kac-Moody algebra. If we introduce local spin densities, $S^a(x)$ and their Fourier transforms,

$$S_q^a = \sum_{k>0} \chi_{\alpha k+q}^\dagger \sigma_{\alpha\beta}^a \chi_{\beta k} \tag{76}$$

we find that their commutators do not close because the sums over k are restricted to $k > 0$. In particular, their commutators are not local, i.e. $[S^a(x), S^b(x')] \propto 1/(x-x')$ rather than $[S^a(x), S^b(x')] \propto \delta(x-x')$.

This might appear to be a death blow to the $c = 1$ theory of the neutral sector. In the underlying quantum mechanics of electrons, these commutators are local, i.e. proportional to δ-functions, so we would expect that in the low-energy theory they would be, at worst, δ-functions smeared out at the scale of the cutoff. However, this argument is a bit too quick. The cutoff in this theory is $O(V_1)$ (see (2)) meaning that our edge theory is an effective field theory for energies less than $O(V_1)$. However, unlike in a Euclidean or relativistic theory, this energy scale does not imply a length scale. While the theory must be local in time (again, modulo non-localities at scales smaller than the cutoff), it does not necessarily have commutators which are local in space.

But do the spin-densities, projected into the low-energy subspace actually have such a non-local algebra? If not, the $c = 1$ theory must be ruled out. If so – and, as we argued above this would not contradict any fundamental principle which is dear to our hearts – then the $c = 1$ theory is a viable candidate to describe the neutral sector of the edge of the Haldane-Rezayi state. Consider the following state, where $\mathcal{P}_H$ is the projection operator into the zero-energy subspace of (2):

$$\mathcal{P}_H \, S^+(w) \, \mathcal{P}_H \; \cdot \; \Psi_0 = \tag{77}$$
$$\mathcal{P}_H \, \mathcal{A} \left(e^{(2wz_1 - |w|^2 - |z_1|^2)/4\ell_0^2} \frac{u_1 u_2}{(w - z_2)^2} \frac{u_3 v_4 - v_3 u_4}{(z_3 - z_4)^2} \; \cdots \right) \prod_{i>1}(w - z_i)^2 \prod_{k>l>1}(z_k - z_l)^2 \, e^{-\frac{1}{4\ell_0^2}\sum_{i>1}|z_i|^2}$$

which results from acting on the ground state with the local projected $S^+(w)$ operator. It is quite plausible that the right-hand-side vanishes upon projection. This would agree with the

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$c = 1$ theory, where $S^+(x)|0\rangle = \sum_{k>0,q} e^{iqx} \chi^\dagger_{\alpha k} \sigma^+_{\alpha\beta} \chi_{\beta k}|0\rangle = 0$ because $\chi_{\beta k}|0\rangle = 0$ for $k > 0$. For a doublet of Dirac fermions (and presumably for any other theory with a local $SU(2)$ transformation law), on the other hand, the $k < 0$ modes will give a non-zero contribution. Furthermore, suppose we act with this operator on a state with 1 neutral fermion:

$$\mathcal{P}_H \, S^+(w) \, \mathcal{P}_H \; \cdot \; \mathcal{A}\left(z_1^k v_1 \, \frac{u_2 v_3 - v_2 u_3}{(z_2 - z_3)^2} \; \cdots \right) \prod (z_i - z_j)^2 = \tag{78}$$

$$\mathcal{P}_H \mathcal{A}\left(e^{(2wz_1 - |w|^2 - |z_1|^2)/4\ell_0^2} \, w^k u_1 \, \frac{u_2 v_3 - v_2 u_3}{(z_2 - z_3)^2} \; \cdots \right) \prod_{i>1} (w - z_i)^2 \prod_{k>l>1} (z_k - z_l)^2 \, e^{-\frac{1}{4\ell_0^2}\sum_{i>1}|z_i|^2}$$

$+$ *terms in which the spin acts on paired electrons*

If this is non-vanishing, it is plausibly equal to (the a_j are some, possibly w-dependent, coefficients):

$$\mathcal{A}\left(\sum_j a_j z_1^j \, w^k u_1 \, \frac{u_2 v_3 - v_2 u_3}{(z_2 - z_3)^2} \; \cdots \right) \prod_{i>j} (z_i - z_j)^2 \, e^{-\frac{1}{4\ell_0^2}\sum_i |z_i|^2} \tag{79}$$

If so, then the up-spin electron (and its concomitant neutral fermion) is no longer localized at w because of the large powers of z_1 from the Jastrow factor. In such a case, however, when we act with another local spin operator, $S^-(w')$, the commutator, which receives non-vanishing contributions only when the two spin operators act on the same electron, need not vanish for $w \neq w'$ (or, rather, need not decay as a Gaussian in $w - w'$).

Even if our hypothesis is incorrect, and the Haldane-Rezayi edge theory is some other theory, it is difficult to see how the $SU(2)$ symmetry could be local. There are simply 'too few' single fermion states, by a factor of two, to allow for a local $SU(2)$ symmetry. This is quite clear from the formulation as a truncated Dirac doublet. If we were to take an inner product different from the inner product of the $c = 1$ theory, this would not help matters since it could not increase the size of the Hilbert space. Could it be that we have simply chosen the wrong symmetry generators? This is unlikely since the symmetry generators (75) have the desired action: they rotate the spins of the fermions. In principle, there is one other possibility. If there were low-energy excitations in the bulk withh anomalous total derivative terms in their $SU(2)$ algebra, these terms could cancel the anomalous terms at the edge. However, there is no trace of such excitations among the states annihilated by (2).

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VIII. EXPERIMENTAL CONSEQUENCES.

If our hypothesis is correct and the edge theory of the Haldane-Rezayi state is the $c = 1+1$ conformal field theory, there are measurable consequences which could elucidate the nature of the $\nu = \frac{5}{2}$ plateau. The electron annihilation operator is $\psi e^{-i\sqrt{2}\phi}$, so the coupling to a Fermi liquid lead will be $\psi e^{-i\sqrt{2}\phi} \Psi_{\text{lead}}$. This is a dimension 2 operator, so the tunneling conductance, G_t, through a point contact between a Fermi liquid lead and the edge of the Haldane-Rezayi state is

$$G_t \sim T^2 \tag{80}$$

See [10,11] to compare (80) with the corresponding expression for a Laughlin state. If the voltage $V \gg T$, then $I \sim V^3$. For tunneling between two Haldane-Rezayi droplets, $G_t \sim T^4$ for $T \gg V$ and $I \sim V^5$ for $T \ll V$. The tunneling of quasiparticles from one edge of a Haldane-Rezayi droplet to another through the bulk is presumably dominated by the tunneling of half-flux quantum quasiparticles, which are created by $\mu e^{-i\phi/2\sqrt{2}}$ where μ is the Dirac theory twist field. The resulting tunneling conductance between the two edges is $G_t \sim T^{-5/4}$ at high temperatures; at low temperature it is $\frac{1}{2} \frac{e^2}{h}$ with corrections determined by the perturbations of a strong-coupling fixed point.

One thing which is, perhaps, surprising about these predictions is that they are precisely the same as would be expected for the $(3,3,1)$ state and for a simple reason: the edge theories are almost the same. According to [19], the neutral sector of the $(3,3,1)$ state is a $c = 1$ Dirac fermion. The only difference with the edge theory of the Haldane-Rezayi state is that the twisted and untwisted sectors are exchanged, but this does not affect the dimensions of the scaling operators which determine the above power laws. Hence, the Haldane-Rezayi and $(3,3,1)$ states cannot be distinguished from simple tunneling experiments at the edge. However, these states are definitely not in the same universality class. Their bulk excitations have different topological properties, as may be seen from the ground state degeneracy on the torus. In an Aharonov-Bohm experiment with two half-flux quantum quasiholes, the

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phase resulting when one winds around another is $3\pi i/4$ in the Haldane-Rezayi state (from (56)) but $-\pi i/8$ in the $(3,3,1)$ state (and 0 in the Pfaffian state). In experiments with more than two quasiholes, the full structure of the non-Abelian statistics of the Haldane-Rezayi state comes into play and, again, Aharonov-Bohm experiments can resolve it from the $(3,3,1)$ and other candidate quantum Hall states.

ACKNOWLEDGMENTS

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[40] While we were completing this paper, a preprint, hep-th/9612172, by S. Guruswamy and A.W.W. Ludwig appeared in which the relationship between the $c = 1$ and $c = -2$ theories was discussed and the hidden symmetry of the $c = 1$ theory was constructed.