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Singular Vectors in Logarithmic Conformal Field Theories

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Abstract

Null vectors are generalized to the case of indecomposable representations which are one of the main features of logarithmic conformal field theories. This is done by developing a compact formalism with the particular advantage that the stress energy tensor acting on Jordan cells of primary fields and their logarithmic partners can still be represented in form of linear differential operators. Since the existence of singular vectors is subject to much stronger constraints than in regular conformal field theory, they also provide a powerful tool for the classification of logarithmic conformal field theories.

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1 Introduction

It is now nearly fifteen years ago since the concept of rationality of conformal field theory (CFT) made its first appearance through the minimal models of Belavin, Polyakov and Zamolodchikov [1]. Since then rational conformal field theories (RCFTs) established themselves as a main tool in modern theoretical physics.

Now it becomes increasingly apparent that so-called logarithmic conformal field theories (LCFTs), first encountered and shown to be consistent in [21], are not just a peculiarity but merely a generalization of ordinary 2-dimensional CFTs with broad and growing applications. One may well say that LCFTs contain ordinary rational conformal field theories (RCFTs) as just the subset of theories free of logarithmic correlation functions. However, logarithmic divergences are sometimes quite physical, and so there is an increasing interest in these logarithmic conformal field theories.

These logarithms have been found by now in a multitude of models such as the WZNW model on the supergroup $GL(1,1)$ [36], the $c_{p,1}$ models (as well as non-minimal $c_{p,q}$ models) [15, 19, 21, 23, 24, 35], gravitationally dressed conformal field theories [4], WZNW models at level 0 [27, 6], and some critical disordered models [7, 30]. Also, the Yangian structure of WZNW models seems to be connected to LCFTs [3]. The theory of indecomposable representations of the Virasoro algebra (which are a particular feature of LCFTs in general) was developed in [35], and logarithmic correlation functions were considered in general in [20, 25, 26, 32, 38], see also [34] about consequences for Zamolodchikov's C-theorem.

First applications to physical systems include the study of (multi-)critical polymers and percolation in two dimensions [8, 15, 37, 39], two-dimensional turbulence and magneto-hydrodynamics [17, 33], the quantum Hall effect [16, 22, 30, 40], and also gravitational dressing [4, 25, 26] as well as disorder and localization effects [7, 29, 30]. They also play a role in the so called unifying $\mathcal{W}$ algebras [5] and are believed to be important for studying the problem of recoil in the theory of strings and D-branes [2, 10, 27, 28, 31] as well as target-space symmetries in string theory in general [27].

Although LCFTs are mainly considered with respect to the Virasoro algebra, the concept is more general allowing for Jordan cell structures with respect to extended chiral symmetry algebras (e.g. current algebras) as first introduced in [26]. Let us briefly recall what We mean by Jordan cell structure. Suppose we have two operators $\Phi(z), \Psi(z)$ with the same conformal weight h. As was first realized in [21], this situation leads to logarithmic correlation functions and to the fact that L_0, the zero mode of the Virasoro algebra, can no longer be diagonalized:

$$\begin{aligned} L_0|\Phi\rangle &= h|\Phi\rangle\,, \\ L_0|\Psi\rangle &= h|\Psi\rangle + |\Phi\rangle\,, \end{aligned} \tag{1.1}$$

where we worked with states instead of the fields themselves. The field $\Phi(z)$ is then an ordinary primary field, whereas the field $\Psi(z)$ gives rise to logarithmic correlation functions and is therefore called a *logarithmic partner* of the primary field $\Phi(z)$. We would like to note that two fields of the same conformal dimension *do not not automatically* lead to LCFTs with respect to the Virasoro algebra. Either, they differ in some other quantum numbers (for examples of such CFTs see [14, 18]), or they form a Jordan cell structure with respect to an extended chiral symmetry only (see [25] for a description of the different possible cases).

This paper aims in generalizing the concept of singular vectors to the case of LCFTs, and concentrates – for the sake of simplicity – on the case of LCFTs with respect to the Virasoro algebra. A singular or null vector $|\chi\rangle$ is a state which is orthogonal to all states,

$$\langle\psi|\chi\rangle = 0 \ \forall\psi\,, \tag{1.2}$$

where in our case the scalar product is given by the Shapovalov form. Such states can be considered to be identically zero, and it is precisely the existence of such states which "makes" CFTs (quasi-)rational by dividing the ideals generated by them out of the Verma modules. This is, of course, well known since [1, 11], leading to degenerate conformal families etc.

A pair of fields $\Phi(z), \Psi(z)$ forming a Jordan cell structure brings the problem of off-diagonal terms produced by the action of the Virasoro field, such that the corresponding representation is indecomposable. Therefore, if $|\chi_\Phi\rangle$ is a null vector in the Verma module on the highest weight state $|\Phi\rangle$ of the primary field, we cannot just replace $|\Phi\rangle$ by $|\Psi\rangle$ and obtain another null vector.

Before we define general null vectors for Jordan cell structures, we present a formalism which might be useful in the future for all kinds of explicit calculations in the LCFT setting. This formalism, which we introduce in section two, has the advantage that the Virasoro modes are still represented as linear differential operators, and that it is compact and elegant allowing for arbitrary rank Jordan cell structures. Moreover, the connection between LCFTs and supersymmetric CFTs, which one could glimpse here and there [15, 6, 36, 37] (see also [9]), seems to be a quite fundamental one. The second half of section two is then devoted to apply our formalism to the definition of logarithmic null vectors.

Section three entirely consists of one very explicit example, our other explicit results are presented in the Appendix. This example treats the $c_{3,1} = -7$ model, which is the next model in the series of $c_{p,1}$ LCFTs after the well known $c_{2,1} = -2$ theory. One Jordan cell is spanned by two fields of conformal weight $h = -\frac{1}{4}$. The primary field alone corresponds to the irreducible sub-representation with a null vector at level two divided out, but the complete Jordan cell should have a logarithmic null vector at level 4, which is then explicitly constructed.

Next, we consider the consequences of the existence of logarithmic null vectors in section four: Their existence is subject to much stronger constraints as the ordinary null vectors on primary fields are. However, they do exist also in CFTs with central charge $c = c_{p,q}$ from the minimal series, if the minimal models are augmented by including certain fields from outside the conformal grid, as first argued in [15]. As far as LCFTs with respect to the Virasoro algebra are concerned, we achieve a classification of all possible cases: Perhaps surprisingly, only models from the generalized minimal series

$$c_{p,q} = 1 - 6\frac{(p-q)^2}{pq} \quad p, q \in \mathbb{Z} - \{0\} \text{ coprime} \tag{1.3}$$

can feature logarithmic null vectors. On one side, these include rational models with $c < 1$, i.e. augmented minimal models and $c_{p,1}$ models, as well as a certain $c = 1$ model with the conformal weights determined by the general formula

$$h_{r,s}(c_{p,q}) = \frac{(pr-qs)^2-(p-q)^2}{4pq} \;\; r,s \in \mathbb{Z}_+ \tag{1.4}$$

with $p = q = 1$ (this corresponds to the Gaussian $c = 1$ model at the self-dual radius $R = 1/\sqrt{2}$). On the other side, we also find LCFTs for $c \geq 25$ which formally amounts to replacing $p \mapsto -p$. These theories are certainly not rational with respect to the Virasoro algebra alone, nor are they unitary, but may be insofar interesting as this is the realm of Liouville theory with its puncture operator [4, 10, 25]. We conclude this section with two plots of CFT spectra in the (h, c) plane, one showing the spectra of ordinary primary fields, the other showing the spectra of Jordan cells of primary fields and their logarithmic partners. They nicely visualize arguments of our earlier work (second reference of [15]) on the origin of LCFTs as limiting points in the space of RCFTs.

2 Setup of Problem and Formalism

LCFTs are characterized by the fact that some of their highest weight representations are indecomposable. This is usually described by saying that two (or more) highest weight states with the same highest weight span a non-trivial Jordan cell. In the following we call the dimension of such a Jordan cell the *rank* of the indecomposable representation.

Therefore, let us assume that a given LCFT has an indecomposable representation of rank r with respect to its maximally extended chiral symmetry algebra $\mathcal{W}$. This Jordan cell is spanned by r states $|w_0, w_1, \ldots; n\rangle$, $n = 0, \ldots, r-1$ such that the modes of the generators of the chiral symmetry algebra act as

$$\Phi_0^{(i)}|w_0, w_1, \ldots; n\rangle \;=\; w_i|w_0, w_1, \ldots; n\rangle + \sum_{k=0}^{n-1} a_{i,k}|w_0, w_1, \ldots; k\rangle\,, \tag{2.1}$$

$$\Phi_m^{(i)}|w_0, w_1, \ldots; n\rangle \;=\; 0 \text{ for } m > 0\,, \tag{2.2}$$

where usually $\Phi^{(0)}(z) = T(z)$ is the stress energy tensor which gives rise to the Virasoro field, i.e. $\Phi_0^{(0)} = L_0$, and $w_0 = h$ is the conformal weight. For the sake of simplicity, we concentrate in this paper on the representation theory of LCFTs with respect to the pure Virasoro algbra such that (2.1) reduces to

$$L_0|h; n\rangle \;=\; h|h; n\rangle + (1 - \delta_{n,0})|h; n-1\rangle\,, \tag{2.3}$$

$$L_m|h; n\rangle \;=\; 0 \text{ for } m > 0\,, \tag{2.4}$$

where we have normalized the off-diagonal contribution to 1. As in ordinary CFTs, we have an isomorphism between states and fields. Thus, the state $|h; 0\rangle$, which is the highest weight state of the irreducible subrepresentation contained in every Jordan cell, corresponds to an ordinary primary field $\Psi_{(h;0)}(z) \equiv \Phi_h(z)$, thereas states $|h; n\rangle$ with $n > 0$ correspond to the so-called logarithmic partners $\Psi_{(h;n)}(z)$ of the primary field. The action of the modes of the Virasoro field on these primary fields and their logarithmic partners is given by

$$\Lambda_{-k}(z)\Psi_{(h;n)}(w) = \frac{(1-k)h}{(z-w)^k}\Psi_{(h;n)}(w) - \frac{1}{(z-w)^{k-1}}\frac{\partial}{\partial w}\Psi_{(h;n)}(w) - (1-\delta_{n,0})\frac{\lambda(1-k)}{(z-w)^k}\Psi_{(h;n-1)}(w)\,, \tag{2.5}$$

with λ normalized to 1 in the follwing. As it stands, the off-diagonal term spoils writing the modes $\Lambda_{-k}(z)$ as linear differential operators.

The aim of this section is mainly to prepare a formalism in which the Virasoro modes are expressed as linear differential operators. To this end, we introduce a new – up to now purely formal – variable θ with the property $\theta^r = 0$. We may then view an arbitrary state in the Jordan cell, i.e. a particular linear combination

$$\Psi_h(\mathbf{a})(z) = \sum_{n=0}^{r-1} a_n \Psi_{(h;n)}(z)\,, \tag{2.6}$$

as a formal series expansion describing an arbitrary function $a(\theta)$ in θ, namely

$$\Psi_h(a(\theta))(z) = \sum_n a_n \frac{\theta^n}{n!}\Psi_h(z)\,. \tag{2.7}$$

This means that the space of all states in a Jordan cell can be described by tensoring the primary state with the space of power series in θ, i.e. $\Theta_r(\Psi_h) \equiv \Psi_h(z) \otimes \mathbb{C}[\![\theta]\!]/\mathcal{I}$, where we devided out the ideal generated by the relation $\mathcal{I} = \langle \theta^r = 0\rangle$. In fact, the action of the Virasoro algebra is now simply given by

$$\Lambda_{-k}(z)\Psi_h(a(\theta))(w) = \left(\frac{(1-k)h}{(z-w)^k} - \frac{1}{(z-w)^{k-1}}\frac{\partial}{\partial w} - \frac{\lambda(1-k)}{(z-w)^k}\frac{\partial}{\partial \theta}\right)\Psi_h(a(\theta))(w)\,. \tag{2.8}$$

Clearly, $\Psi_{(h;n)}(z) = \Psi_h(\theta^n/n!)(z)$, but we will often simplify notation and just write $\Psi_h(\theta)(z)$ for a generic element in $\Theta_r(\Psi_h)$. However, the context should always make it clear, whether we mean a generic element or really $\Psi_{(h;1)}(z)$. The corresponding states are denoted by $|h; a(\theta)\rangle$ or simply $|h;\theta\rangle$. To project onto the $k^{\rm th}$ highest weight state of the Jordan cell, we just use $a_k|h;k\rangle = \partial_\theta^k |h; a(\theta)\rangle\big|_{\theta=0}$. In order to avoid confusion with $|h;1\rangle$ we write $|h;\mathbb{I}\rangle$ if the function $a(\theta) \equiv 1$.

It has become apparent by now that LCFTs are somehow closely linked to supersymmetric CFTs [15, 6, 36, 37] (see also [9]). We suggestively denoted our formal variable by θ, since it can easily be constructed with the help of Grassmannian variables as they appear in supersymmetry. Taking $N = r-1$ supersymmetry with Grassmann variables θ_i subject to $\theta_i^2 = 0$, we may define $\theta = \sum_{i=1}^{r-1}\theta_i$. More generally, θ and its powers constitute a basis of the totally symmetric, homogenous polynomials in the Grassmannians θ_i.

Finally, we remark that the θ variables are associated *not* with the coordinates the fields are localized in coordinate space, but with the positions the fields are localized in h-space (the Jordan cells). Therefore, the θ variables will be labeled by the conformal weight they refer to, whenever the context makes it necessary.

Operator Product Expansion and Correlation Functions.

Now we would like to recover n-point correlation functions within our formalism. To this end we consider the general operator product expansion (OPE) of fields with Jordan cell structure. First, we note that according to [15], the fusion rules of indecomposable representations can be split with respect to the Jordan cell structure, despite the fact that these representations are indecomposable. Hence, the fusion rules can be given in the general form

$$[(h_i;l)] \times [(h_j;m)] = \sum_{k,n} N^{(k,n)}_{(i,l)(j,m)} [(h_k;n)]\,. \tag{2.9}$$

Since the vanishing of any fusion coefficient implies the vanishing of the corresponding structure function in the operator product expansion, we can factorize the structure functions as $\mathcal{C}^{\cdot}_{\cdot\cdot} = \tilde{\mathcal{C}}^{\cdot}_{\cdot\cdot} N^{\cdot}_{\cdot\cdot}$. Then, the OPE of two fields of a LCFT reads

$$\Psi_{(h_i;l)}(z)\Psi_{(h_j;m)}(z') = \sum_{k,n} N^{(k,n)}_{(i,l)(j,m)} (\partial_{h_i} + \partial_{h_j})^{n-|l-m|} \tilde{\mathcal{C}}^k_{ij}(z-z')\Psi_{(h_k;n)}(z')\,. \tag{2.10}$$

As in the ordinary case, the homogenous functions turn out to be $\tilde{\mathcal{C}}^k_{ij}(z-z') = \tilde{C}^k_{ij}\cdot(z-z')^{h_k-h_i-h_j}$. The appearance of the derivatives is due to the behavior of correlation functions with logarithmic fields as in [20, 32]. It is very helpful to view the logarithmic partners of a primary field as producing the additional solutions for differential equations (for the correlation functions) in case of a degenerate solution space. The latter occurs if several points in the spectra of CFTs flow together if one moves to an LCFT point in CFT space (see the second work in [15] as well as the last section of this paper for more details). It is then natural that the operator product expansion of two fields, which on the right hand side has fields whose highest weights would flow together at the LCFT point, must exhibit logarithmic corrections. A nice and precise explanation of this can be found in [25]. Essentially, if we momentarily suppose that we have some kind of a free field representation of our CFT, the fields are $\Phi_{h(\alpha)}(z) = \,:\exp(i\alpha\phi(z)):$ with a certain function $h = h(\alpha)$, and in case that two of these fields have conformal weights which flow together, we may write

$$\begin{aligned}\Phi_{h(\alpha)+\varepsilon}(z) - \Phi_{h(\alpha)}(z) &= \varepsilon\frac{\partial}{\partial h}\Phi_{h(\alpha)}(z) + \mathcal{O}(\varepsilon^2) = \varepsilon\left(\frac{h(\alpha)}{\alpha}\right)^{-1}\frac{\partial}{\partial\alpha}\Phi_{h(\alpha)}(z) + \mathcal{O}(\varepsilon^2)\\ &= \varepsilon\left(\frac{h(\alpha)}{\alpha}\right)^{-1} i\phi(z)\Phi_{h(\alpha)}(z) + \mathcal{O}(\varepsilon^2)\,.\end{aligned} \tag{2.11}$$

It is then easy to see that the appearance of $\phi(z)$ does precisely cause the logarithmic corrections in the correlation functions. Plugged into OPEs, it follows that these can be written as derivatives with respect to h, acting on the structure functions.

The precise power of the logarithmic corrections (i.e. the precise power of the derivative terms) is fixed by the possible "logarithmic defects" on the right hand side of the OPE (or equivalently the fusion rules), as the power scaling law of the ordinary structure constants is fixed by the dimensional defect on the right hand side of the OPE. However, it can be infered from the fusion rules that, contrary to the dimensional defect, the "logarithmicity" is insofar not additive as, e.g., the fusion rules of two representations at the same level in their respective Jordan cells start with the ordinary irreducible sub-representation. More generally, the power of the logarithms in the structure functions of an OPE depends on the *difference* of the levels in the Jordan cells on the left hand side, which results in the above power for the derivatives [20]. In general we end up with an OPE expression as

$$\Psi_{h_i}(a(\theta_i))(z)\Psi_{h_j}(b(\theta_j))(z') = \sum_k \tilde{C}_{ij}^k \Psi_{h_k}\left(\sum_n \left(\sum_{l,m} \frac{a_l}{l!}\frac{b_m}{m!} N_{(i,l)(j,m)}^{(k,n)} (\partial_{h_i} + \partial_{h_j})^{n-|l-m|}\right)\frac{\theta_k^n}{n!}\right)(z')(z-z')^{h_k-h_i-h_j} \tag{2.12}$$

for $|z-z'| \ll 1$. It is convenient to write the cumbersome expression for the precise state within the h_k-Jordan cell as $f(\theta_k) = (a(\theta_i) \times b(\theta_j))(\theta_k)$ as the expansion of the product $a(\theta_i)b(\theta_j)$ for $[h_i] \times [h_j] - [h_k] \to 0$ such that the right hand side can be rewritten as a sum over certain fields $\Psi_{h_k}(f(\theta_k))(z')$. These fields become operators of the form

$$\Psi_{h_k}(f(\theta_k))(z') = \sum_s \Psi_{h_k}(f_s(\theta_k))(z')\left(-2\frac{\partial}{\partial h_k}\right)^s , \tag{2.13}$$

where $f_s(\theta_k)$ denotes the corresponding part in the expansion $(a(\theta_i) \times b(\theta_j))(\theta_k)$. This resembles precisely the way, the Virasoro field became an differential operator with respect to the conformal weight [32]. If r_i, r_j denote the ranks of the Jordan cell fields of the left hand side, we see that the maximal difference which can occur is $r-1$ with $r = \max(r_i, r_j)$. Hence, if the rank of the Jordan cell on the right hand side is $r_k < r$, negative powers for the derivatives might occur. However, as a matter of fact, the corresponding fusion coefficients will vanish.

From this all n-point functions can be obtained, if the standard n-point functions of ordinary primary fields are known. As an example, we consider the 2-point functions. Plugging in the OPE from above, we have simply

$$\left\langle \Psi_{(h_i;l)}(z)\Psi_{(h_j;m)}(z')\right\rangle = \delta_{ij}\tilde{C}_{ij}^0 \sum_n N_{(i;l)(j;m)}^{(0;n)} (\partial_{h_i} + \partial_{h_j})^{n-|l-m|}(z-z')^{-h_i-h_j}\left\langle \Psi_{(0;n)}(z')\right\rangle . \tag{2.14}$$

Now, we have to note the important fact that in LCFT, the only non-vanishing 1-point function is *not* the ordinary identity, but instead we have

$$\left\langle \Psi_{(0;n)}(z)\right\rangle = \delta_{n,r_0-1} . \tag{2.15}$$

For example, the LCFT with $c = -2$ is known to have a realization via a system of two anticommuting fields $\theta(z), \bar{\theta}(z)$ with spin zero (do not confuse these with our θ variables). It is well known that correlation functions of this theory are only non-trivial if a term $\theta(z)\bar{\theta}(z)$ is included in the measure (and hence in the correlation functions), e.g. $\langle \mathbb{I}(z)\rangle = 0$, but $\langle \theta(z)\bar{\theta}(z')\rangle = 1$. A nice and detailed discussion of this can be found in [22]. Similar facts hold for general LCFTs. As a consequence, 2-point functions are only non-zero, if their degree in θ_0 is maximal. Since 2-point functions vanish if the fields are from different Jordan cells, the maximal degree is the same for both fields and so unambigous. From this follows that all LCFTs must in particular have the identity in a Jordan cell whose rank determines the maximal rank of all other fields.* For completeness, we mention that under field conjugation (with respect to the fusion rules) $\Psi \mapsto \Psi^*$, the Jordan cell structure is simply mapped as $a(\theta) \mapsto a^*(\theta) = \theta^{r-1}a(\theta^{-1})$, such that e.g. the conjugate of the primary field is the top field in the Jordan cell. This ensures that the 2-point functions of a field and its conjugate always yield the usual (non-logarithmic) result. Hence, we define in- and out-states as $|\Psi_h(a(\theta))\rangle = \lim_{z\to 0}\Psi_h(a(\theta))(z)|\rangle$ and $\langle\Psi_h(a(\theta))| = \langle|\lim_{z\to\infty} z^{2h}\Psi_h(a^*(\theta))(z)$. Note that the out-state of the identity is therefore its top level logarithmic partner – as it should be.

These considerations generalize to the case of n-point functions. Therefore, we may say that a LCFT is of rank r if its identity operator forms a Jordan cell of rank r, and it is then convenient to include θ_0^{r-1} into the measure for correlation functions

$$\left\langle \prod_{i=1}^{n} \Psi_{h_i}(a_i(\theta_i))(z_i) \right\rangle = \int \mathcal{D}\Psi\theta_0^{r-1} \prod_{i=1}^{n} \Psi_{h_i}(a_i(\theta_i))(z_i)\exp(-S(\Psi))\,, \tag{2.16}$$

such that only correlation functions of maximal degree in θ_0 are non-zero. Here θ_0 denotes the θ variable of the $h = 0$ Jordan cell, since we assumed that this is the only Jordan cell representation which yields a non-zero correlcation function after contracting the n-point function via insertion of OPEs down to a 1-point function. Hence, θ_0 behaves in much the same way, as $\theta(z)\bar{\theta}(z')$ does in the $c = -2$ model. Actually, in this case, we can just view the θ_0 variable as the product of the zero modes of the anticommuting scalar fields, from which $c = -2$ is constructed. However, no such construction is known for higher rank LCFTs, nor do the other models of the $c_{p,1}$ series allow such a realization. The latter are presumably constructed from twisted parafermionic fields, i.e. fields of integer spin which satisfy parafermionic anticommutation relations, since their 4-point function resemble the ones of $\mathbb{Z}_p$ orbifold CFTs. Explicit realizations of this kind are left for future work.

Logarithmic Null Vectors.

Next, we derive the consequences of our formalism. An arbitrary state in a LCFT of level n is a linear combination of descendants of the form

$$|\psi(\theta)\rangle = \sum_k \sum_{\{n_1+n_2+\ldots+n_m=n\}} b_k^{\{n_1,n_2,\ldots,n_m\}} L_{-n_m}\ldots L_{-n_2}L_{-n_1}|h;k\rangle \tag{2.17}$$

which we often abbreviate as

$$|\psi(\theta)\rangle = \sum_{|\mathbf{n}|=n} L_{-\mathbf{n}} b^{\mathbf{n}}(\theta)|h\rangle\,. \tag{2.18}$$

We will mainly be concerned with calculating Shapovalov forms $\langle\psi'(\theta')|\psi(\theta)\rangle$ which ultimatively cook down (by commuting Virasoro modes through) to expressions of the form

$$\langle\psi'(\theta')|\psi(\theta)\rangle = \langle h'; a'(\theta')| \sum_m f_m(c)(L_0)^m |h; a(\theta)\rangle\,, \tag{2.19}$$

*In the whole discussion we assumed that 1-point functions of fields with $h \neq 0$ vanish, which is not necessarily true for non-unitary theories [41]. The more general case of non-zero vacuum expectation values leads to considerable modifications of all correlation functions, which we will consider in a later work.

where we explicitly noted the dependence of the coefficients on the central charge c. Combining (2.19) with (2.18) we write $\langle\psi'(\theta')|\psi(\theta)\rangle = \langle h';a'(\theta')|f_{\mathbf{n}',\mathbf{n}}(L_0,C)|h;a(\theta)\rangle$ for the Shapovalov form between two *monomial* descendants, i.e.

$$\langle h';a'(\theta')|f_{\mathbf{n}',\mathbf{n}}(L_0,C)|h;a(\theta)\rangle = \langle h';a'(\theta')|L_{n'_1}L_{n'_2}\dots L_{-n_2}L_{-n_1}|h;a(\theta)\rangle\,. \tag{2.20}$$

More generally, since $L_0|h;a(\theta)\rangle = (h+\partial_\theta)|h;a(\theta)\rangle$, it is easy to see that an arbitrary function $f(L_0,C)\in\mathbb{C}[\![L_0,C]\!]$ acts as

$$f(L_0,C)|h;n\rangle = \sum_k \frac{1}{k!}\left(\frac{\partial^k}{\partial h^k}f(h,c)\right)|h;n-k\rangle\,, \tag{2.21}$$

and therefore $f(L_0,C)|h;a(\theta)\rangle = |h;\tilde{a}(\theta)\rangle$, where with $a(\theta)=\sum_n a_n\frac{\theta^n}{n!}$ we have

$$\tilde{a}_n = \sum_k \frac{a_{n+k}}{k!}\frac{\partial^k}{\partial h^k}f(h,c)\,. \tag{2.22}$$

This puts the convenient way of expressing the action of L_0 on Jordan cells by derivatives with respect to the conformal weight h, which appeared earlier in the literature, on a firm ground. Moreover, from now on we do not worry about the range of summations, since all series automatically truncate in the right way due to the condition $\theta^r=0$.

It is evident that choosing $a(\theta)=\mathbb{I}$ extracts the irreducible subrepresentation which is invariant under the action of L_0. All other non-trivial choices of $a(\theta)$ yield states which are not invariant under the action of L_0. The existence of null vectors of level n on such a particular state is subject to the conditions that

$$\sum_{|\mathbf{n}|=n} f_{\mathbf{n}',\mathbf{n}}(L_0,C)b^{\mathbf{n}}(\theta,h,c)|h\rangle \equiv \sum_{|\mathbf{n}|=n} f_{\mathbf{n}',\mathbf{n}}(L_0,C)\sum_k b^{\mathbf{n}}_k(h,c)|h;k\rangle = 0\ \forall\ \mathbf{n}':|\mathbf{n}'|=n\,. \tag{2.23}$$

Notice that we have the freedom that each highest weight state of the Jordan cell comes with its own descendants. These conditions determine the $b^{\mathbf{n}}_k(h,c)$ as functions in the conformal weight and the central charge. Clearly, for $a(\theta)=\mathbb{I}$ this would just yield the ordinary results as knonw since BPZ [1], i.e. the solutions for $b^{\mathbf{n}}_0(h,c)$. The question of this paper is, under which circumstances null vectors exist on the whole Jordan cell, i.e. for non-trivial choices of $a(\theta)$. Obviously, these null vectors, which we call *logarithmic null vectors* can only constitute a subset of the ordinary null vectors. From (2.21) we immediately learn that the conditions imply

$$\sum_{k=0}^{s-1}\sum_{|\mathbf{n}|=n} b^{\mathbf{n}}_k(h,c)\frac{1}{(s-1-k)!}\frac{\partial^{s-1-k}}{\partial h^{s-1-k}}f_{\mathbf{n}',\mathbf{n}}(h,c) = 0\ \ \forall\ \mathbf{n}':|\mathbf{n}'|=n\,,\ 1\le s\le r\,, \tag{2.24}$$

which can be satisfied if we put

$$b^{\mathbf{n}}_k(h,c) = \frac{1}{k!}\frac{\partial^k}{\partial h^k}b^{\mathbf{n}}_0(h,c)\,. \tag{2.25}$$

In fact, choosing the $b_k^{\mathbf{n}}(h,c)$ in this way allows one to rewrite the conditions as total derivatives of the standard condition for $b_0^{\mathbf{n}}(h,c)$. Keeping in mind that each Jordan cell module of rank r has Jordan cells of ranks r', $1 \le r' \le r$, as submodules, we can find intermediate null vector conditions, where the null vector only lies in the rank r' submodule (think of $r' = 1$ as a trivial example), if we restrict the range of s in (2.24) accordingly. Of course, this determines the $b_k^{\mathbf{n}}(h,c)$ only up to terms of lower order in the derivatives such that the conditions finally take the general form

$$\sum_k \frac{\lambda_k}{k!}\frac{\partial^k}{\partial h^k}\left(\sum_{|\mathbf{n}|=n} f_{\mathbf{n}',\mathbf{n}}(h,c) b_0^{\mathbf{n}}(h,c)\right) = 0 \;\forall\; \mathbf{n}' : |\mathbf{n}'| = n\,, \tag{2.26}$$

which, however, does not yield any different results. Moreover, the coefficients $b_k^{\mathbf{n}}(h,c)$ can only be determined up to an overall normalization. Clearly, there are $p(n)$ coefficients, where $p(n)$ denotes the number of partitions of n into positive integers. This means that only $p(n)-1$ of the standard coefficients $b_0^{\mathbf{n}}(h,c)$ are determined to be functions in h,c multiplied by the remaining coefficient, e.g. $b_0^{\{1,1,\ldots,1\}}$ (if this coefficient is not predetermined to vanish). In order to be able to write the coefficients $b_k^{\mathbf{n}}(h,c)$ with $k>0$ as derivatives with respect to h, one needs to fix the remaining free coefficient $b_0^{\{1,1,\ldots,1\}} = h^{p(n)}$ as a function of h. The choice given here ensures that all coefficients are always of sufficient high degree in h.* Clearly, this works only for $h \ne 0$. To find null vectors with $h=0$ needs some extra care. One foolproof choice is to put the remaining free coefficient to $\exp(h)$. The problem is that the Hilbert space of states is a projective space due to the freedom of normalization, and that we used h as a projective coordinate in this space, which only works for $h \ne 0$.

It is important to understand that the above is only a necessary condition due to the following subtelty: The derivatives with respect to h are done in a purely formal way. But already determining the standard solution $b_0^{\mathbf{n}}(h,c)$ is not sufficient in itself, and the conditions for the existence of standard null vectors yield one more constraint, namely $h = h_i(c)$ or vice versa $c = c_i(h)$ (the index i denotes possible different solutions, since the resulting equations are higher degree polynomials $\in \mathbb{C}[h,c]$). These constraints must be plugged in *after* performing the derivatives and, as it will turn out, this will severely restrict the existence of logarithmic null vectors, yielding only some *discrete* pairs (h,c) for each level n. Moreover, the set of solutions gets rapidly smaller if for a given level n the rank r of the assumed Jordan cell is increased. Since there are $p(n)$ linearly independent conditions for the $b_0^{\mathbf{n}}(h,c)$ of a standard null vector of level n, a necessary condition is $r \le p(n)$. As mentioned above, h is not a good coordinate for $h=0$, but $c_i(h)$ still is**. Therefore, for $h=0$ we should use c for normalization, meaning that for $h=0$, the $c_i(h)$ have to be plugged in *before* doing the derivatives.

The next section gives one rather explicit example, but further details about our calculations (e.g. how to find logarithmic null vectors with $h=0$) can also be found in the Appendix,

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*We usually choose the least common multiple of the denominators of the resulting rational functions in h,c of the other coefficients in order to simplify the calculations. This, however, occasionaly leads to additional – trivial – solutions which are the price we pay for doing all calculations with polynomials only.

**Again, this is only true as long as $c \ne 0$. The special point $(c=0, h=0)$ unfortunately cannot be treated within our scheme, but must be checked by direct calculations.

where we mainly collect and comment our explicit results.

3 An Example

In this section we want to demonstrate what a logarithmic null vector is and under which conditions it exists. Null vectors are of particular importance for rational CFTs. For any CFT given by its maximally extended symmetry algebra $\mathcal{W}$ and a value c for the central chrage we can determine the so-called degenerate $\mathcal{W}$-conformal families which contain at least one null vector. The corresponding heighest weights turn out to be parametrized by certain integer labels, yielding the so-called Kac-table. If $\mathcal{W} = \{T(z)\}$ is just the Virasoro algebra, all degenerate conformal families have highest weights labeled by two integers r, s,

$$h_{r,s}(c) = \frac{1}{4}\left(\frac{1}{24}\left(\sqrt{(1-c)}(r+s) - \sqrt{(25-c)}(r-s)\right)^2 - \frac{1-c}{6}\right) . \tag{3.1}$$

The level of the (first) null vector contained in the conformal families over the highest weight state $|h_{r,s}(c)\rangle$ is then $n = rs$.

LCFTs have the special property that there are at least two conformal families with the same highest weight state, i.e. that we must have $h = h_{r,s}(c) = h_{t,u}(c)$. This does not happen for the so-called minimal models since their truncated conformal grid precisely excludes this. However, LCFTs may be constructed for example for $c = c_{p,1}$, where formally the conformal grid is empty, or by augmenting the field content of a CFT by considering an enlarged conformal grid. However, if we have the situation typical for a LCFT, we have two non-trivial and *different* null vectors, one at level $n = rs$ and one at $n' = tu$ where we assume without loss of generality $n \leq n'$. Then the null vector at level n is an ordinary null vector on the highest weight state of the irreducible sub-representation $|h;0\rangle$ of the rank 2 Jordan cell spanned by $|h;0\rangle$ and $|h;1\rangle$, but what about the null vector at level n'?

Let us consider the particular LCFT with $c = c_{3,1} = -7$. This LCFT admits the highest weights $h \in \{0, \frac{-1}{4}, \frac{-1}{3}, \frac{5}{12}, 1, \frac{7}{4}\}$ which yield the two irreducible representations at $h_{1,3} = \frac{-1}{3}$ and $h_{1,6} = \frac{5}{12}$ as well as two indecomposable representations with so-called staggered module structure (roughly a generalization of Jordan cells to the case that some highest weights differ by integers [19, 35]) constituted by the triples $(h_{1,1}\!=\!0, h_{1,5}\!=\!0, h_{1,7}\!=\!1)$ and $(h_{1,2}\!=\!\frac{-1}{4}, h_{1,4}\!=\!\frac{-1}{4}, h_{1,8}\!=\!\frac{7}{4})$. We note that similar to the case of minimal models we have the identification $h_{1,s} = h_{2,9-s}$ such that the actual level of the null vector might be reduced. In the following we will determine the null vectors at level 2 and 4 for the rank 2 Jordan cell with $h = \frac{-1}{4}$. First, we start with the level 2 null vector, whose general ansatz is

$$|\chi^{(2)}_{h,c}\rangle = \left(b_0^{\{1,1\}}L_{-1}^2 + b_0^{\{2\}}L_{-2}\right)|h;a(\theta)\rangle + \left(b_1^{\{1,1\}}L_{-1}^2 + b_1^{\{2\}}L_{-2}\right)|h;\partial_\theta a(\theta)\rangle\,, \tag{3.2}$$

where we explicitly made clear how we counteract the off-diagonal action of the Virasoro null mode. It is well known that up to an overall normalization we have

$$b_0^{\{1,1\}} = 3h\,, \quad b_0^{\{2\}} = -2h(2h+1)\,, \tag{3.3}$$

such that according to the last section we should put

$$b_1^{\{1,1\}} = 3\,, \qquad b_1^{\{2\}} = -8h - 2\,. \tag{3.4}$$

The matrix elements $\langle h|L_2\,\partial_\theta^k|\chi_{h,c}^{(2)}\rangle\big|_{\theta=0}$, $k = 0, 1$, do give us further constraints, namely

$$c = -2h\frac{8h-5}{2h+1}\,, \qquad 0 = -2h\frac{16h^2+16h-5}{2h+1}\,. \tag{3.5}$$

From these we learn that only for $h \in \{0, \frac{-5}{4}, \frac{1}{4}\}$ we may have a logarithmic null vector (with $c = 0, 25, 1$ respectively). Therefore, the level 2 null vector for $h = \frac{-1}{4}$ of the $c = -7$ LCFT is just an ordinary one.

Next, we look at the level 4 null vector with the general ansatz

$$\begin{aligned} |\chi_{h,c}^{(4)}\rangle &= \left(b_0^{\{1,1,1,1\}}L_{-1}^4 + b_0^{\{2,1,1\}}L_{-2}L_{-1}^2 + b_0^{\{3,1\}}L_{-3}L_{-1} + b_0^{\{2,2\}}L_{-2}^2 + b_0^{\{4\}}L_{-4}\right)|h;a(\theta)\rangle \\ &+ \left(b_1^{\{1,1,1,1\}}L_{-1}^4 + b_1^{\{2,1,1\}}L_{-2}L_{-1}^2 + b_1^{\{3,1\}}L_{-3}L_{-1} + b_1^{\{2,2\}}L_{-2}^2 + b_1^{\{4\}}L_{-4}\right)|h;\partial_\theta a(\theta)\rangle\,. \end{aligned} \tag{3.6}$$

Considering the possible matrix elements determines the coefficients up to overall normalization as

$$\begin{aligned} b_0^{\{1,1,1,1\}} &= h^4(1232h^3 - 2466h^2 - 62h^2c + 1198h - 296hc + 13hc^2 + 5c^3 + 92c^2 + 128c - 144)\,, \\ b_0^{\{2,1,1\}} &= -4h^4(1120h^4 - 2108h^3 + 140h^3c + 428h^2 - 66h^2c + 338h - 323hc + 90hc^2 \\ &\quad + 60c^2 - 78 + 99c)\,, \\ b_0^{\{3,1\}} &= 24h^4(96h^5 - 332h^4 + 44h^4c + 382h^3 - 8h^3c + 4h^3c^2 - 53h^2c + 12h^2c^2 - 235h^2 \\ &\quad + 11hc^2 + 14hc + 65h - 6 + 3c + 3c^2)\,, \\ b_0^{\{2,2\}} &= 24h^4(32h^3 - 36h^2 + 4h^2c + 8hc + 22h + 3c - 3)(3h^2 + hc - 7h + 2 + c)\,, \\ b_0^{\{4\}} &= -4h^4(550h + 3c^3 - 224h^2c + 66hc^2 + 748h^3 - 48 + 2508h^4 + 11hc^3 + 41h^2c^2 \\ &\quad - 40h^3c - 3008h^5 + 12h^2c^3 + 120h^3c^2 - 184h^4c + 102hc + 27c^2 - 1698h^2 \\ &\quad + 18c + 4h^3c^3 + 768h^6 + 448h^5c + 76h^4c^2)\,. \end{aligned} \tag{3.7}$$

Even for ordinary null vectors at level 4 we have $p(4) = 5$ conditions, but due to the freedom of overall normalization only 4 conditions have been used so far. The last, $\langle h|L_4\,|\chi_{h,c}^{(4)}\rangle\big|_{\theta=0} = 0$, fixes the central charge as a function of the conformal weight to

$$c \in \left\{-2\frac{h(8h-5)}{2h+1}, -\frac{2}{5}\frac{8h^2+33-41h}{3+2h}, -\frac{3h^2-7h+2}{h+1}, 1-8h\right\}\,. \tag{3.8}$$

If we again put $b_1^{\mathbf{n}}(h,c) = \partial_h b_0^{\mathbf{n}}(h,c)$ we obtain the additional constraint $\langle h|L_4\,\partial_\theta|\chi_{h,c}^{(4)}\rangle\big|_{\theta=0} = 0$, i.e.

$$\begin{aligned} 0 &= -4h^3(-14308h^3c^2 + 6600h - 528c + 30hc^3 + 1239840h^5 - 113592h^2 + 5290hc + 144c^2 \\ &\quad + 462h^2c^3 + 4368h^3c^3 + 275hc^4 + 360h^2c^4 + 3296h^4c^3 + 74240h^6c + 25632h^5c^2 \\ &\quad + 67584h^7 + 595224h^3 - 25812h^2c - 12712h^3c + 11574h^2c^2 - 2475hc^2 - 1287136h^4 \\ &\quad + 60c^4 - 249408h^5c + 324c^3 - 12192h^4c^2 - 504320h^6 + 187040h^4c + 140h^3c^4)\,, \end{aligned} \tag{3.9}$$

in which we may insert the four solutions for c to obtain sets of discrete conformal weights (and central charges in turn). The Appendix contains the explicit calculations for all possible Virasoro logarithmic null vectors up to level 5. Here, we are only interested in the null vector for $h = \frac{-1}{4}$. And indeed, the first two solutions for c admit (among others) $h = \frac{-1}{4}$ to satisfy (3.9) with the final result for the null vector

$$\begin{aligned} \left|\chi^{(4)}_{h=-1/4,c=-7}\right\rangle &= \left(\tfrac{315}{128}L_{-1}^4 - \tfrac{525}{64}L_{-2}L_{-1}^2 + \tfrac{315}{128}L_{-2}^2 - \tfrac{105}{64}L_{-3}L_{-1} - \tfrac{105}{64}L_{-4}\right)\left|\tfrac{-1}{4};(\alpha_1\theta^1 + \alpha_0\theta^0)\right\rangle \\ &+ \left(-\tfrac{2463}{128}L_{-1}^4 + \tfrac{2485}{64}L_{-2}L_{-1}^2 + \tfrac{1241}{64}L_{-3}L_{-1} - \tfrac{1383}{128}L_{-2}^2 + \tfrac{821}{64}L_{-4}\right)\left|\tfrac{-1}{4};(\alpha_1\theta^0)\right\rangle . \end{aligned} \tag{3.10}$$

This shows explicitly the existence of a non-trivial logarithmic null vector in the rank 2 Jordan cell indecomposable representation with highest weight $h = \frac{-1}{4}$ of the $c_{3,1} = -7$ rational LCFT. Here, α_0, α_1 are arbitrary constants such that we may rotate the null vector arbitrarily within the Jordan cell. However, as long as $\alpha_1 \neq 0$, there is necessarily always a non-zero component of the logarithmic null vector which lies in the irreducible sub-representation. Although there is the ordinary null vector built solely on $|h;0\rangle$, there is therefore no null vector solely built on $|h;1\rangle$, once more demonstrating the fact that these representations are indecomposable.

4 Kac Determinant and Classification of LCFTs

As one might imagine from the Appendix, it is quite a time consuming task to construct logarithmic null vectors explicitly. However, if we are only interested in the pairs (h, c) of conformal weights and central charges for which a CFT is logarithmic and owns a logarithmic null vector, we don't need to work so hard.

As already explained, logarithmic null vectors are subject to the condition that there exist fields in the theory with identical conformal weights. As can be seen from (3.1), there are always fields of identical conformal weights if $c = c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}$ is from the minimal series with $p > q > 1$ coprime integers. However, such fields are to be identified in these cases due to the existence of BRST charges [12, 13]. Equivalently, this means that there are no such pairs of fields within the truncated conformal grid

$$H(p,q) \equiv \{h_{r,s}(c_{p,q}) : 0 < r < |q|, 0 < s < |p|\} . \tag{4.1}$$

It is worth noting that our explicit calculations for the data collected in the Appendix indeed produced "solutions" for the well known null vectors in minimal models, but these "solutions" never had a non-trivial Jordan cell structure. For example, at level 3 we find a solution with $c = c_{2,5} = -\frac{22}{5}$ and $h = h_{2,1} = h_{3,1} = -\frac{1}{5}$) which, however, is just the ordinary one. This was to be expected because each Verma module of a minimal model has precisely two null vectors (this is why all weights h appear twice in the conformal grid, $h_{r,s} = h_{q-r,p-s}$). We conclude that logarithmic null vectors can only occur if fields of equal conformal weight still exist after all possible identifications due to BRST charges (or due to the embedding structure of the Verma modules [11]) have been taken into account. For later convenience, we further define the boundary of the conformal grid as

$$\begin{aligned} \partial H(p,q) &\equiv \{h_{r,p}(c_{p,q}) : 0 < r \leq |q|\} \cup \{h_{q,s}(c_{p,q}) : 0 < s \leq |p|\}\,, \\ \partial^2 H(p,q) &\equiv \{h_{q,p}(c_{p,q})\}\,. \end{aligned} \tag{4.2}$$

These three sets reflect the possible three embedding structures of the corresponding Verma modules which are of type $III_{\pm}$, $III_{\pm}^{\circ}$, and $III_{\pm}^{\circ\circ}$ respectively [11].

In our earlier work we have argued that LCFTs are a very general kind of conformal theories, containing rational CFTs as the special subclass of theories without logarithmic fields. In the case of minimal models we showed that logarithmic versions of a CFT with $c = c_{p,q}$ can be obtained by augmenting the conformal grid. This can formally be achieved by considering the theory with $c = c_{\alpha p,\alpha q}$. The explicit calculations of null vectors in the present paper, however, did not show the existence of logarithmic fields for minimal models, the reason being simply that the levels of null vectors considered here are too small. Let us look at minimal $c_{2n-1,2}$ models, $n > 1$. Fields within the conformal grid are ordinary primary fields which do not posses logarithmic partners. Therefore, pairs of primary fields with logarithmic partners have to be found outside the conformal grid, and according to our earlier work [15] and [19] must lie on the boundary $\partial H(p,q)$ (note that the corner point is not an element). Notice that for $c_{p,1}$ models this condition is easily met because the conformal grid $H(p,1) = \emptyset$. Fields outside the boundary region which have the property that their conformal weights are $h' = h + k$ with $h \in H(p,q)$, $k \in \mathbb{Z}_+$ do not lead to Jordan cells (they are just descendants of the primary fields). For example, the $c_{5,2} = -\frac{22}{5}$ model admits representations with $h = h_{1,8} = h_{3,2} = \frac{14}{5}$ which do not form a logarithmic pair and are just descendants of the $h = -\frac{1}{5}$ representation. Therefore, even for the $c_{2n-1,2}$ models with their relatively small conformal grid, the lowest level of a logarithmic null vector easily can get quite large. In fact, the smallest minimal model, the trivial $c_{3,2} = 0$ model, can be augmented to a LCFT with formally $c = c_{9,6}$ which has a Jordan cell representation for $h = h_{2,2} = h_{2,4} = \frac{1}{8}$. The logarithmic null vector already has level 8 and reads explicitly

$$\begin{aligned}
&\left|\chi^{(8)}_{h=1/8,c=0}\right\rangle = \\
&\quad \big(10800L_{-1}^8 - 208800L_{-2}L_{-1}^6 + 928200L_{-2}^2L_{-1}^4 - 1060200L_{-2}^3L_{-1}^2 + 151875L_{-2}^4 + 252000L_{-3}L_{-1}^5 \\
&\quad - 631200L_{-3}L_{-2}L_{-1}^3 + 207000L_{-3}L_{-2}^2L_{-1} - 1033200L_{-3}^2L_{-1}^2 + 360000L_{-3}^2L_{-2} - 1249200L_{-4}L_{-1}^4 \\
&\quad + 4165200L_{-4}L_{-2}L_{-1}^2 - 1133100L_{-4}L_{-2}^2 + 176400L_{-4}L_{-3}L_{-1} + 593100L_{-4}^2 + 624000L_{-5}L_{-1}^3 \\
&\quad - 720000L_{-5}L_{-2}L_{-1} - 429300L_{-5}L_{-3} + 1206000L_{-6}L_{-1}^2 - 455400L_{-6}L_{-2} - 206100L_{-7}L_{-1} \\
&\quad - 779400L_{-8}\big)\left|\tfrac{1}{8}, a(\theta)\right\rangle \\
&+ \big(76800L_{-3}L_{-2}L_{-1}^3 + 755200L_{-3}L_{-2}^2L_{-1} - 2596800L_{-3}^2L_{-1}^2 + 106400L_{-3}^2L_{-2} + 179712L_{-4}L_{-1}^4 \\
&\quad + 123648L_{-4}L_{-2}L_{-1}^2 + 3621120L_{-4}L_{-3}L_{-1} - 857856L_{-4}^2 + 739200L_{-5}L_{-1}^3 - 5832000L_{-5}L_{-2}L_{-1} \\
&\quad + 992800L_{-5}L_{-3} + 3444000L_{-6}L_{-1}^2 - 154800L_{-6}L_{-2} - 2210400L_{-7}L_{-1} + 488000L_{-8}\big)\left|\tfrac{1}{8}, \partial a(\theta)\right\rangle\,,
\end{aligned}$$

up to an arbitrary state proportional to the ordinary level 4 null vector. This shows that minimal models can indeed be augmented to logarithmic conformal theories. Level 8 is actually the smallest possible level for logarithmic null vectors of augmented minimal models.

On the other hand, descendants of logarithmic fields are also logarithmic, giving rise to the more complicated staggered module structure [35]. Thus, whenever for $c = c_{p,q}$ the conformal weight $h = h_{r,s}$ with either $r \equiv 0 \bmod p$, $s \not\equiv 0 \bmod q$, or $r \not\equiv 0 \bmod p$, $s \equiv 0$, the corresponding representation is part of a Jordan cell (or a staggered module structure).

The question of whether a CFT is logarithmic really makes sense only in the framework of (quasi-)rationality. Therefore, we can assume that c and all conformal weights are rational numbers. It can then be shown that the only possible LCFTs with $c \leq 1$ are the "minimal" LCFTs with $c = c_{p,q}$. Using the correspondence between the Verma modules $V_{h,c} \leftrightarrow V_{-1-h,26-c}$ one can further show that LCFTs with $c \geq 25$ might exist with (formally) $c = c_{-p,q}$. Again, due to an analogous (dual) BRST structure of these models, pairs of primary fields with logarithmic partners can only be found outside the conformal grid $H(-p,q) = \{h_{r,s}(c_{-p,q}) : 0 < r < q, 0 < s < p\}$, a fact that can also be observed in our direct calculations. For example, at level 4 we found a candidate solution with $c_{-3,2} = 26$ and $h_{4,1} = h_{1,3} = -4$. But again, the explicit calculation of the null vector did not show any logarithmic part.

The existence of null vectors can be seen from the Kac determinant of the Shapovalov form $M^{(n)} = \langle h|L_{\mathbf{n}'}L_{-\mathbf{n}}|h\rangle$, which factorizes into contributions for each level n. The Kac determinant has the well known form

$$\det M^{(n)} = \prod_{k=1}^{n} \prod_{rs=k} (h - h_{r,s}(c))^{p(n-rs)} . \tag{4.3}$$

A consequence of Section 2 is that a necessary condition for the existence of logarithmic null vectors in rank r Jordan cell representations of LCFTs is that $\frac{\partial^k}{\partial h^k}\left(\det M^{(n)}\right) = 0$ for $k = 0, \ldots, r-1$. It follows immediately from (4.3) that non-trivial common zeros of the Kac determinant and its derivatives at level n only can come from the factors whose powers $p(n-rs) = 1$, i.e. $rs = n$ and $rs = n-1$. For example

$$\begin{aligned} \frac{\partial}{\partial h}\left(\det M^{(n)}\right) &= \sum_{n-1 \leq rs \leq n} \frac{1}{(h - h_{r,s}(c))} \det M^{(n)} \\ &+ \sum_{1 \leq rs \leq n-2} \frac{p(n-rs)}{(h - h_{r,s}(c))} \det M^{(n)} , \end{aligned} \tag{4.4}$$

whose first part indeed yields a non-trivial constraint, whereas the second part is zero whenever $\det M^{(n)}$ is. Clearly (4.4) vanishes at $h = h_{r,s}(c)$ up-to one term which is zero precisely if there is one other $h_{t,u}(c) = h$. This is the condition stated earlier. Solving it for the central charge c we obtain

$$c = \begin{cases} -\dfrac{(2t-3u+3s-2r)(3t-2u+2s-3r)}{(u-s)(t-r)} \\ -\dfrac{(2t-3u-3s+2r)(3t-2u-2s+3r)}{(u+s)(t+r)} \end{cases} . \tag{4.5}$$

With an ansatz $c(x) \equiv 1 - 6\frac{1}{x(x+1)}$ we find

$$x \in \left\{\frac{u-s}{t-r+s-u}, \frac{r-t}{t-r+s-u}, \frac{s+u}{t+r-u-s}, \frac{t+r}{u+s-t-r}\right\} , \tag{4.6}$$

i.e. $x \in \mathbb{Q}$. This proves our first claim that logarithmic null vectors only appear in the framework of (quasi-)rational CFTs. The further claims follow then from the well known embedding structure of Verma modules for central charges with rational x (which by the way ensures $c \leq 1$ or $c \geq 25$, where at the limiting points $x_{c\to 1} \to \infty$ and $x_{c\to 25} \to -\frac{1}{2}$).

Obviously, null vectors in rank r Jordan cells with conformal weight h require the existence of r different solutions (r_i, s_i) such that $h_{r_i,s_i}(c) = h$. Up to level 5 there is only one case with $r > 2$, namely the rank 3 logarithmic null vector of the $c = c_{-1,1} = 25$ theory with $h = h_{2,2} = h_{1,3} = h_{3,1} = -3$.

What remains is to find the numbers r, s, t, u (or more generally r_i, s_i). The allowed solutions must satisfy the conditions stated above: A quadruple (r, s, t, u) parametrizes a logarithmic null vector, if with $c = c(r, s, t, u)$ one of the solutions (4.5) for the central charge, both $h_{r,s}(c), h_{t,u}(c) \in \partial H(c)$ where $H(c) \equiv H(x, x + 1)$ is the conformal grid of the Virasoro CFT with central charge $c = c(x)$. This gives the conformal weights of the "primary" logarithmic pairs, the other possibilities are of the form $h \in \partial H(c) \bmod \mathbb{Z}_+$ and belong to "descendant" logarithmic pairs. We use quotation marks because the logarithmic partner of a primary field is not primary in the usual sense.

As an example, we consider the by now well known models with $c = c_{p,1}$, $p > 1$. Precisely all fields in the extended conformal grid (obtained by formally considering $c = c_{3p,3}$) except $h_{1,p}$ and $h_{1,2p}$ as well as their "duals" $h_{2,2p}$ and $h_{2,p}$ form tripels $(h_{1,r} = h_{1,2p-r}, h_{1,2p+r})$ which constitute a rank 2 Jordan cell with an additional Jordan cell like module staggerd into it (for details see [35]). The excluded fields form irreducible representations without any null vectors and are all $\in \partial^2 H(p, 1) \bmod \mathbb{Z}_+$. Similar results hold for the $c = c_{-p,1}$, $p > 1$, models. However, all these LCFTs are only of rank 2. The only cases of higher rank LCFTs seem to be particular $c = 1$ and $c = 25$ theories. Notice that such theories are necessarily *nonunitary*, i.e. the Shapovalov form is necessarily not positive definite. However, since we are able to explicitly construct these theories, e.g. the explicit null vectors in the Appendix, there is no doubt that these theories exist. The reason is that the $c_{\pm p,1}$, $p > 1$, theories still have additional symmetries such that a truncation of the conformal grid to finite size still can be constructed, while the $c = 1$ and $c = 25$ theories presumably are only quasirational, their conformal grid being infinite in at least one direction.

To support our general statements, we give all non-trivial solutions up to level 20, which are *not* $c_{p,\pm 1}$ models, in the following table. This means that we list only logarithmic extensions of minimal models. The levels $r_i s_i$ of the null vectors are given in decreasing order for each central charge. As a general fact, the null vector with the lowest level in a given Jordan cell belongs to the irreducible subrepresentation and is an ordinary one, all others are logarithmic.

One further comment is in order here: Minimal models may be augmented by including the fields from the boundary of the conformal grid. However, this alone does not suffice to get a rational LCFT. The staggered module structure [35] suggests that we also must include the fields from the "next" boundaries modulo p, q, i.e. fields with conformal weights in $\partial^k H(2p, 2q)$, $k = 1, 2$, as well as $h_{q,s}, h_{r,p}$ with $q < r < 2q, p < s < 2p$. This is also supported by analogous results for $c_{p,1}$ models and their fusion rules [15, 19], where fusion closes if the full (staggered) modules are considered. However, the question whether this really leads to a closed fusion product and therefore to rational models of augmented minimal models is left to future work. Nonetheless, our formalism suggests that the operator product expansion (OPE) of logarithmic fields, as discussed in section two, closes within logarithmic fields such that there is a maximal rank for all Jordan cell structures. However, as concluded in section two, the identity operator of a CFT determines the degree of its "logarithmiticity" because its Jordan cell structure determines which correlation functions can be non-zero. Augmented minimal models do not seem to have logarithmic partners of the identity field itself, but they do have a degenerate vacuum (which thus forms a trivial diagonal Jordan cell). This degeneracy is presumably sufficient to ensure that correlation functions of logarithmic fields do not vanish. At least, our formalism is constructed in such a way that it can smoothly be applied to operators Φ with Jordan cell structure as e.g. $\begin{pmatrix}\Phi & \lambda\\ 0 & \Phi\end{pmatrix}\begin{pmatrix}|h;1\rangle\\ |h;0\rangle\end{pmatrix}$ even in the case $\lambda \to 0$, i.e. in the case that representations just have a multiplicity > 1.

c	h_{r_i,s_i}		
$c_{9,2} = \frac{-46}{3}$	$h_{2,10} = h_{2,8} = \frac{-5}{8}$		
$c_{7,2} = \frac{-68}{7}$	$h_{2,8} = h_{2,6} = \frac{-3}{8}$	$h_{2,9} = h_{2,5} = \frac{-9}{56}$	$h_{2,10} = h_{2,4} = \frac{11}{56}$
$c_{5,2} = \frac{-22}{5}$	$h_{2,6} = h_{2,4} = \frac{-1}{8}$	$h_{2,7} = h_{2,3} = \frac{7}{40}$	$h_{3,5} = h_{1,5} = \frac{2}{5}$
	$h_{2,8} = h_{2,2} = \frac{27}{40}$	$h_{2,9} = h_{2,1} = \frac{11}{8}$	$h_{2,10} = h_{4,5} = \frac{91}{40}$
$c_{5,3} = \frac{-3}{5}$	$h_{3,6} = h_{3,4} = \frac{1}{12}$	$h_{4,5} = h_{2,5} = \frac{7}{20}$	
$c_{3,2} = 0$	$h_{4,5} = h_{2,4} = h_{2,2} = \frac{1}{8}$	$h_{4,4} = h_{2,5} = h_{2,1} = \frac{5}{8}$	$h_{3,6} = h_{3,3} = h_{1,3} = \frac{1}{3}$
	$h_{2,6} = h_{4,3} = \frac{35}{24}$	$h_{2,7} = h_{4,2} = \frac{21}{8}$	$h_{5,3} = h_{1,6} = \frac{10}{3}$
	$h_{2,8} = h_{4,1} = \frac{33}{8}$	$h_{2,9} = h_{6,3} = \frac{143}{24}$	$h_{2,10} = h_{6,2} = \frac{65}{8}$
$c_{4,3} = \frac{1}{2}$	$h_{3,5} = h_{3,3} = \frac{1}{6}$	$h_{4,4} = h_{2,4} = \frac{5}{16}$	$h_{3,6} = h_{3,2} = \frac{35}{48}$
	$h_{5,4} = h_{1,4} = \frac{21}{16}$		
$c_{4,-3} = \frac{51}{2}$	$h_{5,4} = h_{2,8} = \frac{-325}{16}$	$h_{3,6} = h_{6,2} = \frac{-851}{48}$	$h_{4,4} = h_{1,8} = \frac{-245}{16}$
	$h_{3,5} = h_{6,1} = \frac{-85}{6}$		
$c_{3,-2} = 26$	$h_{2,10} = h_{8,1} = \frac{-217}{8}$	$h_{2,9} = h_{6,3} = \frac{-551}{24}$	$h_{4,5} = h_{2,8} = h_{6,2} = \frac{-153}{8}$
	$h_{3,6} = h_{5,3} = h_{1,9} = \frac{-52}{3}$	$h_{4,4} = h_{2,7} = h_{6,1} = \frac{-125}{8}$	$h_{2,6} = h_{4,3} = \frac{-299}{24}$
	$h_{2,5} = h_{4,2} = \frac{-77}{8}$	$h_{3,3} = h_{1,6} = \frac{-25}{3}$	$h_{2,4} = h_{4,1} = \frac{-57}{8}$
$c_{5,-3} = \frac{133}{5}$	$h_{4,5} = h_{1,10} = \frac{-387}{20}$	$h_{3,6} = h_{6,1} = \frac{-205}{12}$	
$c_{5,-2} = \frac{152}{5}$	$h_{2,10} = h_{4,5} = \frac{-851}{40}$	$h_{2,9} = h_{4,4} = \frac{-147}{8}$	$h_{2,8} = h_{4,3} = \frac{-627}{40}$
	$h_{3,5} = h_{1,10} = \frac{-72}{5}$	$h_{2,7} = h_{4,2} = \frac{-527}{40}$	$h_{2,6} = h_{4,1} = \frac{-87}{8}$
$c_{7,-2} = \frac{250}{7}$	$h_{2,10} = h_{4,3} = \frac{-1075}{56}$	$h_{2,9} = h_{4,2} = \frac{-943}{56}$	$h_{2,8} = h_{4,1} = \frac{-117}{8}$
$c_{9,-2} = \frac{124}{3}$	$h_{2,10} = h_{4,1} = \frac{-147}{8}$		

The (h,c) Plane: Null Vectors with Level $rs \leq 400$.

It might be illuminating, and the author is fond of plots anyway, to plot the sets $\partial^k H(p,q)$, $k = 0, 1, 2$, for a variety of CFTs. The product pq is roughly a measure for the size of the

CFT since the size of the conformal grid and thus the field content is determined by it. Thus, it seems reasonable to plot all sets with $pq \leq n$ where we have chosen $n = 400$.

To make the structure of the (h, c) plane better visible, we transformed the variables via

$$x \mapsto \mathrm{sign}(x) \log(|x| + 1) \text{ for } x = h, c\,, \tag{4.7}$$

which amounts in a double logarithmic scaling of the axes both, in positive as well as in negative direction. The conformal weights are plotted in horizontal direction, the central charges along the vertical direction. The following plots show only the part of the (h, c) plane which belongs to $c \leq 1$ CFTs, i.e. minimal models and $c_{p,1}$ LCFTs, $p > 0$. The other "half" with $c \geq 25$ shares analogous features. Due to the map (4.7) the vertical range of roughly $[-5.5, 1.0]$ coresponds to $-240 \leq c \leq 1$, whereas the horizontal range $[-5.5, 5.0]$ does roughly correspond to $-240 \leq h \leq 148$. To guide the eye for better orientation, we give here for the lables $\pm\{0, 1, 2, 3, 4, 5\}$ the corresponding values of h, c, which are in the same oder $\pm\{0, 1.718, 6.389, 19.086, 53.598, 147.413\}$.

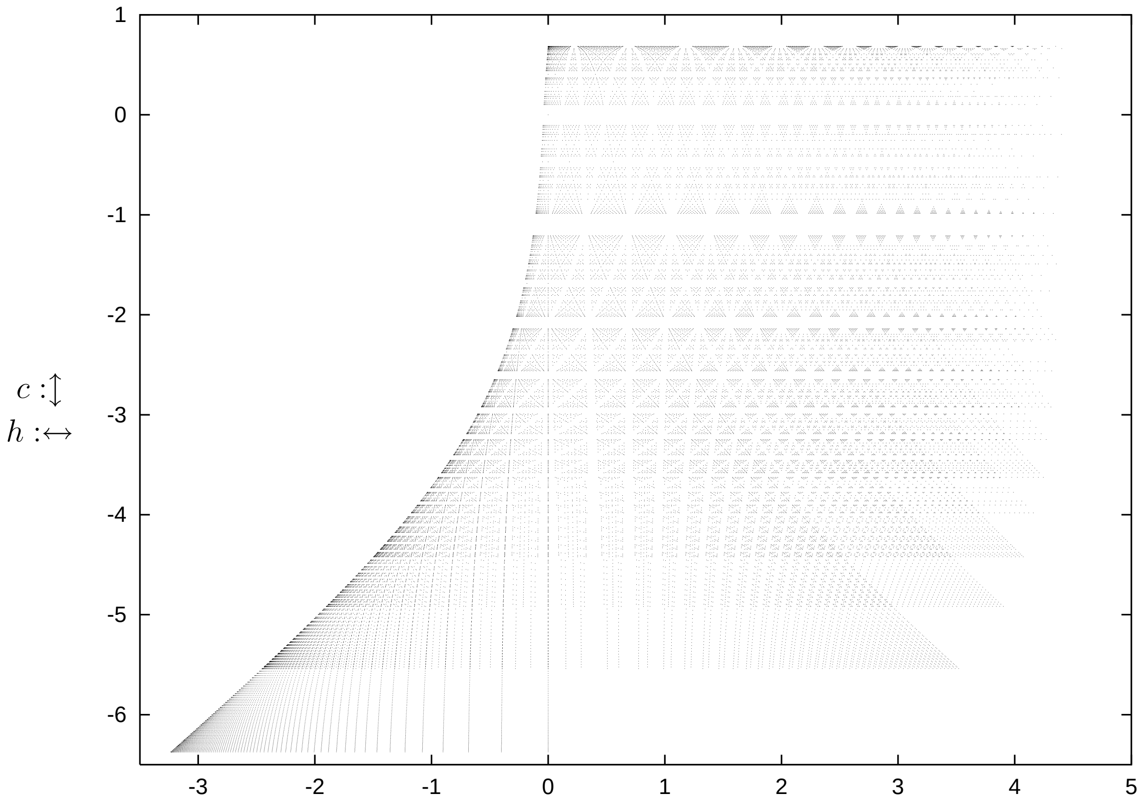

Figure 1: *All the spectra $(H(p, q), c_{p,q})$ for all $p, q > 0$ coprime such that $pq \leq 400$, which constitutes the set of all irreducible highest weight representations of minimal models. This means that for each central charge c we plotted all conformal weights h from within the truncated conformal grid $H(p, q) = \{h_{r,s} : 0 < r < q, 0 < s < p\}$. See text for details about the logarithmic scaling $x \mapsto \mathrm{sign}(x) \log(|x| + 1)$ for $x = h, c$, to make the pattern structure of the spectra of minimal models better visible (cross ref. equation 4.7).*

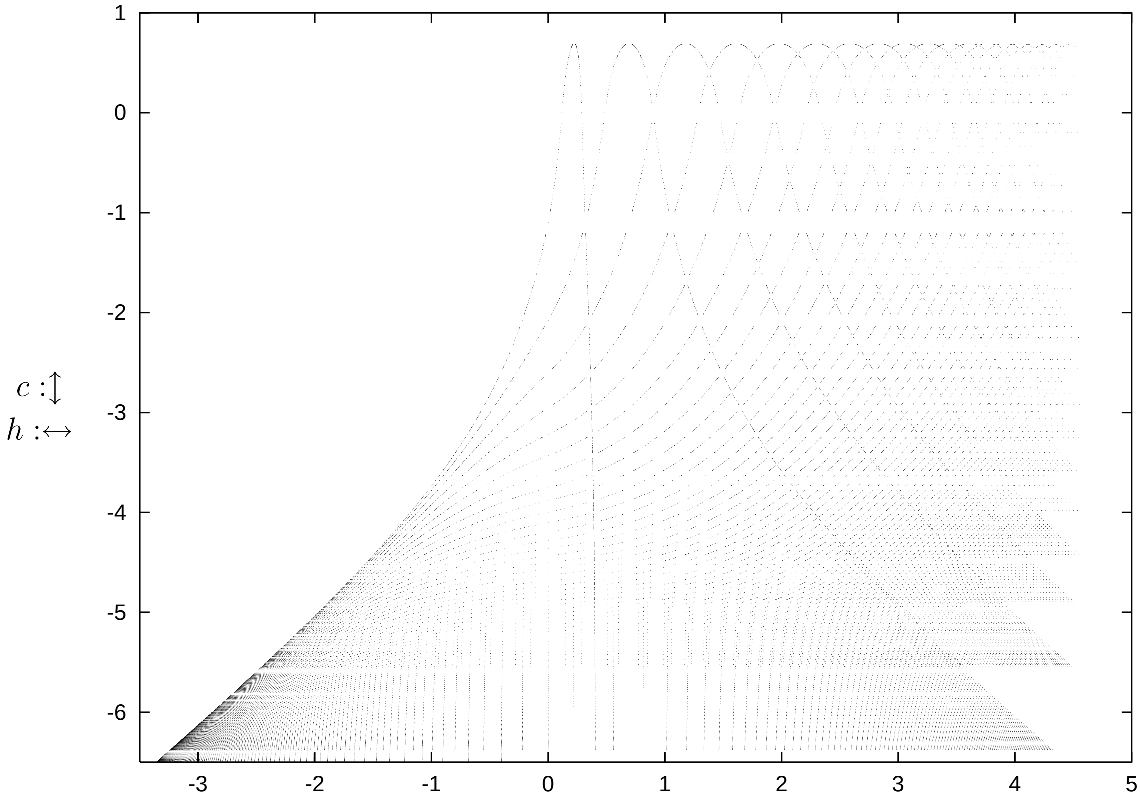

Figure 2: *All the spectra $(\partial H(p,q), c_{p,q})$ for all $p, q > 0$ coprime such that $pq \leq 400$, which constitutes the set of all Jordan cell representations, i.e. all conformal weights where fields with logarithmic partners exist. This means that for each central charge c we plotted all conformal weights h from the boundary of the truncated conformal grid $\partial H(p,q) = \{h_{q,s} : 0 < s \leq p\} \cup \{h_{r,p} : 0 < r \leq q\}$. The logarithmic scaling is the same as in Figure 1, see text for details about it (equation 4.7).*

We did not plot $(\partial^2 H(p,q), c_{p,q})$ since these points just lie at the left border of the pointset in figure 1, they belong to the highest weight representations with type $III_{\pm}^{\circ\circ}$ embedding structure of Verma modules. If one would put both plots above each other, one might infer from them that the set of logarithmic representations precisely lies on the "forbidden" curves of the pointset of ordinary highest weight representations. This illustrates the fact that logarithmic representations appear, if the conformal weights of two highest weight representations become identical.

As discussed in our earlier work [15], this situation arises in the limit of series of minimal models $c_{p_1,q_1}, c_{p_2,q_2}, c_{p_3,q_3}, \ldots$ with $\lim_{i\to\infty} p_i q_i = \infty$. Usually, the field content of these theories increases with i, but it might happen that in the limit p_i and q_i become almost coprime. More precisely, a sequence such as for example $\{c_{\alpha p,(\alpha+1)q}\}_{\alpha\in\mathbb{Z}_+}$ converges to a limiting theory with central charge $\lim_{\alpha\to\infty} c_{\alpha p,(\alpha+1)q} = c_{p,q}$. Therefore, we expect a rather small field content at the limit point since the conformal weights of the $c_{\alpha p,(\alpha+1)q}$ theories also approach the ones of the $c_{p,q}$ model (modulo $\mathbb{Z}$). A more detailed analysis (second reference in [15]) reveals that indeed conformal weights approach each other giving rise for Jordan cells. Hence, the theory at the limit point, while having central chrage $c_{p,q}$ actually is a LCFT. The plots presented here clearly visualize this topology of the space of CFTs in the (h, c) plane of their spectra.

To summarize, our results strongly suggest that augmented minimal models form *rational* logarithmic conformal field theories in the same sense as the $c_{p,1}$ models do. The only difference between the former and the latter is that for the $c_{p,1}$ models $H(p, 1) = \emptyset$. We know since BPZ [1] that under fusion $H(p, q) \times H(p, q) \to H(p, q)$, and since [19, 15] that under fusion $H'(p, q) \times H'(p, q) \to H'(p, q)$ with $H'(p, q) = \partial H(p, q) \cup \partial^2 H(p, q)$, if we deal with the full indecomposable representations. Therefore, the only difficulty can come from mixed fusion products of type $H(p, q) \times H'(p, q)$ which traditionally (without logarithmic operators) are zero due to decoupling. The formalism presented in section 2, however, yields non-zero fusion products by paying the price that representations from $H(p, q)$ appear with non-trivial multiplicities (because of the fact that the corresponding OPEs yield fields on the right hand side with $h \in H(p, q) \bmod \mathbb{Z}$, which have a non-trivial dependence on the formal θ variables. As mentioned before, a more detailed analysis of augmented minimal models is left for future work.

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A Appendix

We present explicit results for all possible Virasoro logarithmic null vectors up to level 5. The null vector at level 1 is trivial, $|\chi^{(1)}_{h=0,c}\rangle = L_{-1}|0\rangle$. There can be no logarithmic null vector at this level. For null vectors of level $n > 1$ we make the general ansatz

$$|\chi^{(n)}_{h,c}\rangle = \sum_j \sum_{|\mathbf{n}|=n} b^{\mathbf{n}}_j(h,c) L_{-\mathbf{n}} \left|h; \partial^j_\theta a(\theta)\right\rangle \tag{A.1}$$

and define matrix elements

$$\begin{aligned} N^{(n)}_{k,l} &= \frac{\partial^k}{\partial\theta^k}\left(\sum_j \sum_{|\mathbf{n}|=n} b^{\mathbf{n}}_j(h,c)\left\langle h\right| L_{\mathbf{n}'_l} L_{-\mathbf{n}} \left|h; \partial^j_\theta a(\theta)\right\rangle\right)\Bigg|_{\theta=0} \\ &= \sum_{j=0}^{k} \sum_{|\mathbf{n}|=n} b^{\mathbf{n}}_j(h,c) \frac{1}{j!}\frac{\partial^j}{\partial h^j} \langle h| L_{\mathbf{n}'_l} L_{-\mathbf{n}} |h\rangle \ , \end{aligned} \tag{A.2}$$

where $\mathbf{n}'_l$ is some enumeration of the $p(n)$ different partitions of n. Since the maximal possible rank of a Jordan cell representation which may contain a logarithmic null vector is $r \leq p(n)$, we consider $N^{(n)}$ to be a $p(n) \times p(n)$ square matrix. Our particular ansatz is conveniently chosen to simplify the action of the Virasoro modes on Jordan cells. Notice, that the derivatives with respect to the conformal weight h do not act on the coefficients $b^{\mathbf{n}}_j(h,c)$. Of course, we assume that $a(\theta)$ has maximal degree in θ, i.e. $\deg(a(\theta)) = r - 1$.

The Logarithmic Null Vectors at Level 2.

As an example, at level 2 we have $p(2) = 2$ and the matrix $N^{(2)}$ reads

$$N^{(2)} = \begin{bmatrix} b^{\{1,1\}}_0(8h^2+4h) + 6b^{\{2\}}_0 h & b^{\{1,1\}}_0(16h+4) + 6b^{\{2\}}_0 + b^{\{1,1\}}_1(8h^2+4h) + 6b^{\{2\}}_1 h \\ 6b^{\{1,1\}}_0 h + b^{\{2\}}_1\left(4h+\frac{1}{2}c\right) & 6b^{\{1,1\}}_0 + 4b^{\{2\}}_1 + 6b^{\{1,1\}}_1 h + b^{\{2\}}_1\left(4h+\frac{1}{2}c\right) \end{bmatrix}. \tag{A.3}$$

A null vector is logarithmic of rank $k \geq 0$ if the first $k+1$ columns of $N^{(n)}$ are zero, where $k = 0$ means an ordinary null vector. As described in the text, one first solves for ordinary null vectors (such that the first column vanishes up to one entry). This determines the $b^{\mathbf{n}}_0(h,c)$. Then one puts $b^{\mathbf{n}}_k(h,c) = \frac{1}{k!}\partial^k_h b^{\mathbf{n}}_0(h,c)$. Without loss of generality we may then assume that all entries except the last row are zero. In our example, this procedure results in

$$N^{(2)} = \begin{bmatrix} 0 & 0 \\ 10h^2 - 16h^3 - 2h^2c - hc & 20h - 48h^2 - 4hc - c \end{bmatrix}, \tag{A.4}$$

where $b^{\{1,1\}}_k = \frac{1}{k!}\partial^k_h(3h)$ and $b^{\{2\}}_k = \frac{1}{k!}\partial^k_h(-2h(2h+1))$ upto an overall normalization. The last step is trying to find simultanous solutions for the last row, i.e. common zeros of polynomials $\in \mathbb{C}[h,c]$. In our example, $N^{(2)}_{2,1} = 0$ yields $c = 2h(5-8h)/(2h+1)$. Then, the last condition becomes $N^{(2)}_{2,2} = -2h(16h^2+16h-5)/(2h+1) = 0$ which can be satisfied for $h \in \{0, \frac{-5}{4}, \frac{1}{4}\}$. From this we finally obtain the explicit logarithmic null vectors at level 2:

(h,c)	$\vert\chi^{(2)}_{h,c}\rangle$
$(0,0)$	$(3L_{-1}^2 - 2L_{-2})\,\vert 0; a(\theta)\rangle$
$(\frac{1}{4},1)$	$(3L_{-1}^2 - 3L_{-2})\left\vert\frac{1}{4}; a(\theta)\right\rangle - 4L_{-2}\left\vert\frac{1}{4}; \partial_\theta a(\theta)\right\rangle$
$(\frac{-5}{4},25)$	$(3L_{-1}^2 + 3L_{-2})\left\vert\frac{-5}{4}; a(\theta)\right\rangle - 4L_{-2}\left\vert\frac{-5}{4}; \partial_\theta a(\theta)\right\rangle$

Note, that according to our formalism, $h=0, c=0$ does not turn out to be a logarithmic null vector at level 2. Here and in the following the highest order derivative $\partial_\theta^k a(\theta)$ indicates the maximal rank of a logarithmic null vector to be k (and hence the maximal rank of the corresponding Jordan cell representation to be $r=k+1$). It is implicitly understood that $a(\theta)$ is then chosen such that the highest order derivative yields a non-vanishing constant.

All null vectors are normalized such that all coefficients are integers. Clearly, they are not unique since with $|\chi(\theta)\rangle = \sum_k \left|\chi_k; \partial_\theta^k a(\theta)\right\rangle$ every vector

$$|\chi'(\theta)\rangle = \sum_k \left|\chi_k; \sum_{l\geq 0} \lambda_{k,l}\partial_\theta^{k+l} a(\theta)\right\rangle \tag{A.5}$$

is also a null vector.

The Logarithmic Null Vectors at Level 3.

Following the same procedure for level 3 null vectors, we first obtain two branches of solutions $c = c_i(h)$, namely

$$c_1(h) = -2h\frac{8h-5}{1+2h}\,, \qquad c_2(h) = -\frac{3h^2-7h+2}{h+1}\,. \tag{A.6}$$

Clearly, $c_1(h)$ is the same solution as obtained at level 2, meaning the trivial fact that each level 2 null vector is also a null vector at level 3. On the other hand, these solutions need not all be redundant, because it might happen that a null vector at level n, which turned out to be just an ordinary one, becomes a logarithmic null vector at a higher level $n' > n$. The results for level 3 are:

(h,c)	$\vert\chi^{(3)}_{h,c}\rangle$ (trivial solutions)
$(\frac{-1}{5},\frac{-22}{5})$	$(175L_{-1}^3 + 350L_{-2}L_{-1} - 154L_{-3})\left\vert\frac{-1}{5}; a(\theta)\right\rangle$
$(0,0)$	$(-3L_{-1}^3 + 2L_{-2}L_{-1} + 2L_{-3})\,\vert 0; a(\theta)\rangle$
$(\frac{1}{2},\frac{1}{2})$	$(14L_{-1}^3 - 28L_{-2}L_{-1} - 7L_{-3})\left\vert\frac{1}{2}; a(\theta)\right\rangle$

(h,c)	$\lvert\chi^{(3)}_{h,c}\rangle$ (logarithmic null vectors)
$(0,-2)$	$(-L_{-1}^3 + 2L_{-2}L_{-1})\,\lvert 0; a(\theta)\rangle - L_{-3}\,\lvert 0; \partial_\theta a(\theta)\rangle$
$(\frac{1}{4},1)$	$(-9L_{-1}^3 + 9L_{-2}L_{-1} + 9L_{-3})\left\lvert \frac{1}{4}; a(\theta)\right\rangle$
	$+ (28L_{-1}^3 - 16L_{-2}L_{-1} - 16L_{-3})\left\lvert \frac{1}{4}; \partial_\theta a(\theta)\right\rangle$
$(1,1)$	$(3L_{-1}^3 - 12L_{-2}L_{-1} + 6L_{-3})\,\lvert 1; a(\theta)\rangle$
	$+ (7L_{-1}^3 - 34L_{-2}L_{-1} + 23L_{-3})\,\lvert 1; \partial_\theta a(\theta)\rangle$
$(-3,25)$	$(-49L_{-1}^3 - 196L_{-2}L_{-1} - 294L_{-3})\,\lvert -3; a(\theta)\rangle$
	$+ (7L_{-1}^3 + 126L_{-2}L_{-1} + 287L_{-3})\,\lvert -3; \partial_\theta a(\theta)\rangle$
$(\frac{-5}{4},25)$	$(-147L_{-1}^3 - 147L_{-2}L_{-1} - 147L_{-3})\left\lvert \frac{-5}{4}; a(\theta)\right\rangle$
	$+ (28L_{-1}^3 + 224L_{-2}L_{-1} + 224L_{-3})\left\lvert \frac{-5}{4}; \partial_\theta a(\theta)\right\rangle$
$(-2,28)$	$(-45L_{-1}^3 - 90L_{-2}L_{-1} - 90L_{-3})\,\lvert -2; a(\theta)\rangle$
	$+ (7L_{-1}^3 + 86L_{-2}L_{-1} + 104L_{-3})\,\lvert -2; \partial_\theta a(\theta)\rangle$

From this we can infer that the first three solutions are only ordinary null vectors, and that all remaining solutions correspond to rank 2 Jordan cell highest weight representations. We also find the solutions already obtained at level 2, as is to be expected. Some care has to be taken with the solution $h = 0, c = -2$. The $c = c_{2,1} = -2$ model is well known and has a rank 2 Jordan cell structure for the $h = 0$ representation. So, we would expect a logarithmic null vector of level 3, since the logarithmic partner of the primary field $\Psi_{(h=0;0)}(z) \equiv \Phi_{1,1}(z) = \mathbb{I}$ is the field $\Psi_{(h=0;1)}(z) \equiv \Phi_{1,3}(z)$. However, our standard ansatz does not reveal this null vector. The secret of this special solution is that the irreducible sub-representation at $h = 0$ is the *vacuum* representation, which always contains a level 1 null vector. Therefore, we have the additional freedom to add a "zero" via

$$\left\lvert\chi^{(3)}_{h=0,c}\right\rangle \mapsto \left\lvert\chi^{(3)}_{h=0,c}\right\rangle + \left(\alpha L_{-1}^2 + \beta L_{-2}\right) L_{-1}\,\lvert 0; \partial_\theta a(\theta)\rangle\ . \tag{A.7}$$

Moreover, such special solutions with $h=0$ do not satisfy our proposed general relation that $b_1^{\mathbf{n}}(h,c)|_{h=0,c=c_i(0)} = \partial_h b_0^{\mathbf{n}}(h,c)|_{h=0,c=c_i(0)}$ with $c = c_i(h)$ the appropriate general solution for the central charge for generic h (note that the level 1 null vector in the vacuum representation is independent of c). The reason for this is that for $h \neq 0$ we can use h as a homogeneous coordinate in the projective space of states (due to the freedom of normalization). For $h = 0$ the only coordinate left is c and hence we must calculate

$$b_k^{\mathbf{n}}(h,c)|_{h=0,c=c_i(0)} = \frac{1}{k!}\partial_h^k b_0^{\mathbf{n}}(h, c_i(h))\Big|_{h=0} \tag{A.8}$$

instead, where the central charge has been replaced in advance by the appropriate solution $c = c_i(h)$ for generic h.

It follows, that under certain circumstances $h = 0$ representations may be extended to Jordan cell structure of rank at least 2. This is expected due to the fact that all $c_{p,1}$ rational LCFTs have a rank 2 Jordan cell representation containing the vacuum (identity) representation. However, these representations have a more complicated structure, because they form together with a representation on $h=1$ a staggered indecomposable module [35].

The recipe to find such null vectors with $h=0$ is simple. One observes that $L_0|0;k\rangle = (1-\delta_{k,0})|0;k-1\rangle$. It is then easy to see that the only non-trivially vanishing matrix elements at level n are of the form $\langle 0;a'(\theta)|L_{\mathbf{n}'}L_{-\mathbf{n}}|0;a(\theta)\rangle$ where both partitions $\mathbf{n}' = \{n'_1, n'_2, \ldots\}$, $\mathbf{n} = \{n_1, n_2, \ldots\}$ do not contain 1. Restricting ourselves to the case of rank 2 Jordan cells, and putting the coefficients $b_0^{\mathbf{n}}(0,c)$ to zero where $\mathbf{n}$ does not contain 1, we are left solely with conditions for $b_1^{\mathbf{n}}(0,c)$, $1 \notin \mathbf{n}$, and c, which depend on the $b_0^{\mathbf{n}}(0,c)$, $1 \in \mathbf{n}$. For $n=2$ we would have to find a solution for $b_1^{\{2\}}$ and c which both depend only on one variable, $b_0^{\{1,1\}}$, so it is not surprising that we didn't find any. For $n=3$ however, we again have two constraints ($b_1^{\{3\}}$, c), but now also two variables ($b_0^{\{1,1,1\}}$, $b_0^{\{2,1\}}$). For $n=4$ we have three constraints and three variables etc., in general there are $p_2(n)+1$ constraints and $p(n)-p_2(n)$ variables, where $p_2(n)$ is the number of partitions of n not containing 1.

The Logarithmic Null Vectors at Level 4.

The next results are given without further comments and with trivial (non logarithmic) solutions omitted. Level $n=4$ is the smallest level where we find a logarithmic null vector of rank 3, namely with $h=-3, c=25$.

(h,c)	$\lvert\chi_{h,c}^{(4)}\rangle$
$(-\frac{1}{4},-7)$	$(315L_{-1}^4+315L_{-2}^2-210L_{-3}L_{-1}-210L_{-4}-1050L_{-2}L_{-1}^2)\,\lvert\frac{-1}{4};a(\theta)\rangle$
	$+(-878L_{-3}L_{-1}+2577L_{-1}^4-11830L_{-2}L_{-1}^2+3657L_{-2}^2-1718L_{-4})\,\lvert\frac{-1}{4};\partial_\theta a(\theta)\rangle$
$(0,-2)$	$(L_{-1}^4-2L_{-2}L_{-1}^2-2L_{-3}L_{-1})\,\lvert 0;a(\theta)\rangle+2L_{-4}\,\lvert 0;\partial_\theta a(\theta)\rangle$
$(\frac{3}{8},-2)$	$(1260L_{-1}^4+2835L_{-2}^2+1260L_{-3}L_{-1}-1890L_{-4}-6300L_{-2}L_{-1}^2)\,\lvert\frac{3}{8};a(\theta)\rangle$
	$+(3832L_{-3}L_{-1}+2152L_{-1}^4-14120L_{-2}L_{-1}^2+9882L_{-2}^2-7008L_{-4})\,\lvert\frac{3}{8};\partial_\theta a(\theta)\rangle$
$(0,1)$	$(-3L_{-1}^4+12L_{-2}L_{-1}^2-6L_{-3}L_{-1})\,\lvert 0;a(\theta)\rangle+(-16L_{-2}^2+12L_{-4})\,\lvert 0;\partial_\theta a(\theta)\rangle$
$(1,1)$	$(-60L_{-1}^4+240L_{-2}L_{-1}^2+120L_{-3}L_{-1}-240L_{-4})\,\lvert 1;a(\theta)\rangle$
	$+(-89L_{-1}^4+476L_{-2}L_{-1}^2+118L_{-3}L_{-1}-716L_{-4})\,\lvert 1;\partial_\theta a(\theta)\rangle$
$(\frac{9}{4},1)$	$(45L_{-1}^4+405L_{-2}^2+630L_{-3}L_{-1}-810L_{-4}-450L_{-2}L_{-1}^2)\,\lvert\frac{9}{4};a(\theta)\rangle$
	$+(1996L_{-3}L_{-1}+110L_{-1}^4-1220L_{-2}L_{-1}^2+1206L_{-2}^2-2772L_{-4})\,\lvert\frac{9}{4};\partial_\theta a(\theta)\rangle$
$(-\frac{21}{4},25)$	$(-990L_{-1}^4-8910L_{-2}^2-33660L_{-3}L_{-1}-65340L_{-4}-9900L_{-2}L_{-1}^2)\,\lvert\frac{-21}{4};a(\theta)\rangle$
	$+(45946L_{-3}L_{-1}+901L_{-1}^4+11650L_{-2}L_{-1}^2+12861L_{-2}^2+102234L_{-4})\,\lvert\frac{-21}{4};\partial_\theta a(\theta)\rangle$
$(-3,25)$	$(63504L_{-1}^4+254016L_{-2}L_{-1}^2+635040L_{-3}L_{-1}+762048L_{-4})\,\lvert -3;a(\theta)\rangle$
	$+(59283L_{-1}^4+110124L_{-2}L_{-1}^2+148302L_{-3}L_{-1}+76356L_{-4})\,\lvert -3;\partial_\theta a(\theta)\rangle$
	$+(-15104L_{-1}^4-186920L_{-2}L_{-1}^2-63504L_{-2}^2-450920L_{-3}L_{-1}-575628L_{-4})\,\lvert -3;\partial_\theta^2 a(\theta)\rangle$
$(-\frac{27}{8},28)$	$(77220L_{-1}^4+173745L_{-2}^2+849420L_{-3}L_{-1}+1042470L_{-4}+386100L_{-2}L_{-1}^2)\,\lvert\frac{-27}{8};a(\theta)\rangle$
	$+(269896L_{-3}L_{-1}+71336L_{-1}^4+150760L_{-2}L_{-1}^2-148374L_{-2}^2+113616L_{-4})\,\lvert\frac{-27}{8};\partial_\theta a(\theta)\rangle$
$(-2,28)$	$(13860L_{-2}L_{-1}^2+27720L_{-3}L_{-1}+27720L_{-4}+6930L_{-1}^4)\,\lvert -2;a(\theta))\rangle$
	$+(1577L_{-1}^4-9716L_{-2}L_{-1}^2-3564L_{-2}^2-18640L_{-3}L_{-1}-21412L_{-4})\,\lvert -2;\partial_\theta a(\theta)\rangle$
$(-\frac{11}{4},33)$	$(208845L_{-1}^4+696150L_{-2}L_{-1}^2+208845L_{-2}^2+1253070L_{-3}L_{-1}+1253070L_{-4})\,\lvert\frac{-11}{4};a(\theta)\rangle$
	$+(58354L_{-1}^4-244540L_{-2}L_{-1}^2-304086L_{-2}^2-525036L_{-3}L_{-1}-684156L_{-4})\,\lvert\frac{-11}{4};\partial_\theta a(\theta)\rangle$

The Logarithmic Null Vectors at Level 5.

Again, we only list the non-trivial results. Searching for logarithmic null vectors produces a high amount of trivial solutions (mostly from minimal models). Indeed, although minimal models contain pairs of operators $\Phi_{r,s}(z)$, $\Phi_{t,u}(z)$ with $h_{r,s}(c) = h_{t,u}(c)$, these operators are not different ones, but are identified with each other. Therefore, the existence of logarithmic operators is bound to the non-existence of a BRST operator such that these pairs of operators get identified in the BRST invariant representation modules. It has been shown for the $c_{p,1}$ models that one of the BRST charges becomes a local operator and thus an element of the field content of the theory itself, spoiling the usual field identifications in minimal models.

(h,c)	$\vert\chi^{(5)}_{h,c}\rangle$
$(-\frac{1}{2},-\frac{25}{2})$	$(168L^5_{-1} - 840L_{-2}L^3_{-1} - 378L_{-3}L^2_{-1} + (-504L_{-4} + 672L^2_{-2})L_{-1} + 168L_{-3}L_{-2} - 84L_{-5})\,\vert\tfrac{-1}{2}; a(\theta)\rangle + (-260L^5_{-1} - 6316L_{-2}L^3_{-1} + 1425L_{-3}L^2_{-1} + (6576L^2_{-2} + 864L_{-4})L_{-1} + 2076L_{-5} + 1308L_{-3}L_{-2})\,\vert\tfrac{-1}{2}; \partial_\theta a(\theta)\rangle$
$(-\frac{1}{4},-7)$	$(-4095L^5_{-1} + 13650L_{-2}L^3_{-1} + 16380L_{-3}L^2_{-1} + (8190L_{-4} - 4095L^2_{-2})L_{-1} - 8190L_{-3}L_{-2} + 4095L_{-5})\,\vert\tfrac{-1}{4}; a(\theta)\rangle + (-13641L^5_{-1} + 86150L_{-2}L^3_{-1} + 78564L_{-3}L^2_{-1} + (-27201L^2_{-2} + 5442L_{-4})L_{-1} + 81L_{-5} - 52482L_{-3}L_{-2})\,\vert\tfrac{-1}{4}; \partial_\theta a(\theta)\rangle$
$(0,-7)$	$(9L^5_{-1} - 60L_{-2}L^3_{-1} + 64L^2_{-2}L_{-1} - 6L_{-3}L^2_{-1} - 36L_{-4}L_{-1})\,\vert 0; a(\theta)\rangle + (-32L_{-3}L_{-2} + 12L_{-5})\,\vert 0; \partial_\theta a(\theta)\rangle$
$(0,-2)$	$(-L^5_{-1} + 9L_{-2}L^3_{-1} - 14L^2_{-2}L_{-1} + 4L_{-3}L^2_{-1} + 4L_{-4}L_{-1})\,\vert 0; a(\theta)\rangle + (7L_{-3}L_{-2} + L_{-5})\,\vert 0; \partial_\theta a(\theta)\rangle$
$(\frac{3}{8},-2)$	$(-13860L^5_{-1} + 69300L_{-2}L^3_{-1} + 55440L_{-3}L^2_{-1} + (-6930L_{-4} - 31185L^2_{-2})L_{-1} - 62370L_{-3}L_{-2} + 31185L_{-5})\,\vert\tfrac{3}{8}; a(\theta)\rangle + (32L^5_{-1} + 36800L_{-2}L^3_{-1} + 18352L_{-3}L^2_{-1} + (-55368L^2_{-2} + 4636L_{-4})L_{-1} + 69228L_{-5} - 110736L_{-3}L_{-2})\,\vert\tfrac{3}{8}; \partial_\theta a(\theta)\rangle$
$(1,-2)$	$(90L^5_{-1} - 900L_{-2}L^3_{-1} + 540L_{-3}L^2_{-1} + (-1080L_{-4} + 1440L^2_{-2})L_{-1} - 720L_{-3}L_{-2} + 360L_{-5})\,\vert 1; a(\theta)\rangle + (433L^5_{-1} - 3910L_{-2}L^3_{-1} + 1878L_{-3}L^2_{-1} + (6568L^2_{-2} - 6276L_{-4})L_{-1} + 1012L_{-5} - 3464L_{-3}L_{-2})\,\vert 1; \partial_\theta a(\theta)\rangle$
$(0,1)$	$(-3L^5_{-1} + 12L_{-2}L^3_{-1} + 6L_{-3}L^2_{-1} - 12L_{-4}L_{-1})\,\vert 0; a(\theta)\rangle + (-32L_{-3}L_{-2} + 20L_{-5})\,\vert 0; \partial_\theta a(\theta)\rangle$
$(\frac{1}{4},1)$	$(-7785L^5_{-1} + 9450L_{-2}L^3_{-1} + 48060L_{-3}L^2_{-1} + (46710L_{-4} - 1665L^2_{-2})L_{-1} - 26370L_{-3}L_{-2} + 45045L_{-5})\,\vert\tfrac{1}{4}; a(\theta)\rangle + (18058L^5_{-1} - 49260L_{-2}L^3_{-1} - 24168L_{-3}L^2_{-1} + (39362L^2_{-2} - 46068L_{-4})L_{-1} - 6706L_{-5} + 7556L_{-3}L_{-2})\,\vert\tfrac{1}{4}; \partial_\theta a(\theta)\rangle$
$(1,1)$	$(3600L_{-5} + 300L^5_{-1} - 1200L_{-2}L^3_{-1} - 1800L_{-3}L^2_{-1})\,\vert 1; a(\theta)\rangle + (157L^5_{-1} - 1100L_{-2}L^3_{-1} - 1242L_{-3}L^2_{-1} + (-512L^2_{-2} + 2400L_{-4})L_{-1} + 256L_{-3}L_{-2} + 7540L_{-5})\,\vert 1; \partial_\theta a(\theta)\rangle$

(h,c)	$\lvert\chi^{(5)}_{h,c}\rangle$ (continued)
$(\frac{9}{4},1)$	$(-525L_{-1}^5+5250L_{-2}L_{-1}^3-2100L_{-3}L_{-1}^2+(-5250L_{-4}-4725L_{-2}^2)L_{-1}-9450L_{-3}L_{-2}$
	$+\,23625L_{-5})\,\lvert\tfrac{9}{4};a(\theta)\rangle$
	$+\,(-646L_{-1}^5+7860L_{-2}L_{-1}^3-6504L_{-3}L_{-1}^2+(-8334L_{-2}^2-7860L_{-4})L_{-1}+54270L_{-5}$
	$-\,16668L_{-3}L_{-2})\,\lvert\tfrac{9}{4};\partial_\theta a(\theta)\rangle$
$(4,1)$	$(10920L_{-1}^5-218400L_{-2}L_{-1}^3+589680L_{-3}L_{-1}^2+(-1703520L_{-4}+698880L_{-2}^2)L_{-1}$
	$-\,1397760L_{-3}L_{-2}+2271360L_{-5})\,\lvert 4;a(\theta)\rangle$
	$+\,(22973L_{-1}^5-495860L_{-2}L_{-1}^3+1491702L_{-3}L_{-1}^2+(1703232L_{-2}^2-4610268L_{-4})L_{-1}$
	$+\,6714864L_{-5}-3755904L_{-3}L_{-2})\,\lvert 4;\partial_\theta a(\theta)\rangle$
$(-\frac{21}{4},25)$	$(-22770L_{-1}^5-227700L_{-2}L_{-1}^3-1001880L_{-3}L_{-1}^2+(-3051180L_{-4}-204930L_{-2}^2)L_{-1}$
	$-\,409860L_{-3}L_{-2}-4713390L_{-5})\,\lvert\tfrac{-21}{4};a(\theta)\rangle$
	$+\,(-14173L_{-1}^5-81010L_{-2}L_{-1}^3-210716L_{-3}L_{-1}^2+(-18261L_{-2}^2-211166L_{-4})L_{-1}$
	$+\,126477L_{-5}-36522L_{-3}L_{-2})\,\lvert\tfrac{-21}{4};\partial_\theta a(\theta)\rangle$
$(-8,25)$	$(243540L_{-1}^5+4870800L_{-2}L_{-1}^3+27763560L_{-3}L_{-1}^2+(119821680L_{-4}+15586560L_{-2}^2)L_{-1}$
	$+\,62346240L_{-3}L_{-2}+319524480L_{-5})\,\lvert -8;a(\theta)\rangle$
	$+\,(-201067L_{-1}^5-4833140L_{-2}L_{-1}^3-30958458L_{-3}L_{-1}^2+(-18063808L_{-2}^2$
	$-\,149094204L_{-4})L_{-1}-437525104L_{-5}-80048512L_{-3}L_{-2})\,\lvert -8;\partial_\theta a(\theta)\rangle$
$(-3,25)$	$(494629L_{-1}^5+166300L_{-2}L_{-1}^3-8361514L_{-3}L_{-1}^2+(-8372784L_{-2}^2-42795892L_{-4})L_{-1}$
	$-\,86217192L_{-5}-15363936L_{-3}L_{-2})\,\lvert -3;a(\theta)\rangle$
	$+\,(-254016000L_{-1}L_{-4}-31752000L_{-2}L_{-1}^3-111132000L_{-3}L_{-1}^2-285768000L_{-5}$
	$-\,7938000L_{-1}^5)\,\lvert -3;\partial_\theta a(\theta)\rangle$
	$+\,(210105L_{-1}^5+20103300L_{-2}L_{-1}^3+74383470L_{-3}L_{-1}^2+(197235360L_{-4}+13547520L_{-2}^2)L_{-1}$
	$+\,20321280L_{-3}L_{-2}+266025060L_{-5})\,\lvert -3;\partial_\theta^2 a(\theta)\rangle$
$(-\frac{5}{4},25)$	$(-522018L_{-1}^5-200340L_{-2}L_{-1}^3-1437912L_{-3}L_{-1}^2+(-3132108L_{-4}+321678L_{-2}^2)L_{-1}$
	$+\,449820L_{-3}L_{-2}-2810430L_{-5})\,\lvert\tfrac{-5}{4};a(\theta)\rangle$
	$+\,(112097L_{-1}^5+886810L_{-2}L_{-1}^3+2533260L_{-3}L_{-1}^2+(-350215L_{-2}^2+4848726L_{-4})L_{-1}$
	$+\,4498511L_{-5}-412174L_{-3}L_{-2})\,\lvert\tfrac{-5}{4};\partial_\theta a(\theta)\rangle$
$(-4,26)$	$(-658350L_{-1}^5-4389000L_{-2}L_{-1}^3-15800400L_{-3}L_{-1}^2+(-39501000L_{-4}-2633400L_{-2}^2)L_{-1}$
	$-\,5266800L_{-3}L_{-2}-52668000L_{-5})\,\lvert -4;a(\theta)\rangle$
	$+\,(-66567L_{-1}^5+1411570L_{-2}L_{-1}^3+6502092L_{-3}L_{-1}^2+(3205032L_{-2}^2+21063180L_{-4})L_{-1}$
	$+\,37048440L_{-5}+7128264L_{-3}L_{-2})\,\lvert -4;\partial_\theta a(\theta)\rangle$
$(-2,28)$	$(-235620L_{-1}^5-415800L_{-2}L_{-1}^3-1413720L_{-3}L_{-1}^2+(-2827440L_{-4}+110880L_{-2}^2)L_{-1}$
	$+\,110880L_{-3}L_{-2}-2716560L_{-5})\,\lvert -2;a(\theta)\rangle$
	$+\,(33601L_{-1}^5+535350L_{-2}L_{-1}^3+1481478L_{-3}L_{-1}^2+(93608L_{-2}^2+3042156L_{-4})L_{-1}$
	$+\,3302084L_{-5}+180728L_{-3}L_{-2})\,\lvert -2;\partial_\theta a(\theta)\rangle$
$(-\frac{27}{8},28)$	$(-554268L_{-1}^5-2771340L_{-2}L_{-1}^3-8868288L_{-3}L_{-1}^2+(-19676514L_{-4}-1247103L_{-2}^2)L_{-1}$
	$-\,2494206L_{-3}L_{-2}-23694957L_{-5})\,\lvert\tfrac{-27}{8};a(\theta)\rangle$
	$+\,(9632L_{-1}^5+1526208L_{-2}L_{-1}^3+5327280L_{-3}L_{-1}^2+(2238744L_{-2}^2+13829124L_{-4})L_{-1}$
	$+\,0919684L_{-5}+4477488L_{-3}L_{-2})\,\lvert\tfrac{-27}{8};\partial_\theta a(\theta)\rangle$

(h,c)	$\vert\chi^{(5)}_{h,c}\rangle$ (continued)
$(-5,28)$	$(-24024L_{-1}^5 - 240240L_{-2}L_{-1}^3 - 864864L_{-3}L_{-1}^2 + (-2306304L_{-4} - 384384L_{-2}^2)L_{-1}$ $- 960960L_{-3}L_{-2} - 3843840L_{-5})\,\vert -5; a(\theta)\rangle$ $+ (-5792L_{-1}^5 + 28595L_{-2}L_{-1}^3 + 165576L_{-3}L_{-1}^2 + (269976L_{-2}^2 + 713808L_{-4})L_{-1}$ $+ 2087148L_{-5} + 867132L_{-3}L_{-2})\,\vert -5; \partial_\theta a(\theta)\rangle$
$(-4,33)$	$(-12186720L_{-1}^5 - 81244800L_{-2}L_{-1}^3 - 235609920L_{-3}L_{-1}^2 + (-503717760L_{-4}$ $- 86661120L_{-2}^2)L_{-1} - 173322240L_{-3}L_{-2} - 671623680L_{-5})\,\vert -4; a(\theta)\rangle$ $+ (81693L_{-1}^5 + 37104780L_{-2}L_{-1}^3 + 121415478L_{-3}L_{-1}^2 + (98074688L_{-2}^2$ $+ 308044644L_{-4})L_{-1} + 546134192L_{-5} + 217814656L_{-3}L_{-2})\,\vert -4; \partial_\theta a(\theta)\rangle$
$(-\frac{11}{4},33)$	$(-30421755L_{-1}^5 - 101405850L_{-2}L_{-1}^3 - 283936380L_{-3}L_{-1}^2 + (-547591590L_{-4}$ $- 30421755L_{-2}^2)L_{-1} - 60843510L_{-3}L_{-2} - 578013345L_{-5})\,\vert \frac{-11}{4}; a(\theta)\rangle$ $+ (2904546L_{-1}^5 + 81047140L_{-2}L_{-1}^3 + 219449816L_{-3}L_{-1}^2 + (77929626L_{-2}^2$ $+ 457905228L_{-4})L_{-1} + 582191814L_{-5} + 136340532L_{-3}L_{-2})\,\vert \frac{-11}{4}; \partial_\theta a(\theta)\rangle$
$(-\frac{7}{2},\frac{77}{2})$	$(-66512160L_{-1}^5 - 332560800L_{-2}L_{-1}^3 - 848030040L_{-3}L_{-1}^2 + (-1596291840L_{-4}$ $- 266048640L_{-2}^2)L_{-1} - 465585120L_{-3}L_{-2} - 1862340480L_{-5})\,\vert \frac{-7}{2}; a(\theta)\rangle$ $+ (4370107L_{-1}^5 + 196391660L_{-2}L_{-1}^3 + 526060563L_{-3}L_{-1}^2 + (360913408L_{-2}^2$ $+ 1102564968L_{-4})L_{-1} + 1629758776L_{-5} + 659098684L_{-3}L_{-2})\,\vert \frac{-7}{2}; \partial_\theta a(\theta)\rangle$

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