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Curiosities at c-effective = 1

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Abstract

The moduli space of all rational conformal quantum field theories with effective central charge $c_{eff} = 1$ is considered. Whereas the space of unitary theories essentially forms a manifold, the non unitary ones form a fractal which lies dense in the parameter plane. Moreover, the points of this set are shown to be in one-to-one correspondence with the elements of the modular group for which an action on this set is defined.

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1 Introduction

Since the invention of two dimensional conformal quantum field theory by BPZ [1], one of the great outstanding problems is the classification of all rational conformal field theories (RCFTs). BPZ have found the discrete series of minimal models, and later all unitary RCFTs with $c = 1$ were be classified [2, 3, 4]. The study of $\mathcal{W}$-algebras proved to be fruitful for the search of new rational models [5, 6]. In particular, $\mathcal{W}$-algebras were found which exist only for some special values of the central charge. In a recent work [7] we have completed the classification of all theories with $c_{\mathit{eff}} = c - 24h_{min} \leq 1$. Here h_{min} denotes the energy of the minimal highest weight state. There we constructed a series of new rational models (coming from certain $\mathcal{W}$-algebras). These theories exist for $c = 1 - 8k, k \in \mathbb{Z}$ and are $c_{\mathit{eff}} = 1$ theories.
In the following we will study the moduli space $\mathcal{M}$ of all theories with $c_{\mathit{eff}} = 1$. We write $\mathcal{M} = M \cup M'$, where M denotes the well known space of unitary $c = 1$ theories and M' denotes the space of non unitary $c_{\mathit{eff}} = 1$ theories. With $\mathcal{R}(X)$ we denote the set of RCFTs in a moduli space X.
Considering the moduli space of non unitary RCFTs, $\mathcal{R}(M')$, one obtains a strange result, see figure 2. In this paper we will answer several questions which naturally arise from this picture: Is this set $\mathcal{R}(M')$ dense in $(\mathbb{R}_+)^2$? Is there an explanation of the quadric curves? Most important, are the points of $\mathcal{R}(M')$ rational points in a manifold of generically non rational conformal field theories (CFTs), or are they isolated? Finally, will the space of all CFTs look like that even for $c_{\mathit{eff}} > 1$?
The last question cannot be answered yet. The others will be answered here. Surprisingly, the structure of the set $\mathcal{R}(M')$ is highly "chaotic" and forms a self similar fractal, nonetheless dense in $\mathbb{R}^2$. Moreover, its points are in one-to-one correspondence to the elements of the modular group $\mathrm{PSL}(2, \mathbb{Z})$. This correspondence suggests an application of these theories to phenomena like the fractional quantum Hall effect.

2 Preliminaries

In a recent work [7] new RCFTs with effective central charge $c_{\mathit{eff}} = 1$ have been discovered and classified which possess extended symmetry algebras ($\mathcal{W}$-algebras). In the following, let $\varepsilon \in \{0, 1\}$ and let $\min\lambda$ denote the smallest representative of $\lambda \in \mathbb{Z}/m\mathbb{Z}$. Then we have shown that for every k there exist two RCFTs, one with

$$\begin{aligned} c &= 1 - 24k\,, & k \in \mathbb{Z}_+/2 \\ h_{\frac{\lambda}{2k+2\varepsilon},(-)^\varepsilon\frac{\lambda}{2k+2\varepsilon}} &= \left[\left(\frac{\min\lambda}{2k+2\varepsilon}\right)^2 - 1\right]k + \varepsilon\left(\frac{\min\lambda}{2k+2\varepsilon}\right)^2, & \lambda \in \mathbb{Z}/(k+\varepsilon)\mathbb{Z} \\ h_{1,1} &= 0 \end{aligned} \tag{2.1}$$

which has the extended chiral symmetry algebra $\mathcal{W}(2,3k)$, and its $\mathbb{Z}_2$ orbifold

$$\begin{array}{rcll} c &=& 1-24k\,, & k\in\mathbb{Z}_+/4 \\ h_{\frac{\lambda}{4k+4\varepsilon},(-)^{\varepsilon}\frac{\lambda}{4k+4\varepsilon}} &=& \left[\left(\dfrac{\min\lambda}{4k+4\varepsilon}\right)^2-1\right]k+\varepsilon\left(\dfrac{\min\lambda}{4k+4\varepsilon}\right)^2, & \lambda\in\mathbb{Z}/(4k+4\varepsilon)\mathbb{Z} \\ h_{1,1} &=& 0 & \\ h_{2,2} &=& 3k & \end{array} \tag{2.2}$$

with chiral symmetry algebra $\mathcal{W}(2,8k)$. Here $h_{r,s}$ denotes the Virasoro highest weight analogous to the Virasoro highest weights of degenerate models to generic central charge $c=1-24\alpha_0^2$ given by $h_{r,s}=\frac{1}{4}\left((r\alpha_-+s\alpha_+)^2-(\alpha_-+\alpha_+)^2\right)$, where $\alpha_\pm=\alpha_0\pm\sqrt{1+\alpha_0^2}$. Note that in contrast to the generic degenerate Virasoso model $s=\pm r$ and r is not restricted to integers only. For further details see [7]. All these RCFTs indeed have effective central charge $c_{\mathit{eff}}=c-24h_{min}=1$.

The finitely many highest weight representations are highest weight representations with respect to the extended symmetry algebra. The characters which are infinite sums of Virasoro characters can be expressed in terms of Jacobi-Riemann Θ-functions divided by the usual Dedekind η-function. For example the vacuum character of the $\mathcal{W}(2,3k)$ theories is given by

$$\chi_{vac}(\tau) = \sum_{n\in\mathbb{Z}_+}\chi^{Vir}_{h_{n,n}}(\tau)=q^{\frac{1-c}{24}}\sum_{n\in\mathbb{Z}_+}\frac{q^{h_{n,n}}-q^{h_{n,-n}}}{\eta(\tau)} \tag{2.3}$$

$$= \frac{1}{2\eta(\tau)}\left(\Theta_{0.k}(\tau)-\Theta_{0,k+1}(\tau)\right), \tag{2.4}$$

where $\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, $\Theta_{\lambda,k}(\tau)=\sum_{n\in\mathbb{Z}}q^{(2kn+\lambda)^2/4k}$, and $q=e^{2\pi i\tau}$. The other characters can be obtained by the modular transformation $S:\tau\mapsto-\frac{1}{\tau}$. Details including S and T matrix and fusion rules again can be found in [7].

The partition function $Z(\tau,\bar\tau)$ is diagonal in the $\mathcal{W}$-characters $\chi_{\lambda,\varepsilon}$ (obvious notation), if the symmetric theory is chosen. Therefore, the $\mathcal{W}$-algebra is the maximally extended symmetry algebra. In this case one has for the $\mathcal{W}(2,3k)$ theories

$$Z(\tau,\bar\tau) = \sum_{\lambda,\varepsilon}\left|\chi_{\lambda,\varepsilon}\right|^2 \tag{2.5}$$

$$= \frac{1}{2\eta(\tau)\eta(\bar\tau)}\left(\sum_{\lambda=0}^{2k-1}\left|\Theta_{\lambda,k}\right|^2+\sum_{\lambda=0}^{2k}\left|\Theta_{\lambda,k+1}\right|^2\right) \tag{2.6}$$

$$= \frac{1}{2}\left(Z[k]+Z[k+1]\right), \tag{2.7}$$

where we denote with $Z[2R^2]\equiv Z(R)$ the partition function of an U(1) theory of S^1 mappings with fixed compactification radius R. This notation is chosen for later convenience. The U(1) partition function is given by

$$Z[k]=\frac{1}{\eta\bar\eta}\sum_{m,n\in\mathbb{Z}}q^{\frac{1}{4k}(n+km)^2}\bar q^{\frac{1}{4k}(n-km)^2}\,. \tag{2.8}$$

This function is modular invariant and, moreover, possesses the well known duality property $Z[k] = Z[\frac{1}{k}]$. Of course, non diagonal partition functions can be obtained for theories which are not symmetrically composed from the chiral and antichiral part. These non diagonal theories are related to the diagonal ones via automorphisms of the fusion rules.

3 Classification

First, let us recall some important identities for U(1) partition functions for compactification radii which are square roots of rational numbers: Let $k, p, q \in \mathbb{Z}_+$. In that case one has

$$
\begin{aligned}
Z[k] &= \frac{1}{\eta\bar{\eta}} \sum_{1\leq n\leq 2k} |\Theta_{n,k}|^2 \,, && (3.1)\\
Z[\frac{p}{q}] &= \frac{1}{\eta\bar{\eta}} \sum_{n \,\mathrm{mod}\, 2pq} \Theta_{n,pq}(\tau)\Theta_{n',pq}(\bar{\tau})\,, && (3.2)
\end{aligned}
$$

where $n' = qr + ps \bmod 2pq$ if $n = qr - ps \bmod 2pq$. Then all functions of the form

$$
Z = \sum_i a_i Z[x_i], \quad x_i \in \mathbb{Q} \tag{3.3}
$$

are modular invariant and correspond to finite dimensional representations of (congruence subgroups) of PSL(2,$\mathbb{Z}$). But only a small subset of these functions is physically relevant and related to RCFTs. The physical requirements are 1.) that the power series of the characters in q has non negative integer coefficients, 2.) that the first non vanishing coefficient is equal to one and 3.) that the vacuum character does not contain currents (in order to be a Virasoro character). These requirements strongly restrict the possible linear combinations of type (3.3).
One of the main results in [7] is the classification of all RCFTs with effective central charge $c_{\mathit{eff}} = 1$. The unitary case has been treated earlier [2, 3, 4]. All unitary RCFTs with $c = 1$ fit in the A-D-E classification of finite subgroups of SU(2). The corresponding physical relevant partition functions are all given by linear combinations of U(1) partition functions corresponding to certain congruence subgroups of PSL(2,$\mathbb{Z}$). In [4] the completeness of this set of known partition functions (up to non-congruence subgroups of PSL(2,$\mathbb{Z}$)) has been proven with the help of the Serre-Stark theorem. Nevertheless one additional partition function appeared which was rejected there because it could not belong to an unitary theory. It is of the form

$$
Z_{new}[x, y] = \frac{1}{2}\left(Z[x] + Z[y]\right) , \tag{3.4}
$$

where x, y are rational numbers. We have the following classification [7]:

• **1** *Partition functions of type (3.4) are physically relevant, i.e. belong to theories with Virasoro vacuum characters,* $\chi_{vac} = \frac{q^{-c/24}}{\eta(\tau)}(1 - q + \ldots)$*, iff* $x, y \in \mathbb{Q}_+$ *such that*

$$x = \frac{p}{q}, \quad y = \frac{p'}{q'}, \quad p'q' - pq \in \{1, 4\} . \tag{3.5}$$

Then $c = 1 - 24pq$*.*

This matches exactly the series of $\mathcal{W}(2, 3k)$ and $\mathcal{W}(2, 8k)$ algebras for $p'q' - pq = 1$ and $p'q' - pq = 4$, respectively. In the case of x or $y \notin \mathbb{Z}_+$ or, due to duality, $1/x$ or $1/y \notin \mathbb{Z}_+$ we have an automorphism of the fusion rules.
The surprising fact is that only certain pairs of rational numbers yield RCFTs. In the following we will concentrate on the case of $\mathcal{W}(2, 3k)$ theories, since the $\mathcal{W}(2, 8k)$ theories can be understood as their $\mathbb{Z}_2$ orbifolds (for the case of $k \in \mathbb{Z}_+ + \frac{1}{n}$, $n = 2, 4$ see [7]).

4 Moduli Space

In the work [3] the moduli space of the $c = 1$ theories has been considered. The well known picture of the moduli space which also indicates the flow of marginal perturbations has to be modified in the manner shown in figure 1 in order to accomodate the non unitary theories. First, we note that the moduli space is now three dimensional (up to the three exceptional points for the theories with symmetry group E_6, E_7 and E_8 modded out). We used the notation $Z_n \equiv Z(n/\sqrt{2}) = Z[n^2]$. The diagonal in the R_1, R_2 plane is the line of unitarity. The points with $2R^2_{\text{circle}}$ or $2R^2_{\text{orb}} \in \mathbb{Q}_+$ belong to rational unitary $c = 1$ theories, the points in the R_1, R_2 plane which fulfill theorem 1 represent rational non unitary theories.
For the unitary $c = 1$ CFTs one has well known marginal flows which connect the RCFTs via irrational theories. This is not longer true for the non unitary RCFTs with $c_{\mathit{eff}} = 1$. On the contrary we have

• **2** *There exists no marginal flows between the non unitary RCFTs with* $c_{\mathit{eff}} = 1$*.*

This follows, since the $(1, 1)$ field v which could generate the marginal flow, has non vanishing self coupling C^v_{vv}. In fact, the field v belongs to the $\mathcal{W}$-conformal family[1] to the lowest Virasoro eigenvalue, (h_{min}, h_{min}), whose self fusion coefficient $N^{min}_{min,min}$ does not vanish [7]. It has been shown [8] that, for theories with maximally extended symmetry algebra as in our case, this implies the non vanishing of the self coupling of the primary fields contained in the considered $\mathcal{W}$-conformal family, since their self coupling constants must be proportional to $C^{min}_{min,min}$.
Let us now take a closer look at the R_1, R_2 plane. Rational non unitary theories are represented by a certain subset $\mathcal{R}(M') \subset \{x, y \in \mathbb{Q}_+ | x \neq y\}$. For the following, we

[1] the family defined with respect to the whole chiral symmetry algebra, not just with respect to the Virasoro field.

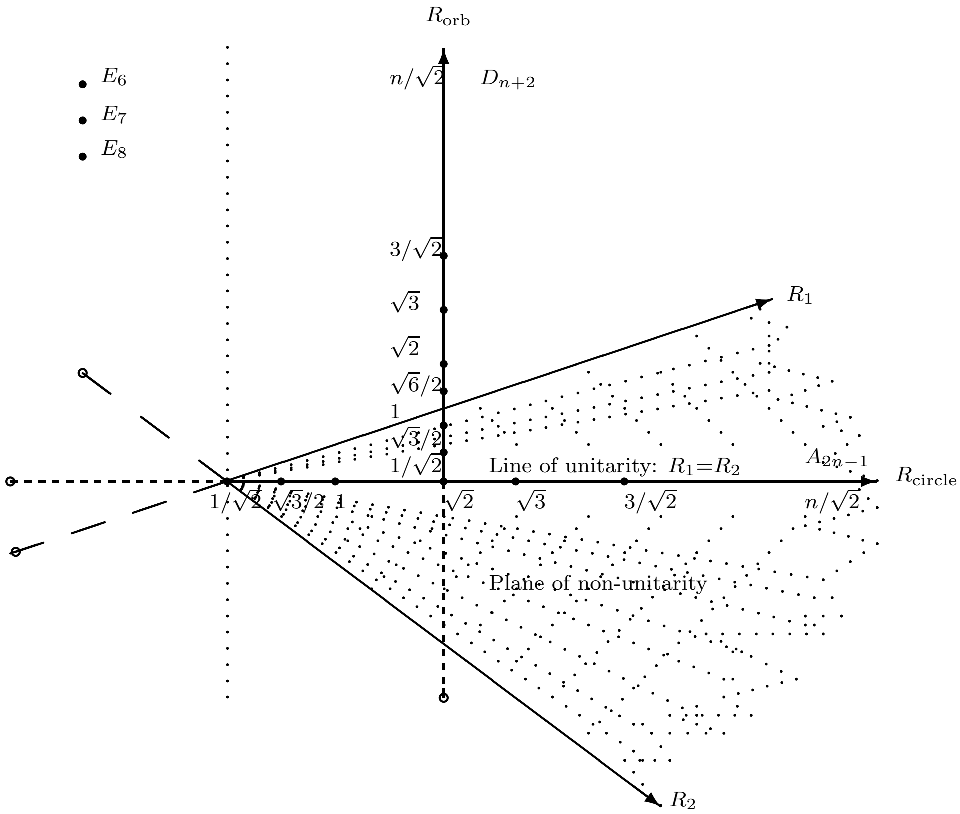

Fig. 1. Survey of $c_{eff} = 1$ models. The diagonal axis in the R_1, R_2 plane represents compactification on a circle S^1 with radius R_{circle}, the vertical axis represents compactifications on the orbifold $S^1/\mathbb{Z}_2$ with radius R_{orb}. The dashed regions of theses lines are determined by the duality $R \leftrightarrow 1/2R$. The non unitary models lie in the cut plane $\{R_1, R_2 | R_1 \neq R_2\}$.

do not assume that all rational numbers are represented in a divisor free fraction. The condition of theorem 1 can be reformulated in the following way: Iff $p, q, p', q' \in \mathbb{Z}_+$ such that

$$\det \begin{pmatrix} p' & p \\ q & q' \end{pmatrix} = 1 \tag{4.1}$$

then a RCFT exists with $c = 1 - 24pq$ and partition function $Z[p/q, p'/q'] = (Z[p/q] + Z[p'/q'])/2$ which is unique up to duality. Thus, there is a correspondence of points in $\mathcal{R}(M')$ to elements of PSL(2,$\mathbb{Z}$). Furthermore, we see immidiately that the line of unitarity, $R_1 = R_2$, would correspond to determinant zero matrices.[2] Hence, non-unitary theories exist arbitrary close to the line of unitarity, but in order to approximate one rational number by two others keeping condition (4.1) the denominators will increase

[2]More precisely, only a small subset of them corresponds to $c = 1$ theories. The other determinat zero matrices correspond to the ordinary (A_{p-1}, A_{q-1}) minimal models. For example the matrix $\binom{1\ \ p}{q\ \ pq}$ corresponds to the partition function $Z = (Z[1/pq] - Z[p/q])/2$. Other types of determinant zero matrices yield automorphisms of the fusion rules.

and in consequence the central charge will decrease. Therefore, the degree of non unitarity increases when approaching the diagonal. Figure 2 shows $\mathcal{R}(M')$ calculated up to maximal denominators of 100 000. Note, that in particular near the diagonal the density of points is rather low. Due to duality we restrict ourselves to the region $[0,1]\times[0,1]$ in the $2R_1^2, 2R_2^2$ plane. Let us now formulate the following statement:

• **3** *The set* $\Gamma \equiv \mathcal{R}(M') \cap \{(x,y)\,|\,0 \le y < x \le 1\}$ *is a fundamental region of all non unitary RCFTs with* $c_{\mathit{eff}} = 1$*.* $\mathcal{R}(M')$ *is in one-to-one correspondence to the group* PSL(2,$\mathbb{Z}$) *which acts on* $\mathcal{R}(M')$*. The correspondence to* Γ *is defined by the following identifications: Let* $A \in \mathrm{PSL}(2,\mathbb{Z})$*,* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$*. Then* A *is identified with* A^T*,* A^{-1}*, and* $(A^T)^{-1}$ *by duality. The symmetry under exchanging* R_1 *with* R_2 *yields the identification* $A \sim SA$ *where* $S \equiv \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$*, i.e. we restrict* A *to matrices with positive integer entries only. This subset of* PSL(2,$\mathbb{Z}$) *forms a semi group which acts on* Γ*.*

The proof of this statement is simple. Γ is obtained by identifying theories which are equivalent under duality. Moreover, since the exchange of the radii does not affect the partition function, we may order them such that $0 \le 2R_1^2 < 2R_2^2 \le 1$. The map from the modular group PSL(2,$\mathbb{Z}$) to $\mathcal{R}(M')$ is just given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\left|\frac{a}{d}\right|, \left|\frac{b}{c}\right| \right) . \tag{4.2}$$

PSL(2,$\mathbb{Z}$) is completely generated by the two generators $S \equiv \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T \equiv \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and the relations $S^2 = (ST)^3 = 1\!\!1$. Together with (4.2) this yields a very fast algorithm to generate pictures of $\mathcal{R}(M')$. The action of S reduces to an exchange of R_1 and R_2 which does not affect the partition function. The set Γ is obtained in the following way: Let $x = |a/d|$, $y = |c/d|$ obtained via (4.2) from any element of PSL(2,$\mathbb{Z}$), i.e. any admissible word in S, T. Then the corresponding point in Γ is given by $(\max\{\min\{x,1/x\},\min\{y,1/y\}\},\min\{\min\{x,1/x\},\min\{y,1/y\}\})$. So far, there are always two elements of PSL(2,$\mathbb{Z}$), which correspond to one point in $\mathcal{R}(M')$. They differ by two signs such that, without loss of generality, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right)$ yield the same point. We define that matrices with negative integer entries correspond to the $\mathbb{Z}_2$-orbifolds (actually, one should multiply all matrix entries with 2 to get determinant 4). This gives the desired one-to-one correspondence.

5 Strange Structure and Quadric Curves

Figure 2 shows a surprisingly strange and complex structure of the set $\mathcal{R}(M')$. First, one observes a "net of curves". We will see later that the structure of $\mathcal{R}(M')$ is also related to continued fractions. The self similarity of $\mathcal{R}(M')$ is a bit non trivial. The set seems to be divided in rhombi by a net of origin lines and symmetric hyperbolas. The self similarity of the rhombi has two directions. First by rescaling $x \mapsto \alpha x$, $\alpha \in \mathbb{Q}_+$,

i.e. moving along origin lines, and second by moving along the hyperbolas, i.e. by the mapping $x \mapsto \alpha\frac{1}{x}$. The self similarity is, at least at the crude approximation level of $\mathcal{R}(M')$ shown in the figure, not a perfect one. The "curves", which cross through the rhombi, may differ in shape and number, but the rough structure is the same. Actually, $\mathcal{R}(M')$ forms a so called multi-fractal. A multi-fractal does not have a well defined scaling factor α under which a part of it reproduces the whole. Instead of that there exist different scaling factors α_i for which certain subsets are self similar. In our case every prime number p defines such a scaling and thus $\mathcal{R}(M')$ is the union of an infinite number of fractals.
The approximation level may be defined by the maximal length of words in S, T, i.e. the number of generators after dividing out the relations $S^2 = (ST)^3 = \mathbb{1}$, which have been used to generate the picture. In the following we will assume that every element of PSL(2,$\mathbb{Z}$) is represented by a word of minimal length. Since T acts by translation of either the numerators or the denominators of x and y, but S acts by exchanging x and y, we see that every occurence of S in a word $A \in \mathrm{PSL}(2,\mathbb{Z})$ marks a node where the movement of a point of $\mathcal{R}(M')$ under the action of T changes the direction. From this the self similarity stems, since this generates the tree like structere, i.e. that the curves seem to spread out from bundle points. Since the action of S alone does not change the picture at all, a better measure for the approximation level is just the maximal number of T generators in a word.
Let us now discuss these curves. Let $(\frac{p}{q}, \frac{p'}{q'})$ be a point in $\mathcal{R}(M')$. If p, q are coprime then $a, b \in \mathbb{Z}$ exist such that we may write

$$\frac{p'}{q'} = \frac{(p')^2}{p'q'} = \frac{(ap+bq)^2}{pq \pm 1} = \frac{(ap+bq)^2}{pq}\left(1 \mp \frac{1}{pq \pm 1}\right) . \tag{5.1}$$

We see that for $p, q \gg 1$ the deviation from simple algebraic curves of the form $y = a^2x + 2ab + b^2\frac{1}{x}$ gets very small. On the other hand every point of $\mathcal{R}(M')$ lies (more or less) close to such a curve. More generally, every point of $\mathcal{R}(M')$ lies close to at least one curve with its graph being of the form $(x, \alpha x + \beta\frac{1}{x} + \gamma)$ or $(\alpha y + \beta\frac{1}{y} + \gamma, y)$ where $\alpha, \beta, \gamma \in \mathbb{Q}$. To see this just write

$$\frac{p'}{q'} = \frac{(ap+bq)(cp+dq)}{epq+f} \tag{5.2}$$

$$= \left(ac\frac{p}{q} + bd\frac{q}{p} + (ad+bc)\right)\left(e + \frac{f}{pq}\right)^{-1} \tag{5.3}$$

or with $p, q \leftrightarrow p', q'$, where $a, b, c, d, e, f \in \mathbb{Z}$. Let now $p, q \gg f$. Then there is a high probability that the numerator divides the denomintor. In particular in the case a, b or c, d coprime there are infinitely many p, q such that $(ap+bq)$ or $(cp+dq)$ divides $pq+f$ for small f. Thus we have proven

• **4** *Let (x, y) be an arbitrary point of the set $\mathcal{R}(M')$. Then there exist numbers $\alpha, \beta, \gamma \in \mathbb{Q}$ such that $y \approx \alpha x + \beta\frac{1}{x} + \gamma$ (or vice versa) up to order of at least $\frac{1}{pq}$. In particular* *there always exists a solution of the form $y \approx a^2 x + 2ab + b^2 \frac{1}{x}$ if x is given by a rational number with coprime numerator and denominator*[3].

In the sense of this proposition we may view $\mathcal{R}(M')$ as the union of an infinite set of approximated curves, which form the net-like structure of figure 2. The special curves mentioned in the second part of the proposition are the ones which collect the points fastest with increasing approximation depth. Not all parts of the graph of a certain curve have the same density of collected points. This is due to the unequally distributed probability that pairs of rational numbers up to a maximal denominator lie along a curve.

6 Density in *M'*

The probably most interesting question might be whether the set $\mathcal{R}(M')$ of non unitary RCFTs lies dense in the set M'. To answer this question we need some preparation which also enlightens the relationship of $\mathcal{R}(M')$, the modular group and continued fractions. Consider the mapping (see eqn. (4.2))

$$Q : \mathrm{PSL}(2,\mathbb{Z}) \longrightarrow \mathbb{R}^2 \tag{6.1}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(\frac{a}{d}, \frac{b}{c}\right) . \tag{6.2}$$

Define $U_n = (ST^n)^t = \left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & n \end{smallmatrix}\right)$. Then every $A \in \mathrm{PSL}(2,\mathbb{Z})$ can be written as $A = U_{n_k} \ldots U_{n_2} U_{n_1} (T^{n_0})^t$. We say that A is a word of length $\ell(A) = \sum_{i=0}^{k} n_i$. It is well known [9] that the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are given as continued fractions

$$\frac{a}{b} = [n_0, n_1, \ldots, n_{k-1}] , \tag{6.3}$$

$$\frac{c}{d} = [n_0, n_1, \ldots, n_{k-1}, n_k] , \tag{6.4}$$

where we use the notation

$$[n_0, n_1, n_2, \ldots] = n_0 - \cfrac{1}{n_1 - \cfrac{1}{n_2 - \cfrac{1}{\ddots}}} . \tag{6.5}$$

Note, that any number $w \in \mathbb{R}$ has an expansion into a continued fraction. Now let A be an approximation of a certain number $w \in \mathbb{R}$ better than $\varepsilon > 0$, i.e. $\left|\frac{a}{b} - \frac{c}{d}\right| < \varepsilon$. Then every matrix $B = A' \cdot A$ such that $\ell(A' \cdot A) = \ell(A') + \ell(A)$ yields an approximation of w better then ε. We denote matrices B with this property by $B \succ A$. Noting that $\frac{a/d}{b/c} = \frac{a}{d} \cdot \frac{c}{b} \approx w^2$, one then easily sees

[3]In this case $a, b \in \mathbb{Z}$. If the greatest common divisor in the rational number x is, say, m, then a, b are rational numbers over the modulus m.

• **5** *The orbit of all matrices $A' \cdot A \succ A$ under the mapping Q is confined to a band around w^2 with width $\varepsilon' = 2w\varepsilon + \varepsilon^2$. The band is defined by the equations $\frac{x}{y} = \alpha$ or $xy = \alpha$ where $\alpha \in [w^2 - \varepsilon', w^2 + \varepsilon'] \cup [\frac{1}{w^2} - \varepsilon', \frac{1}{w^2} + \varepsilon']$.*

Figure 3 shows an example with $A = \begin{pmatrix} 3 & 2 \\ 11 & 7 \end{pmatrix}$. This matrix yields $w^2 \approx \frac{14}{33}$ with an accuracy $\varepsilon = \left|\frac{2}{3} - \frac{7}{11}\right| = \frac{1}{33}$. The band is therefore confined to a width $\varepsilon' = 2w\varepsilon + \varepsilon^2 \approx 0.04$. The overall broadness at $x = 1$ is then around $\frac{2}{25}$. The dirty dust outside this band visible in figure 3 is due to the fact that the algorithm did not eliminate points with matrices whose length $\ell(A' \cdot A) < \ell(A') + \ell(A)$.
In the limit $\varepsilon \to 0$ the orbit shrinks to a band with zero width, since $\varepsilon' \to 0$. From this it follows that the band is filled dense by mapping all $B = A' \cdot A \succ A$ to $\mathbb{R}^2$ via Q, as $\varepsilon \to 0$ (again $\ell(A' \cdot A) = \ell(A') + \ell(A)$). In fact, the matrices of the whole modular group PSL(2,$\mathbb{Z}$) approximate (via the map Q) every real number, if the points are projected to the, say, x-axis. Let now A be given (and thus w) and consider the set of all matrices $B \succ A$. This set forms a closed semi group. Take the complementary semi group of all matrices B^{-1}. This semi group yields every possible w'-values since A^{-1} is arbitrarily shifted to the left with increasing word length $\ell(B^{-1})$ and thus does not much contribute to the w'-value of B^{-1}. Therefore this in some sense complementary set of matrices approximates every real number under the map Q and projecting onto the x-axis. From our duality relations we then obtain the desired result that the original semi group fills the w-band dense. We arrive at

• **6** *The set $\mathcal{R}(M')$ lies dense in $(\mathbb{R}_+)^2$, i.e. the set of non unitary RCFTs with $c_{\mathit{eff}} = 1$ lies dense in the moduli space of all non unitary $c_{\mathit{eff}} = 1$ CFTs.*

Finally, for completeness we want to comment on the case of the theories with $c = 1 - 24k$, k a half or quarter integer which we did not discuss here in further detail. These theories have more complicated partition functions given as certain linear combinations of (3.4). Their partition functions could not be found by Kiritsis [4] since he concentrated on the bosonic case, i.e. required that the q-power expansion of the partition functions had integer spaced exponents. In these fermionic theories the power series has half integer spaced exponents. For example the partition functions for the $W(2, 3k)$ RCFTs, $k \in \mathbb{Z}_+ + \frac{1}{2}$, are

$$\begin{aligned} Z_{\mathit{ferm}}(\tau, \bar{\tau}) &= Z[2(2l-1)] + Z[2(2l+1)] + Z[\frac{(2l-1)}{2}] + Z[\frac{(2l+1)}{2}] &\quad (6.6) \\ &= Z_{bos}[2(l-1), 2(2l+1)] + Z_{bos}[\frac{(2l-1)}{2}, \frac{(2l+1)}{2}], &\quad (6.7) \end{aligned}$$

where we have written $k = \frac{2l-1}{2}$ and Z_{bos} denotes our bosonic partition function eqn. (3.4). Of course, one again may factorize $2l \pm 1$ in different ways to obtain twisted partition functions, but this has to be done simultaneously in both bosonic partition functions.

The set $\mathcal{R}(M'_{ferm})$ for these fermionic theories is equivalent to our set $\mathcal{R}(M'_{bos})$ in the sense that there exists a bijection given by multiplying one row or column of a matrix corresponding to a point in $\mathcal{R}(M'_{bos})$ by 2. That this indeed is a bijection follows if the several identifications due to duality and symmetry under exchange of the two radii are taken into account. Similar statements hold for the orbifold theories with $k \in \mathbb{Z}_+ + \frac{1}{4}$.

7 Conclusion

We have shown that the structure of the moduli space $\mathcal{M}$ of $c_{\mathit{eff}} = 1$ theories is highly non trivial. The non unitary RCFTs form a multi-fractal but dense set of isolated points, thus by no means an algebraic variety or orbifold, as it is conjectured for the unitary ones (and as it is true for the unitary theories with $c_{\mathit{eff}} \leq 1$). This may shed a new light of the possible structure of the space of all CFTs.
We proved that the points of the set of non unitary RCFTs are in one-to-one correspondence with the elements of the modular group PSL(2,$\mathbb{Z}$). We constructed an action of PSL(2,$\mathbb{Z}$) on $\mathcal{R}(M')$. Moreover, the approximation of the set $\mathcal{R}(M')$ up to a certain level (which corresponds to continued fractions up to a maximal length) yields an interesting structure of quadric curves.
While these non unitary theories presumably will not have an application in string theories, they very well may help to describe phenomena in statistical physics, where non unitarity might be essential. One example could be the fractional quantum Hall effect (FQHE). One way to describe the FQHE works with quantum fluid droplets. On the border of these droplets lives a CFT which necessarily has $c_{\mathit{eff}} = 1$. If the border is not simply connected (e.g. the border of an annulus), then there can be charge transport between different border components. In this case the CFTs living on the border should be non unitary and therefore should be one of the models considered in this work.
Is has been stated often that only FQHEs to fractional fillings ν with small denominators should be observable. In fact, larger denominators are much more difficult to measure in experiments. This is similar to our models. Points with larger denominators are always close to points with rather small denominators. Of course, if a system is forced to go to one particular theory, it will choose the one with less non unitarity and a smaller number of states (i.e. it will choose the theory nearest to a trivial ground state theory), and this theory is the one with smaller denominators. Thus the "topology" of the set of our theories could very well force a behaviour as it is proposed for the FQHE.
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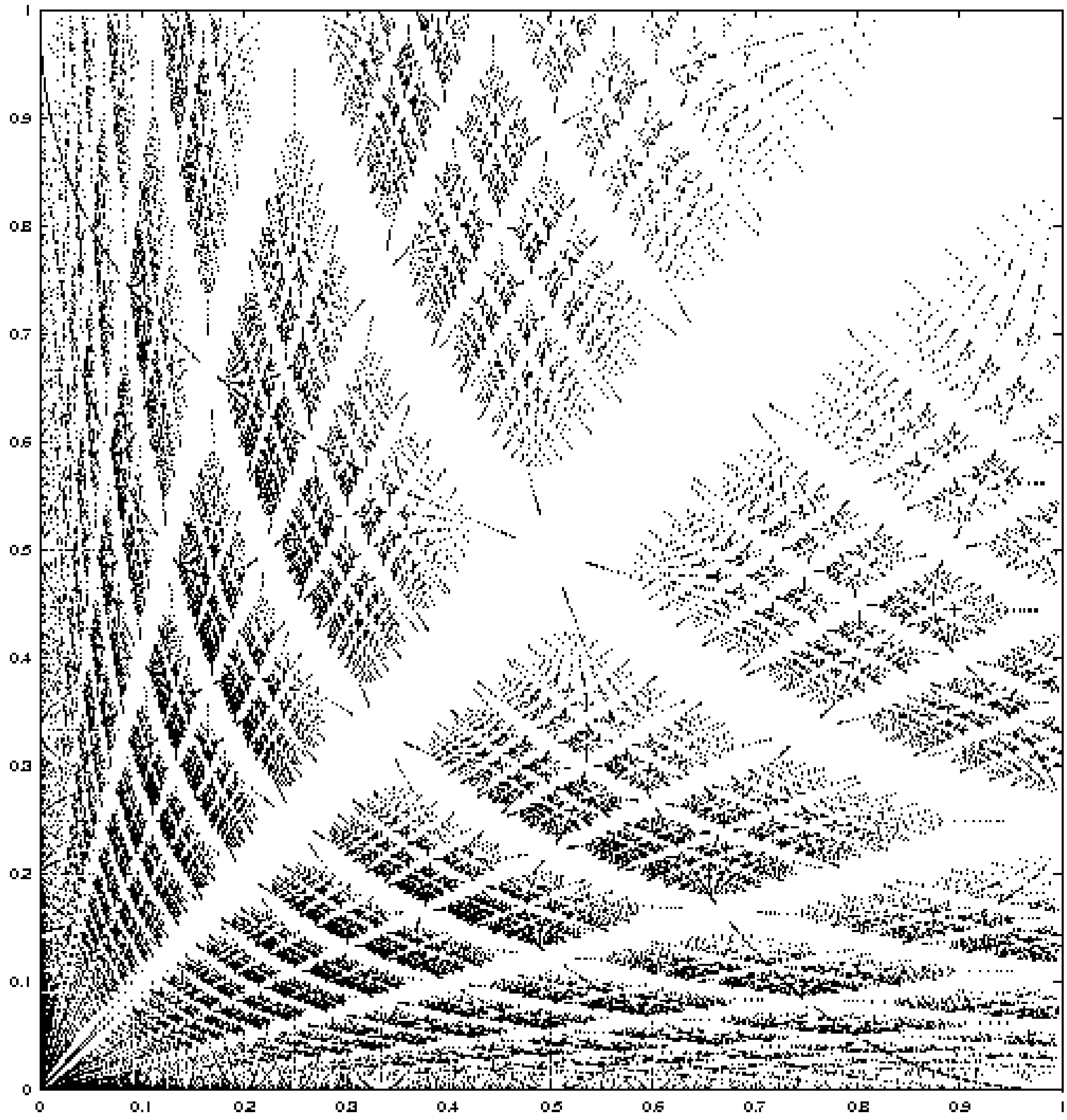

Fig. 2. The moduli space of non unitary $c_{eff} = 1$ theories. The x-axis is $2R_1^2 = p/q$, the y-axis is $2R_2^2 = p'/q'$, both in the range $[0, 1]$. Shown are points with denominators $q, q' \leq 100\,000$. The points outside the shown region are obtained by duality $p/q \mapsto q/p$, $p'/q' \mapsto q'/p'$.

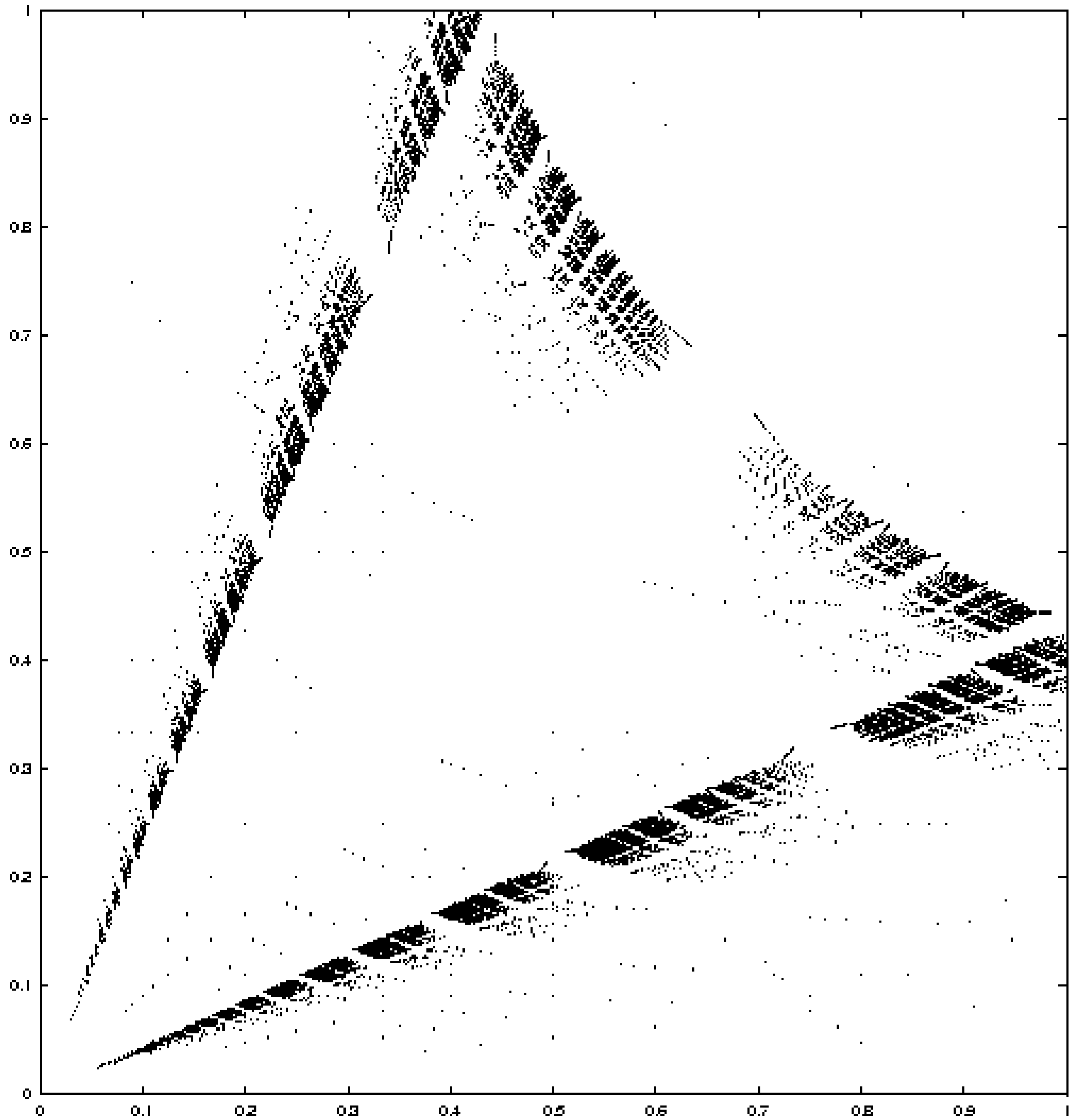

Fig. 3. Orbit of the matrix $A = \binom{3\ 2}{11\ 7}$ under the action of PSL(2,$\mathbb{Z}$) using the map Q. The orbit is confined to a band with width $\varepsilon' \approx 0.04$ around $w^2 \approx 0.42$. Plotted are all matrices $A' \cdot A$ with $\ell(A') \leq 23$.