Introduction: In recent years a number of fundamental ideas and methods of mathematical physics have penetrated through psychological barriers between physics and topology. In the knot theory this development was initiated by V. Jones who used von Neumann algebras to construct a new polynomial invariant of knots and links in the 3-dimensional sphere $S^3$. The Jones’s discovery gave impetus to an enormous development in the knot theory, 3-dimensional topology and related domains. This development was the main subject of the meeting.

Since 1984 when Jones introduced his polynomial the main line of attack was explanation of the nature of this polynomial and its generalization, say, to links in other 3-manifolds. To the moment of writing, several different points of view on the Jones polynomial have been developed basing on various techniques coming from algebra and mathematical physics. Here is a short but impressive list of theories more or less directly involved in the subject: theory of quantum groups, conformal field theory in dimension 2, representation theory of symmetric groups and Hecke algebras, theory of exactly solvable models of statistical mechanics etc..

The Arbeitsgemeinschaft considered 4 different though related lines of study forming the main body of the theory.

I. The first and most algebraic approach stems directly from the original Jones’s paper. It involves Temperley-Lieb algebras, Hecke algebras and their natural modifications due to Birman-Wenzl. The core of this approach is the theory of braid groups and their linear representations.

II. The second approach was concerned with Witten’s ideas, relating the Jones polynomial to the conformal field theories in dimension 2 and Chern-Simons invariants. From the topological viewpoint the important achievement of Witten is the inclusion of the Jones polynomial in a more general picture of topological quantum field theories.

III. The approach based on the statistical mechanical models and the theory of quantum groups: A state sum model for the Jones polynomial was introduced by Kauffman. For other related polynomials one involves more general vertex models associated with $R$-matrices. This line is crowned with a construction of the topological quantum field theory in dimension 3 extending the Jones polynomial (and predicted by Witten).
IV. The last approach was concerned with another kind of state sum models producing topological invariants of knots and 3-manifolds. These are the so-called face models and simplicial models based on quantum $6j$-symbols associated with quantum groups.

Finally, one was concerned with various aspects of the subject which either have drawn considerable attention in the recent time or seem to be good starting points for further research. Of course, quite a number of interesting problems were left outside the schedule.

Vortragsauszüge (Abstracts of the talks):

Uwe Kaiser
Hecke Algebras and the HOMFLY Polynomial

A certain class of finite dimensional (quadratic) group representations of braid groups factors through algebra representations of the classical Hecke algebras $H_n(q) = KB_n/(\sigma_i^2-(q-1)\sigma_i-q)$, where $KB_n$ is the group algebra of the braid group $B_n$, $K$ is a field, $q \in K$ and $B_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$; $\sigma_i^2-(q-1)\sigma_i-q$ is the two-sided ideal generated by the indicated elements for all $1 \leq i \leq n-1$. Ocneanu proved for each $z \in K$ the existence of normed trace functions $tr : H_n(q) \rightarrow K$ which are compatible with the inclusions $H_n(q) \rightarrow H_{n+1}(q)$ and satisfy $tr(x\sigma_n) = ztr(x)$ for $x \in H_n(q)$. Thus, by Markov’s theorem, there are maps $B_n \rightarrow K = C(q, z)\sqrt{w}$, $w = 1 - q + z$ which induce a function on the set of the isotopy classes of links in $S^3$. A change of variables leads to the HOMFLY polynomial which takes values in the Laurent ring $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ which turns out to be the universal linear skein invariant.

Wolfgang Müller
Modular Functors and Conformal Field Theories (CFT)

One important motivation for studying CFT’s in physics comes from string theory where one considers strings (something $\cong S^1$) moving in some background manifold $M$. Whereas a point particle moving in $M$ sweeps out a curve, a string evolving in time sweeps out some Riemann surface $\Sigma$. From the physical point of view, it is highly plausible that physics should not depend on the parametrisation of the string but only on the conformal structure of $\Sigma$. So, thinking of the corresponding quantum mechanical theory, we define a CFT as a functor from the category $C$ (defined below) to the category of Hilbert spaces. The objects of $C$ are the one-dimensional compact manifolds $S_i$ and a morphism from $S_0$ to $S_1$ is a Riemann surface $\Sigma$ with $\delta \Sigma = S_0 \sqcup S_1$. If $\mathcal{H}_S$ denotes the Hilbert space that is attached to a 1-manifold, we postulate that $\mathcal{H}_{S_0 \sqcup S_1} = \mathcal{H}_{S_0} \otimes \mathcal{H}_{S_1}$. Furthermore, let $\Sigma$ be a surface with exactly two boundary circles so that we have an operator $T_\Sigma : \mathcal{H}_{S_0} \rightarrow \mathcal{H}_{S_1}$ and let $\Sigma$ be $\Sigma/S_0 = S_1$ then $T_\Sigma = \text{trace}T_\Sigma$. From these axioms we get that the partition function $Z_T$ of the theory is modular.

Finally, the concept of a modular functor was presented which is constructed similar to a CFT but where one attaches to each boundary circle a representation of some fixed group $G$.

S. Ochanine
The Jones Polynomial

This introductory talk presented the original construction of Jones’ polynomial via von Neumann algebras. Main topics:
M. Kontsevich
Topological Quantum Field Theories (TQFT)

The definition of TQFT is the following: Let \( C_d \) be the category whose objects are the \( d \)-dimensional oriented closed manifolds and whose morphisms are the \( (d+1) \)-dimensional bordisms. Then, a TQFT in \( (d+1) \) dimensions is a \( \otimes \)-functor from \( C_d \) to the category of finite-dimensional vector spaces.

Some examples (and counterexamples) coming from physics were considered: the non-linear \( \sigma \)-model, Witten-Jones theory, Floer-Donaldson theory.

Finally, the equivalence of notions of Witten’s \((2+1)\) dimensional TQFT and the modular functor coming from CFT was presented.

A. Szücs
Kauffman’s state model for the Jones polynomial

First part: The Kauffman bracket in knot theory.
1.) Reidemeister moves. 2.) Kauffman bracket \( \langle K \rangle \) and its state model. 3.) \( f[K](A) = \langle K \rangle (-A)^{-3w(K)} \) is the Jones polynomial if \( A = t^{-1/2} \).
Application: Tait’s conjecture is true, namely, for any two simple alternating links the numbers of crossings are equal.

Second part: The Kauffman bracket in graph theory and statistical mechanics.
Here, we have shown that the Kauffman bracket, the dichromatic polynomial of a graph, and the partition function of Pott’s model are essentially the same.

Jaap Kalkman
Jones-Witten theory ((2+1)-dimensional topological quantum field theory (TQFT))

Also called Chern-Simons (CS) theory, since it is based on the CS functional \( \text{CS}_k(A) = \frac{k}{4\pi} \int_M \text{Tr}(AdA + \frac{1}{3}A[A,A]) \), where \( k \in \mathbb{Z} \) and \( A \in \Omega^1(M) \otimes g \), the space of connections on the bundle \( M \times G \) (\( M \) a 3-manifold, \( G \) a compact Lie group).

The talk was divided in two parts: In the first part, some evidence was presented that the partition function of the CS-theory, \( Z = \int_A D A \exp(i\text{CS}_k(A)) \), gives topological invariants of 3-manifolds. This was done using a stationary phase approximation \( (k \to \infty) \). In the second part, it was shown that one can obtain the Jones polynomial by calculating \( Z_L = \int_A D A W_L(A) \exp(i\text{CS}_k(A)) \), where \( W_L(A) \) is the trace of the holonomy along a link \( L \). The computation was done using surgery properties of TQFTs and additional data from conformal field theories (WZW-models).

Alan Durfee
The Kauffman polynomial and Birman-Wenzl algebras
The Jones polynomial $V(t)$ of an oriented link can be defined in terms of the Kauffman bracket, an invariant of regular isotopy classes of unoriented link diagrams. The Kauffman polynomial $K(l, m)$ is a two-variable generalization of $V(t)$ which is similarly defined in terms of a two-variable generalization $\Lambda$ of the bracket polynomial. The Kauffman polynomial is different from the HOMFLY polynomial $P(l, m)$, another two-variable generalization of $V(t)$. For instance, $K$ is almost independent of the orientation of the link.

Just as $P(l, m)$ can be defined in terms of a trace on the Hecke algebra, so can $K(l, m)$ be defined as a trace on an algebra constructed by Birman and Wenzl.

D. Siersma

Peter Schauenburg

Categories of tangles and their linear representations

Tangles are local versions of links. They form a category which can be modified attaching certain additional structures (orientations, framing) to the tangles. $R$-matrices produce linear representations of these categories, i.e. their covariant functors in the category of vector spaces. For links, this construction yields invariants generalizing the Jones polynomial without having to use the Alexander-Markov reduction. There are also relations with the HOMFLY and Kauffman polynomial.

The linear representations of the category of tangles can be understood and constructed via ribbon quasitriangular Hopf algebras (RQHA). The latter are algebras whose category of representations have the same properties as the category of tangles. Thus, functors from the category of tangles (oriented and 'coloured') into the category of representations of a RQHA $A$ can be constructed in a very natural way: The tensor product of tangles is carried to the tensor product of representations, the braiding of tangles is carried to the braiding of $A$-modules (induced by the comultiplication and by the quasitriangular structure on $A$ respectively), turning of the projection of some tangle corresponds to taking the dual of a representation (defined via the antipode of $A$). The notion of a ribbon Hopf algebra is defined to amend some deficiencies of the natural dual representation.

Christian Kassel

The Yang-Baxter equation and quantum groups

In this talk examples of $R$-matrices, i.e. solutions of the Yang-Baxter equation, were given. All these examples arise from representations of certain Hopf algebras $A$ together with a specific $R \in \text{ISO}(A \otimes A)$. These Hopf algebras introduced by Drinfel’d are called quasitriangular Hopf algebras.

The case of the quantized universal envelopping algebra $U_q(sl(2))$ of $sl(2)$ was presented in some detail, along with its representation theory, and its universal $R$-matrix which can be obtained by Drinfel’d’s double construction (where a quasi-triangular Hopf algebra is associated to any Hopf algebra).

Johannes Huebschmann

Poisson brackets on representation spaces and quantization
Let $\pi$ be the fundamental group of a closed surface $S$ and let $G$ be a Lie group. In a series of papers Goldman introduced and examined certain symplectic structures on the representation space $\text{Rep}(\pi, G) = \text{Hom}(\pi, G)/G$, the $G$-action on $\text{Hom}(\pi, G)$ coming from conjugation in $G$. These symplectic structures give rise to Poisson structures on a suitable smooth submanifold of $\text{Rep}(\pi, G)$. Goldman also introduced certain Lie algebras of closed curves together with homomorphisms of Lie algebras into a Poisson algebra of the kind just mentioned.

Turaev introduced a structure of a non-commutative algebra on certain skein modules defined over $S \times I$, and he showed that these skein algebras furnish a quantization of the Lie-Poisson algebras over the Goldman Lie algebras mentioned before, in a suitable sense. In this way a rigorous notion of quantization of Wilson loop observables in Chern Simons gauge theory over $S \times I$ is obtained.

Johaer van de Leur
Quantum 6j-symbols

The representation theory for the $U_q(sl_2)$ (quantum $sl_2$) algebra is discussed. For the finite dimensional irreducible representations $V^j$, $0 \leq j \in \frac{1}{2}\mathbb{Z}$, a complete orthonormal basis is given. The decomposition of tensor products of two such irreducible representations yields a direct sum of irreducible ones. Especially the $q$-analogs of the Clebsch-Gordan coefficients (CGC) are described. These CGCs and the universal $R$-matrix are presented in some graphical way and all kinds of relations between them are described in this notation.

Decomposing tensor products of three irreducible representations in two different ways into irreducible components, $(V^{j_1} \otimes V^{j_2}) \otimes V^{j_3}$ and $V^{j_1} \otimes (V^{j_2} \otimes V^{j_3})$, gives two complete orthonormal bases in $V^{j_1} \otimes V^{j_2} \otimes V^{j_3}$. The matrix elements connecting these bases are the $q$-analogs of the $gl$-symbols. Introducing the shadow world, there is a way to attach a graphical presentation for these $q - 6j$-symbols.

At the end of the talk it was shown, that any graphical configuration which describes relations between CGCs and $R$-matrices can be transformed into the shadow world. So any identity between CGCs and $R$-matrices gives an identity between $6j$-symbols.

T. tom Dieck
Quantum invariants of three-manifolds

The talk was a report on the work of Reshetikhin and Turaev. The invariants are constructed via the following scheme:

(1) An oriented, connected, closed three-manifold $M$ can be constructed by Dehn-surgery on framed links in $S^3$. The Kirby moves tell one under which conditions two framed links give the same manifold.

(2) The representation theory of a suitable quasi-triangular Hopf algebra yields a functor $F$ from the category of Ribbon Tangles to the category of vector spaces.

(3) Under suitable extra conditions and by applying this functor $F$ to the framed links in $S^3$ the invariant of the manifold is constructed. The problem is the invariance under the Kirby moves.

(4) The extra conditions under (3) lead to the notion of a modular Hopf algebra. Specific modular Hopf algebras are constructed from the quantum group $U_q(sl(2))$ by specializing
the generic parameter $q$ to a root of unity. The main intention of this work is to relate representation theory of quantum groups to geometry of three-manifolds, links in three-manifolds and topological quantum field theories.

G. Masbaum
A construction of 3-manifold invariants from the Kauffman bracket

Reshitikhin and Turaev have constructed new non-trivial invariants from any modular Hopf algebra (such algebras can be constructed from any classical Lie algebra). Recently, Lickorish has shown how to express these invariants, in the special case of $SU(2)$, in terms of Kauffman’s one variable bracket

$$\times = A \bigotimes + A^{-1} \bigotimes , \quad L \cup \bigcirc = (-A^2 - A^{-2})L.$$ 

More precisely, if $M^3$ is obtained by surgery on a bounded link $L$ in $S^3$, then the invariants can be expressed as linear combinations of Kauffman brackets of cablings of $L$, where $A$ is a primitive root of unity of order divisible by 4.

The talk presented recent work of C. Blanchet, N. Habegger, P. Vogel and the speaker, who developed Lickorish’s approach further. It discussed how to find all those linear combinations of brackets of cablings which yield 3-manifold invariants. It was shown that $A$ can also be a primitive root of unity of any even order, yielding new non-trivial invariants, but that no other evaluations are possible within this approach. (The invariants at roots of order $\equiv 2$ modulo 4 should correspond, in some sense, to quantum $SO(3)$.) The proof uses only Kirby’s calculus and elementary linear algebra, but no representation theory of quantum groups or Temperley-Lieb algebras. Moreover, this approach gives an example of the explicit geometric meaning of colors and Verlinde algebras.

Maxim Kontsevich
Higher associativity, higher invariants and cohomology of the moduli space of curves

Invariants of manifolds are functions from the set of manifolds to some field $k$. Higher invariants are proposed to be elements of $\bigoplus_{n=0}^{\infty} H^{2n}(B Diff(X), k)$ where $X$ is a manifold.

Usual invariants are just 0-components of higher invariants. It is possible to describe a machinery giving higher invariants for the case of oriented surfaces with boundaries. Let $A$ be any finite-dimensional associative algebra with a non-degenerate scalar product $(\cdot, \cdot)$ such that $(xy, z) = (x, yz)$. Then it is possible to define some invariant of surfaces with boundaries in the following way: Any such surface can be cut into pieces looking like a ribboned three-star. This defines a way to convolute the tensor of structure constants of $A$ and the scalar product. There is some homotopy analog to associative algebras, the so called $A_{\infty}$-algebras. The main statement is that homotopy analogs of the notion of associative algebras with scalar product produce elements of the cohomology of the moduli space of curves with marked points.

Günther Harder
The Turaev-Viro invariant of a three dimensional manifold
Using the $q - 6j$ symbol Turaev and Vira attach an invariant to any three dimensional manifold $M$ without (or with triangulated ) boundary. The invariant depends on the choice of an integer $r \geq 3$ and a choice of a $2r$-th root of unity $q_0$ such that $q = q_0^2$ is a primitive $r$-th root of unity.

To any triangulation of $M$ one defines admissible colourings

$$\phi : \{\text{edges}\} \rightarrow I = \left\{0, \frac{1}{2}, \ldots, \frac{r-2}{2}\right\}$$

and to each such colouring one defines a number

$$|M_\phi| = w^{-a} \prod_{E \in \{\text{edges}\}} w_\phi(E) \cdot \prod_{T \in \{\text{tetrahedra}\}} |T^\phi|$$

where the $w_i$ are certain weights and

$$|T^\phi| = \text{weight factor } \left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\}$$

and where $\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\}$ is the $q - 6j$ symbol build from the colours of the tetrahedra.

The theorem of Turaev-Vira says that this number is an invariant of $M$. The proof depends on the well known identities among the $q - 6j$ symbols.

**Jens Hoppe**

**The Tetrahedron Equation and Zamolodchikov’s solution**

Starting with a quick passage from Newtonian mechanics to n-particle states of a relativistic quantum field theory, the Yang-Baxter equation was reviewed as a consistency condition for the S-matrix factorisation in a (1+1)-dimensional relativistic quantum field theory. In analogy, the tetrahedron equations arise as consistency conditions in the (2+1)-dimensional scattering theory of ‘straight strings’. The (very large) number of independent functions appearing in these functional equations can be substantially reduced by an appropriate ansatz, due to Zamolodchikov. Following the proof of Baxter - involving a variety of non-trivial observations, and spherical trigonometry - this particular ansatz can be shown to lead to an explicit solution. The general $d$-simplex equations were shortly mentioned.

(P.S. Will membrane theories be related to invariants of 3-manifolds?)