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2-Dimensional Turbulence: A Novel Approach via Logarithmic Conformal Field Theory

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Abstract

A new conformal field theory description of two-dimensional turbulence is proposed. The recently established class of rational logarithmic conformal field theories provides a unique candidate solution which resolves many of the drawbacks of former approaches via minimal models. This new model automatically includes magneto-hydrodynamic turbulence and the Alf'ven effect.

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1 Setting the Stage for Conformal Turbulence

Once upon a time* A. Polyakov proposed a novel way of treating two-dimensional fluid mechanics: the correlation functions of certain conformal field theories (CFTs) satisfy the Hopf chains arising from the Navier Stokes equations [25]. His story has been retold in numerous variations [2, 3, 5, 6, 7, 8, 10, 20, 22, 24, 26, 27] and it soon became clear that there are actually infinitely many stories of minimal models solving two-dimensional turbulence. But all these could not lift all of the mysteries. Moreover, different stories had different happy ends, contradicting each other in their answers to lasting secrets as, for instance, the spectrum of turbulence.

What most the stories told so far have in common is that the solution to conformal turbulence was sought among the minimal models [1], that the infrared problem was not really solved, and – last but not least – that the solution did not conserve any member of the infinite set of integrals of motion, except the very first ones. Though, to be honest, some of them did at least notice the gaps.

I will try to tell another fairy-tale, one in which the hero is not a minimal model but one of the recently established rational logarithmic conformal field theories. I will try to convince you that my hero might have a better chance to survive the adventures settled by the above mentioned problems. But let us now go into medias res ...

The central point of the CFT approach to turbulence in two dimensions [25] is to consider the Hopf equations for the correlation functions of the velocity field $v_\alpha(x)$,

$$\sum_{p=1}^{n} \Big\langle v_{\alpha_1}(x_1)\ldots \dot{v}_{\alpha_p}(x_p)\ldots v_{\alpha_n}(x_n)\Big\rangle = 0\,, \tag{1.1}$$

where $\dot{v}_\alpha = \partial v_\alpha/\partial t$ is expressed in terms of v_α via the equations of motion. It is convenient to introduce the vorticity ω and the stream function ψ given by

$$v_\alpha(x) = e_{\alpha\beta}\partial_\beta\psi\,, \qquad \omega(x) = e_{\alpha\beta}\partial_\alpha v_\beta = \Delta\psi\,. \tag{1.2}$$

They satisfy the Navier-Stokes equations

$$\dot{\omega} + e_{\alpha\beta}\partial_\alpha\psi\partial_\beta\Delta\psi = \nu\Delta\omega\,, \tag{1.3}$$

where ν is the viscosity and $\dot{\omega} = \partial\omega/\partial t$. In the case that a stirring force is present, it would appear on the right hand side of equation (1.3). As usual, it is assumed that for large Reynolds numbers there exists an inertial range of scales where both viscosity and stirring force can be neglected, i.e. all distances are much smaller than the scale of an external pump L, and much larger than the viscous scale a. In this case, one has the inviscid Hopf equation

$$\begin{aligned} &\sum_{p=1}^{n} \langle\omega(x_1)\ldots\dot{\omega}(x_p)\ldots\omega(x_n)\rangle = 0\,, \\ &\dot{\omega}(x) = -e_{\alpha\beta}\partial_\alpha\psi(x)\partial_\beta\Delta\psi(x)\,. \end{aligned} \tag{1.4}$$

*to be precise, September 1992

The ultraviolet cut-off $1/a$ in momentum space prohibits to go on without a careful point splitting of the non-linear term $\dot{\omega}$,

$$\dot{\omega}(x) = -\overline{\lim_{a\to 0}}\, e_{\alpha\beta}\partial_\alpha\psi(x+a)\partial_\beta\Delta\psi(x)\,, \tag{1.5}$$

where the limiting procedure includes an averaging over directions before $|a|$ is taken to be smaller as all other distances. The key of using CFT techniques is now to assume that ψ is a primary operator of some CFT, and to replace point split expressions by the appropriate operator product expansion (OPE). The structure of the CFT dictates then via the fusion rules [1] which primary fields may appear on the right hand side of the OPE. If

$$\psi(x+a)\psi(x) = |a|^{2(h(\phi)-2h(\psi))}(\phi(x) + \text{descendants})\,, \tag{1.6}$$

or in more compact notation of fusion rules

$$[\psi]\times[\psi] = [\phi] + \dots\,, \tag{1.7}$$

where ϕ is the field with minimal conformal dimension $h(\phi)$ showing up in the OPE, one has

$$\dot{\omega}(x) \propto |a|^{2(h(\phi)-2h(\psi))}\left[L_{-2}\bar{L}_{-1}^2 - \bar{L}_{-2}L_{-1}^2\right]\phi(x)\,. \tag{1.8}$$

Since turbulence is characterized by flux states from which conserved quantities leak away, the CFT in question must be non-unitary, and therefore ϕ is non-trivial, since only Gibbs states are described by unitary theories.

I will now briefly summarize, what conditions on the parameters determining the CFT are known [25]. Firstly, the energy density in wave number space can be expressed by $|\psi|^2$ which yields

$$E(k) \propto k^{4h(\psi)+1}\,, \tag{1.9}$$

and is expected to be somewhere in the range of the predictions as of Moffat [23], $E(k) \propto k^{-11/3}$, Kraichnan [19], $E(k) \propto k^{-3}$, or Saffman [29], $E(k) \propto k^{-4}$. There exist an infinite number of (within the inertial range) conserved integrals of the type

$$H_n = \int \omega^n(x)\mathrm{d}^2x\,. \tag{1.10}$$

The first non-trivial, H_2, is called enstrophy. Polyakov [25] obtained from the requirement for a steady turbulence spectrum that the enstrophy flux be conserved (i.e. $\langle\dot{\omega}(z)\omega(0)\rangle \propto |z|^{-h(\omega)-h(\dot{\omega})}\langle\mathbb{1}\rangle$) the condition

$$(h(\phi)+2) + (h(\psi)+1) = 0\,. \tag{1.11}$$

The issue of higher-order integrals and their fluxes has been raised first by [8, 9]. Requiring the H_n dissipation to remain constant while the viscosity $\nu \propto a^{2h(\psi)-2h(\phi)}$ tends to zero, one obtains [8] the condition

$$\dot{H}_n = \nu\int^{1/a} k^2 h_n(k)\mathrm{d}k \propto \nu^{\frac{(n-1)(h(\psi)+1)+h(\phi)+2}{h(\phi)-h(\psi)}}\,. \tag{1.12}$$

Obviously, this can only be satisfied for all n if $2h(\psi) = h(\phi) = -2$ which gives Kraichnan's spectrum. This corresponds to $\psi(x) \propto |x|^2$ so that the vorticity ω has zero scaling dimension, maybe logarithmic. Then, all powers of the vorticity can have constant fluxes in k-space simultaneously. Therefore, Kraichnan's spectrum would satisfy all conservation laws if it were local.

On the other hand, Polyakov [25] introduced an additional constraint to ensure that the set of conformal correlators is a steady solution, namely that the OPE (1.6) vanishes in the ultraviolet limit $a \to 0$, i.e.

$$h(\phi) > 2h(\psi)\,, \tag{1.13}$$

and thus with (1.11) $h(\psi) < -1$. Note that Kraichnan's spectrum is at the limit point of the allowed values of the conformal scaling dimensions of ψ, ϕ. In [8], Falkovich and Hanany showed that this limit solution cannot be obtained by a minimal model. Unfortunately, this limit solution also produces a logarithmic infrared divergence after substitution into the equations for the correlation functions.

To summarize, any candidate CFT solution carrying constant flux should satisfy two conditions: i) the solution should be local in k-space, i.e. the integral determining the flux should converge, and ii) this constant flux from the converging integral should be non-zero and of correct sign to satisfy the k-space boundary conditions of pumping and damping. But there is the problem, because any CFT solution satisfying (1.11) and (1.13) does violate both of these conditions! To be more precise, (1.13) tells that the energy spectrum (1.9) is steeper than Kraichnan's solution, yielding a power-law infrared divergence. Next, the second condition of non-zero flux is violated, because the three-point function $\langle\psi\psi\psi\rangle \sim \langle\psi\phi\rangle$ is zero since primary fields with different dimensions $h(\psi) \neq h(\phi)$ are always orthogonal*.

So far, all considerations have been done under the assumption that the vacuum expectation values of the fields (i.e. the one-point functions) are all zero. If this is not the case, then there are straight-forward modifications, e.g. $\langle\dot\omega(z)\omega(0)\rangle \propto |z|^{h(\chi)-h(\omega)-h(\dot\omega)}\langle\chi(0)\rangle$, where χ is the minimal-dimension operator in the fusion product $[\psi]\times[\psi]\times[\psi]$. Thus, (1.11) had to be replaced by $(h(\phi)+2)+(h(\psi)+1)-h(\chi) = 0$.

2 Opening the Curtain for a new Solution

After paraphrasing what the problems of conformal turbulence are, I come now to the point to introduce my candidate solution. Neither is it a minimal model, nor is it a CFT in the usual sense – it is one of the recently established rational logarithmic CFTs [12, 14], i.e. a CFT where correlation functions might be logarithmic in behavior. The fact that a CFT can be consistently defined even in the case of logarithmic correlators was first pointed out by Gurarie [15]. Further progress in understanding such logarithmic CFTs was provided by several people, see e.g. [4, 12, 14, 17, 21, 28, 30].

*Solutions with $\psi = \phi$ or at least $h(\psi) = h(\phi)$ were shown in [8] not to work.

I propose that two-dimensional turbulence can be described by the rational logarithmic CFT with central charge $c = c_{6,1} = -24$ and maximally extended symmetry algebra $\mathcal{W}(2, 11, 11, 11)$. Actually, I will show that the tensor product of two of these theories (having then total central charge $c = -48$) is able to overcome most of the difficulties of the last section. Moreover, the logarithmic behavior of certain correlation functions, and the fact that logarithmic CFTs can posses *different* fields with equal conformal dimensions will be quite helpful in the following.

First, I will provide the reader with the ingredients of this theory needed below. The field content is as follows: there are 17 fields, splitting into 12 ordinary primary fields and 5 "logarithmic" operators, whose $\mathcal{W}$ families we denote by $[h]$ and $[\tilde{h}]$ respectively, h being the conformal dimension of the field. According to [12] the field content of a $c_{p,1}$ model is given by the conformal grid of the "would-be minimal model" $\mathcal{M}(3p, 3)$, i.e. the conformal weights can be expressed by $h_{1,r} = \frac{1}{4p}\left[(p-r)^2 - (p-1)^2\right]$ yielding the set

$$\Big\{[0], [-\tfrac{3}{8}], [-\tfrac{2}{3}], [-\tfrac{7}{8}], [-1], [-\tfrac{25}{24}], [-\tilde{1}], [-\tilde{\tfrac{7}{8}}], [-\tilde{\tfrac{2}{3}}], [-\tilde{\tfrac{3}{8}}], [\tilde{0}], [\tfrac{11}{24}], [1], [\tfrac{13}{8}], [\tfrac{7}{3}], [\tfrac{25}{8}], [4]\Big\}\,. \tag{2.1}$$

Notice that the conformal dimensions of different representations often are identical or differ just by integers, a characterizing feature of logarithmic CFTs. Actually, all except the representations $[h_{1,p}]$ and $[h_{1,2p}]$ fall into triplets of the form $([h_{1,r}], [\tilde{h}_{1,2p-r}], [h_{1,2p+r}])$, where $h_{1,2p-r} = h_{1,r}$ and $h_{1,2p+r} = h_{1,r} + r$. The reader should consult [12, 14] for details on the special structure of the indecomposable $[\tilde{h}]$ representations. Any of these splits into two not completely reducible representations $[\tilde{h}^{\pm}]$ such that the quantum dimensions of the both subrepresentations add up to zero. Moreover, one has $[h_{1,r}] \subset [\tilde{h}^{+}_{1,2p-r}]$ and $[h_{1,2p+r}] \subset [\tilde{h}^{-}_{1,2p-r}]$ with the additional relation $2[h_{1,r}] + 2[h_{1,2p+r}] = [\tilde{h}_{1,2p-r}]$.

The fusion rules can be calculated according to [12, 14]. Here, I only give the cases, which are important for the following, obtained by the method of [12]. Interpreting the right hand sides one should keep in mind the relations between representations mentioned above.

$$\begin{array}{rcrcl}
[0] & \times & [0] & = & [0]\,,\\
[0] & \times & [\tilde{0}] & = & [\tilde{0}]\,,\\
[0] & \times & [-1] & = & [-1]\,,\\
[0] & \times & [-\tilde{1}] & = & [-\tilde{1}]\,,\\
[\tilde{0}] & \times & [\tilde{0}] & = & 2[\tilde{0}] + 2[-\frac{2}{3}] + 2[-1] + 2[1] + 2[\frac{7}{3}] + 2[4]\,,\\
[\tilde{0}] & \times & [-1] & = & 2[\tilde{0}] + 2[-\frac{2}{3}] + [-\tilde{1}] + 2[1] + 2[\frac{7}{3}]\,,\\
[\tilde{0}] & \times & [-\tilde{1}] & = & 2[\tilde{0}] + 2[-\frac{2}{3}] + 2[-1] + 2[1] + 2[\frac{7}{3}] + 2[4]\,,\\
[-1] & \times & [-1] & = & [0] + 2[-\frac{2}{3}] + 2[-1] + 2[\frac{7}{3}] + 2[4]\,,\\
[-1] & \times & [-\tilde{1}] & = & 2[-\frac{2}{3}] + 2[-1] + [\tilde{0}] + 2[\frac{7}{3}] + 2[4]\,,\\
[-\tilde{1}] & \times & [-\tilde{1}] & = & 2[\tilde{0}] + 2[-\frac{2}{3}] + 2[-1] + 2[1] + 2[\frac{7}{3}] + 2[4]\,,
\end{array} \tag{2.2}$$

The only thing to learn from the fusion rules which is important is that the leading most negative contribution in all non-trivial fusion products is given by the representations $[-1]$ and $[-\tilde{1}]$.

The next step is to consider the tensor product of two copies of my rational logarithmic CFT candidate – not to be confused with the fact that any CFT is a tensor product of a left and right chiral part. As a shorthand, let me introduce the compact notations $\Phi_{(h|h')}(z) = (\Phi_h \otimes \Phi_{h'})(z)$ for fields and $[h|h'] = [h] \otimes [h']$ for conformal families. Of course, $h(\Phi_{(h|h')}) = h + h'$. I propose now the (symmetrized) choices $\psi_1 = \frac{1}{2}(\Phi_{(-1|\tilde{0})} + \Phi_{(\tilde{0}|-1)})$, $\psi_2 = \frac{1}{2}(\Phi_{(-\tilde{1}|\tilde{0})} + \Phi_{(\tilde{0}|-\tilde{1})})$ as well as $\phi_1 = \Phi_{(-1|-1)}$, $\phi_2 = \frac{1}{2}(\Phi_{(-1|-\tilde{1})} + \Phi_{(-\tilde{1}|-1)})$, and finally $\phi_3 = \Phi_{(-\tilde{1}|-\tilde{1})}$. The fields ψ and ϕ which are supposed to describe the turbulence are then linear combinations of the fields ψ_i and ϕ_j, respectively.

The existence of several fields with the correct conformal dimensions $-1, -2$ and fusion products

$$\psi_i \times \psi_j = A_{ij}\phi_1 + B_{ij}\phi_2 + C_{ij}\phi_3 + \ldots \tag{2.3}$$

introduces an additional degree of freedom which might be used to cancel certain divergences in the ultraviolet caused by logarithms. In fact, correlation functions containing "logarithmic" fields (from the $[\tilde{h}]$ representations) differ from the correlation functions of "ordinary" fields (from the $[h]$ representations) by multiplicative logarithms. The two-point functions of a pair of fields Φ_h and $\tilde{\Phi}_h$ with $h = h_{1,r} = h_{1,2p-r}$ are

$$\langle \Phi_h(z)\Phi_h(w)\rangle = 0\,, \tag{2.4}$$

$$\langle \Phi_h(z)\tilde{\Phi}_h(w)\rangle = \frac{C_{h\tilde{h}}}{(z-w)^{2h}}\,, \tag{2.5}$$

$$\langle \tilde{\Phi}_h(z)\tilde{\Phi}_h(w)\rangle = \frac{1}{(z-w)^{2h}}\left(-2C_{h\tilde{h}}\log(z-w) + D_{\tilde{h}\tilde{h}}\right)\,, \tag{2.6}$$

where the constant $D_{\tilde{h}\tilde{h}}$ is arbitrary due to the possible shift $\tilde{\Phi}_h \mapsto \tilde{\Phi}_h + \lambda\Phi_h$. If in the ultraviolet region distances become very small, expressions of the form $(z-w)^{-\alpha}\log(z-w) \sim \frac{1}{h}(z-w)^{-\alpha}$ asymptotically for $\alpha < 0$ and $|z-w| \gg 1$. Therefore, choosing $D_{\tilde{h}\tilde{h}} = -C_{h\tilde{h}}/h$ in the definition, one can eliminate an ultraviolet divergence from the two-point function $\langle \tilde{\Phi}_h(z)\tilde{\Phi}_h(w)\rangle$. Hence, the appearance of logarithmic operators does not spoil the nice ultraviolet behavior of non-unitary CFTs. The only remaining case which might cause difficulties is $\alpha = 0$ which I discuss below.

The situation in the infrared is more complicated. As with ordinary non-unitary CFTs, one has the problem of unphysically diverging correlation functions. However, my candidate solution has the very special property that the fields have *integer* dimensions. This allows non-diagonal combinations of left and right chiral part, and in particular non-trivial chiral states, by preserving the locality condition $h - \bar{h} \in \mathbb{Z}$ for a field $\Phi_h(z) \otimes \bar{\Phi}_{\bar{h}}(\bar{z})$. These chiral fields arise by twisting and are related to non-trivial boundary conditions. In general, their one-point functions do not vanish. Hence, the physical field ψ has a decomposition

$$\begin{aligned}\psi(z,\bar{z}) \;=\;& a_1\Phi_{(-1|0)}(z)\otimes\bar{\Phi}_{(-1|0)}(\bar{z}) + a_1\Phi_{(0|-1)}(z)\otimes\bar{\Phi}_{(0|-1)}(\bar{z})\\ &+ \tilde{a}_1\Phi_{(-1|\tilde{0})}(z)\otimes\bar{\Phi}_{(-1|\tilde{0})}(\bar{z}) + \tilde{a}_1\Phi_{(\tilde{0}|-1)}(z)\otimes\bar{\Phi}_{(\tilde{0}|-1)}(\bar{z})\\ &+ a_{+2}\Phi_{(-1|-1)}(z)\otimes\bar{\Phi}_{(0|0)}(\bar{z}) + a_{-2}\Phi_{(0|0)}(z)\otimes\bar{\Phi}_{(-1|-1)}(\bar{z})\\ &+ \tilde{a}_{+2}\Phi_{(-1|-1)}(z)\otimes\bar{\Phi}_{(\tilde{0}|\tilde{0})}(\bar{z}) + \tilde{a}_{-2}\Phi_{(\tilde{0}|\tilde{0})}(z)\otimes\bar{\Phi}_{(-1|-1)}(\bar{z}) + \ldots\end{aligned} \tag{2.7}$$

and analogous for ϕ. The same holds for other primary fields of higher conformal weights which appear as higher order terms in the OPE. Notice that chiral fields yield powers in z^2 or $\bar{z}^2$ instead of $|z|^4$ where the square is due to the twofold tensor product structure I use. Thus, chiral fields model the additional terms in the infrared, analytical in z^2 or $\bar{z}^2$, which stem from the stirring forces. This is quite natural, since stirring forces which generate vortices and turbulence are chiral. Taking all this into account one gets

$$\begin{aligned}
&\langle\psi(z)\psi(0)\rangle \\
&= |z|^{-4h(\psi)}\langle 1\!\!1\rangle + c_1|z|^{2(h(\phi)-2h(\psi))}\langle\phi\rangle + c_{+2}z^{h(\chi)-4h(\psi)}\langle\chi^+\rangle + c_{-2}\bar{z}^{h(\chi)-4h(\psi)}\langle\chi^-\rangle + \ldots \\
&= |z|^{-4h(\psi)}\left[1 + c_1\left(\frac{|z|}{L}\right)^{2h(\phi)}\langle\phi\rangle + c_{+2}\left(\frac{z}{L}\right)^{h(\chi)}\langle\chi^+\rangle + c_{-2}\left(\frac{\bar{z}}{L}\right)^{h(\chi)}\langle\chi^-\rangle + \ldots\right], \quad (2.8)
\end{aligned}$$

where the last equation reintroduces the infrared scale. Since $h(\psi) = -1$ and $h(\phi) = -2$, the second term is independent of $|z|$. The chiral fields $\chi^\pm$ stand for the next leading orders and have $h(\chi) \in 2\mathbb{Z}$, $h(\chi) \geq -2$. Therefore, after a Fourier transform, one arrives at

$$\langle\psi(k)\psi(-k)\rangle = \frac{\alpha_0}{|k|^{2-4h(\psi)}} + \frac{\alpha_1}{L^{2h(\phi)}}\delta(k) + \frac{\alpha_2}{L^{h(\chi)}}\frac{\partial^{h(\chi)-4h(\psi)}}{\partial k^{h(\chi)-4h(\psi)}}\delta(k) + \ldots \qquad (2.9)$$

In my case of $h(\psi) = -1$, $h(\phi) = -2$, this precisely resembles Polyakov's Bose condensate in momentum space, formed by the large scale motions. The main difference to Polyakov's approach is that the Bose condensate is contained *within* the CFT description. The first non-trivial higher order term comes from the *chiral* primary fields of the $[-1|-1]$ and $[-\tilde{1}|-\tilde{1}]$ representations. The coefficients can arbitrarily be adjusted by rescaling fields. Notice that all fields χ are given as linear combinations of different possible decompositions such that one can always enforce $\langle\chi\rangle = 0$ if so required by the boundary conditions.

This might help to solve the infrared problem, but it seems that now the ultraviolet region spoils the solution, since (1.6) and thus (1.8) do not vanish in the limit $a \to 0$. Luckily, there is an interesting way out. Namely, if the pair of fields Φ_h, $\tilde{\Phi}_h$ has integer valued conformal weights, $h \in \mathbb{Z}$, then the logarithmic CFT furnishes a hidden continuous symmetry [4], and Φ_h can be regarded as a conserved current due to (2.4–2.6). As has been pointed out by Polyakov himself [25], one way to satisfy the inviscid Hopf equations (1.4) is that the operator

$$\Omega(z) = L_0\bar{L}_0\left[L_{-2}\bar{L}_{-1}^2 - \bar{L}_{-2}L_{-1}^2\right]\phi(z) \qquad (2.10)$$

is a symmetry of the underlying CFT. Such a symmetry precisely appears in my example, since ϕ is composed out of $\Phi_{h_{1,5}}$ and $\tilde{\Phi}_{h_{1,7}}$ which form such a pair with conformal weights $h_{1,5} = h_{1,7} = -1$, where $\Phi_{h_{5,1}}$ gives rise to the hidden continuous symmetry. The additional $L_0\bar{L}_0$ term has been introduced to ensure that the Hopf equations only feel $\Phi_{h_{1,5}}$ in the correlators such that the symmetry argument applies. To see this, one notes that [30]

$$L_0\langle\tilde{\Phi}_1(z_1)\tilde{\Phi}_2(z_2)\Phi_3(z_3)\ldots\Phi_n(z_n)\rangle = k_n\langle\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\ldots\Phi_n(z_n)\rangle\,, \qquad (2.11)$$

where Φ_i denote ordinary primary fields, $\tilde{\Phi}_i$ logarithmic operators, and k_n are universal constants. In my point of view it is appealing to find a possible solution of the inviscid Hopf equations, which is not trivial, i.e. provided by a symmetry rather than the fact $\Omega \equiv 0$.

3 Happy End?

Let me summarize what has been accomplished so far. The proposed CFT solution is a rational logarithmic CFT with $c = -48$. It furnishes Kraichnan's picture of two-dimensional turbulence and incorporates automatically the stirring forces in the infrared region via chiral primary fields. The Hopf equations are satisfied due to an exact symmetry of this CFT, and are therefore valid both, in the ultraviolet as well as in the infrared.

However, I do not expect that this is the whole story. Polyakov [25] argued that two-dimensional turbulence is conformally invariant. But in an incompressible fluid it should also be invariant under area-preserving diffeomorphisms, generated by the $\mathcal{W}_{1+\infty}$ algebra. Therefore, I conjecture that my logarithmic CFT can be extended to an non-unitary $\mathcal{W}_{1+\infty}$ model. It has been shown in [30] that at least the Borel subalgebra of $\mathcal{W}_{1+\infty}$ is a symmetry of logarithmic CFTs. This might link two-dimensional turbulence with the transitions between fractional quantum Hall states. For first ideas in this direction see [13].

It has been pointed out in [8] that ordinary rational CFTs, in particular minimal models, can never yield Kraichnan's turbulence, although the latter can be approximated by certain sequences of minimal models. Their sequence of minimal models which approaches $h(\phi) = -2$ has a limit of central charge $c = -47$ under the assumptions that ϕ is the field of (almost) lowest conformal weight in the operator algebra. In general one obtains

$$\left(\frac{p}{q}\right)_{\pm} = (1-2h) \pm \sqrt{4(h^2-2) + \frac{n^2}{q^2}}\,, \tag{3.1}$$

as the condition for a minimal model with $c_{p,q}$ to contain a field of conformal dimension $h = h(p,q)_{r,s}$ such that $(ps-qr)^2 = n^2$. There are, however, two ways to make the right hand side of (3.1) independent of q for $q \gg 1$. Either n is kept always small compared to q, or $n = aq + b$ with a, b small compared to q. The first way yields $p/q = (1-2h) \pm 2\sqrt{h^2 - h}$, which in general is irrational. This means that the limiting point of such a sequence of minimal models is not a rational CFT (although, there exist rational CFTs at the limiting central charges $c = 1 + 24h$, see [11]). On the other hand, the second way might yield for instance $p/q = 2(1-2h)$, thus p, q become non coprime in this limit.

As I have shown in my earlier work [12], this indicates that the limiting point of such a sequence of minimal models is a rational logarithmic CFT. One easily can check that there are such $c_{p,1}$ limiting points for $h = -1$ and $h = -2$. However, there is no $c_{p,1}$ model which allows both conformal weights at the same time. Hence, I was forced to consider tensor products. Considering $\psi \times \psi = \phi$ it is a simple task to convince oneself that the twofold tensor product of the $c_{6,1}$ model is the only possibility.

One of the appealing properties of "Kraichnan's spectrum"* is that it respects all conservation laws given by the H_n integrals of motion. If one would insist on a different spectrum such as $E(k) \propto k^{-4}$, one will not find any suitable candidate within the $c_{p,1}$ models: Using a tensor product construction one is bound to the condition $h(\phi) = 2h(\psi)$ and to Kraichnan's spectrum. In general, a spectrum $E(k) \propto k^{-\alpha}$ means $h(\psi) = -(\alpha+1)/4$. To find such a field in a $c_{p,1}$ model, one needs $h(p,1)_{1,s}$ where

$$s = p \pm \sqrt{p^2 - (3+\alpha)p + 1} \tag{3.2}$$

must be an integer $1 \leq s \leq 3p-1$. If one hopes to find also the field ϕ in the same theory, one had to satisfy the additional Diophantine equation

$$s' = p \pm \sqrt{p^2 - (13-\alpha)p + 1}\,. \tag{3.3}$$

If the case $h(\psi) = h(\phi) = -\frac{3}{2}$ is excluded due to the unphysical spectrum $E(k) \propto k^{-5}$, there are no further simultaneous solutions of both Diophantine equations. From this it is concluded that, if my solution is correct, then it is unique.

I mentioned two problems in the first section, which usually are not solved by a CFT approach to two-dimensional turbulence.

Let me start with the problem of non-zero flux. In my theory, the three-point function $\langle\psi\psi\psi\rangle \sim \langle\psi\phi\rangle$ does not necessarily vanish, because both, ϕ and ψ, are made out of the same basic fields, namely Φ_{-1}, $\tilde{\Phi}_{-1}$, Φ_0 and $\tilde{\Phi}_0$. Therefore, the three-point function reduces to terms (notice the factorization due to the tensor product)

$$\langle\psi\psi\psi\rangle \sim \langle\Phi_{-1}\Phi_{-1}\rangle\langle\Phi_{-1}\Phi_0\rangle + \langle\Phi_{-1}\Phi_{-1}\rangle\langle\Phi_{-1}\tilde{\Phi}_0\rangle + \ldots \tag{3.4}$$

There are several non-vanishing contributions. If the one-point functions do not vanish then e.g. the first term would contribute. In any case, the second term does not vanish due to the properties of logarithmic operators [12, 14, 21, 30].

The other problem is the locality of the spectrum, i.e. the convergence of the integral determining the flux in k-space. As derived e.g. in [8], one has to consider the integral

$$\int_{1/L} k h_2(k) \mathrm{d}k\,, \tag{3.5}$$

where the density $h_2(k) \propto k^{2h(\psi)}$. Kraichnan's spectrum yields a logarithmic divergence for the infrared scale $L \to \infty$. If one keeps in mind that the theory in question has logarithmic operators, one has the modification $h_2(k) \propto k^{2h\psi)}(C + C' \log(k/a))$. Notice that a logarithm is always scale dependent. If the infrared and the ultraviolet scales are related such that their product La remains constant, then the integral converges for $C = 0$ in the infrared region. It seems quite natural to use the scales set by physics to define the scale of the logarithmic terms.

*to be precise, the non-trivial generalization of it given in [9]

One additional feature of my solution is that magneto-hydrodynamics in two dimensions is contained in my theory as a mere corollary. Since all fields in question are degenerated such that there are inequivalent fields of equal conformal weights, the theory has enough "space" to carry the magnetic degrees of freedom. Besides the stream function ψ there is a second independent dynamical variable, the magnetic flux function $\tilde{\psi}$. Due to the Alf'ven effect, kinetic energy and magnetic energy are asymptotically exact equi-partitioned within the inertial range [16, 18]. Therefore, the corresponding fields must have the same conformal weights, i.e. $h(\psi) = h(\tilde{\psi})$. It already has been noted in [26] that the Alf'ven effect brings logarithmic correlators onto the stage, naturally accounted for in my theory. Details are left for future work.

To conclude, I would like to mention one possible serious drawback of the CFT approach to two-dimensional turbulence: there is no time. This means that a CFT approach is blind for any renormalization due to spectral transfer of vorticity flux to smaller scales, driven by the strain of velocity shear. Fortunately, it turns out [9] that the renormalization law, determined from time correlations between velocity gradients from different spectral intervals, is just provided by the mean stretching rate (Lyapunov exponent) $\bar{\lambda}$. In my approach $\xi = \log(|z|/L)$ is the fundamental variable, since $\omega(z)$ has scaling dimension zero. Notice that any logarithmic CFT can only yield integer powers of $\log(z)$ terms. The renormalization is then simply given by $\xi \mapsto \xi/\bar{\lambda}(\xi)$, where one has as a result [9] that $\bar{\lambda}(\xi) \propto \xi^{1/3}$. With this replacement of the fundamental variable ξ, my approach reproduces the results for n-point correlators, e.g. $\langle \omega_1 \omega_2 \ldots \omega_{2n} \rangle \propto \xi^{2n/3}$, as given in [9]. The energy spectrum renormalizes then to $E(k) \propto k^{-3} \log^{-1/3}(kL)$ which was derived by Kraichnan [19] assuming weak time correlations and using a one-loop approximation.

I have now to leave it to the estimated reader whether they lived happily ever after ...

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