Problem 1: Relativistic Kinematics

In a two-body scattering event, \( A + B \rightarrow C + D \), it is convenient to introduce the Mandelstam variables

\[
\begin{align*}
    s &= \frac{(p_A + p_B)^2}{c^2}, \\
    t &= \frac{(p_A - p_C)^2}{c^2}, \\
    u &= \frac{(p_A - p_D)^2}{c^2}.
\end{align*}
\] (1)

The theoretical virtue of the Mandelstam variables is that they are Lorentz invariants, with the same value in any inertial system. Experimentally, though, the more accessible parameters are energies and scattering angles.

a) Show that \( s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2 \).

b) Find the centre-of-mass energy of \( A \), in terms of \( s, t, u \), and the masses.

c) Find the Lab (B at rest) energy of \( A \).

d) Find the total centre-of-mass energy \( (E_{\text{tot}} = E_A + E_B = E_C + E_D) \).

Problem 2: Lie algebras

Let \( V \) be a vector space with a product \( [\cdot,\cdot] : V \times V \rightarrow V \) that satisfies the conditions

\[
\begin{align*}
    \text{Antisymmetry:} & \quad [v,w] + [w,v] = 0, \\
    \text{Jacob identity:} & \quad [v,[w,u]] + [u,[v,w]] + [w,[u,v]] = 0,
\end{align*}
\] (2)

for any \( w,v \in V \). Then \( (V, [\cdot , \cdot]) \) is a Lie algebra. Evaluation of the product \( [\cdot,\cdot] \) on a basis \( \{T_a\} \) of \( V \) (the generators) gives rise to the structure constants \( f_{ab}^c \) via

\[
[T_a, T_b] = f_{ab}^c T_c. \] (4)

a) Consider a three-dimensional vector space \( V \) with the basis elements \( T_1, T_2, T_3 \). Define an antisymmetric product \( [\cdot,\cdot] \) as follows:

\[
\begin{align*}
    [T_1, T_2] &= -iT_2 \quad (5a) \\
    [T_2, T_3] &= -iT_3 \quad (5b) \\
    [T_1, T_3] &= -iT_1 \quad (5c)
\end{align*}
\]

Does this define a Lie algebra?

b) What relations do, respectively, the antisymmetry and the Jacobi identity impose on the structure constants \( f_{ab}^c \) of a Lie algebra?

c) The exponential map allows to map a Lie algebra element \( X \) into a group element \( \exp(X) \) of the corresponding Lie group. Show that

\[
\det(\exp(X)) = \exp(\text{Tr}(X)), \] (6)

for any matrix \( X \in \text{Mat}(n, \mathbb{C}) \). (Hint: proceed in three steps: diagonal \( X \), diagonalisable \( X \), and generic \( X \).)
d) Determine the properties of Lie algebra elements for the Lie groups

\[
\begin{align*}
SU(n) &= \{ U \in \text{Mat}(n, \mathbb{C}) \mid UU^\dagger = U^\dagger U = 1, \det(U) = 1 \}, \\
SO(n) &= \{ O \in \text{Mat}(n, \mathbb{R}) \mid OO^T = O^T O = 1, \det(O) = 1 \},
\end{align*}
\]

by employing (6) and \( \exp(X) \approx 1 + X + \ldots \).

**Problem 3: Gauge Transformations**  
Consider a scalar field \( \phi \) and a gauge field \( A_\mu \) subject to the (non-abelian) gauge transformations

\[
\phi \mapsto \phi^U := U\phi, \quad A_\mu \mapsto A_\mu^U := U A_\mu U^{-1} - \frac{i}{g} (\partial_\mu U) U^{-1},
\]

for \( U \) a smooth map on space-time taking values in a representation of a Lie group \( G \).

a) Show that the transformations (8) form a group.

b) Show that the covariant derivative \( D_\mu \phi := (\partial_\mu - igA_\mu) \phi \) transforms covariantly under (8), i.e. \( (D_\mu \phi)^U = U(D_\mu \phi) \).

c) Derive the explicit expression of the field strength \( F \), which is defined as

\[
F_{\mu\nu} \phi := \frac{i}{g} [D_\mu, D_\nu] \phi, \quad \text{for any } \phi.
\]

Derive the transformation behaviour under (8). What happens if \( G \) is abelian?

d) Derive the infinitesimal versions of (8) by writing

\[
U = \exp \left( i \sum_\lambda \omega_\lambda T^\lambda \right),
\]

wherein \( \omega_\lambda \) are gauge parameters, i.e. smooths functions on the space-time. (N.B. the sum over \( \lambda \) extends from 1 to the rank of the corresponding gauge group \( G \).) Here, \( T^\lambda \) are representation matrices for the generators of the Lie algebra.

e) (Bonus) Rewrite the infinitesimal gauge transformation of \( A_\mu \) by using the covariant derivative of the adjoint representation.