Olaf Lechtenfeld
Marcus Sperling

Fundamental Interactions

Problem 8: Two-particle decay For the decay $1 \to 2+3$, where particle 1 is assumed to be at rest, the decay rate is given by

$$\Gamma = \frac{S}{32\pi^2 m_1} \int |\mathcal{M}|^2 \frac{\delta^4(p_1 - p_2 - p_3)}{\sqrt{\vec{p}_2^{\,2} + m_2^2}\sqrt{\vec{p}_3^{\,2} + m_3^2}} \, d^3\vec{p}_2 \, d^3\vec{p}_3 \; , \tag{1}$$

where m_i is the mass of the ith particle, p_i is its four-momentum, and $\vec{p}_i$ is its spatial momentum. S is a statistical factor that corrects double-counting when there are identical particles in the final state: i.e. if particle 2 and 3 are identical then $S = 1/2!$. The dynamics of the decay process is contained in the amplitude $\mathcal{M}(p_1, p_2, p_3)$, which we assume to be averaged over spin degrees of freedom.

(i) Express the 4-dimensional delta function into the temporal and the 3-dimensional spatial delta function. Employing that particle 1 is at rest, perform the integral over $\vec{p}_3$.

(ii) The amplitude originally depended on all three four-momenta. However, p_1 is constant for the integration, and p_3 has been taken care of in the previous step. Moreover, $\mathcal{M}$ must be a Lorentz scalar, such that it can only depend on $\vec{p}_2^{\,2}$.

For the remaining integral change to spherical coordinates (r, θ, ϕ) for $\vec{p}_2$ and perform the angular integration.

(iii) Simplify the argument of the remaining 1-dimensional delta function by a change of variable to

$$u = \sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2} \; . \tag{2}$$

Now, evaluate the final integral and verify that

$$\Gamma = \frac{S\,|\vec{p}_2|}{8\pi m_1^2} |\mathcal{M}(\vec{p}_2^{\,2})|^2 \; , \tag{3}$$

where the formula is evaluated at the particular value

$$|\vec{p}_2| = \frac{1}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2} \tag{4}$$

determined from the conservation laws.

Without ever knowing the functional form of $\mathcal{M}$ we have been able to carry out the integrals for the 2-body decay. Formula (3) is sometimes referred to as *golden rule* for a 2-body decay.

Problem 9: Z width We recall that the Standard Model Lagrangian contains the following interaction vertex

$$\frac{-i\, g_2}{2\cos\theta_w} \gamma^\mu (c_V(f) - c_A(f)\gamma^5) \tag{5}$$

between the Z boson and a fermion anti-fermion pair $f\bar{f}$. c_V (c_A) denotes the vector (axial-vector) coupling of the fermion to the Z-boson and are given by

$$c_V(f) = T_3(f) - 2Q(f)\sin^2\theta_W \tag{6a}$$
$$c_A(f) = T_3(f) \; . \tag{6b}$$

Standard Model fermions		Q	T_3	m_f (approximate)
leptons	e^-, μ^-, τ^-	-1	$-\frac{1}{2}$	$511\,\mathrm{keV},\ 105\,\mathrm{MeV},\ 1776\,\mathrm{MeV}$
neutrinos	ν_e, ν_μ, ν_τ	0	$\frac{1}{2}$	$< 2\,\mathrm{eV}$
up-type quarks	u, c, t	$\frac{2}{3}$	$\frac{1}{2}$	$2{,}3\,\mathrm{MeV},\ 1{,}275\,\mathrm{GeV},\ 173{,}5\,\mathrm{GeV}$
down-type quarks	d, s, b	$-\frac{1}{3}$	$-\frac{1}{2}$	$4{,}8\,\mathrm{MeV},\ 95\,\mathrm{MeV},\ 4{,}2\,\mathrm{GeV}$

Table 1: Particle properties, cf. Particle data group `http://pdg.lbl.gov`.

Here, T_3 denotes the third component of the weak isospin and Q the electric charge. These properties are provided in Tab. 1 for the Standard Model fermions. The amplitude $\mathcal{M}_{Z\to f\bar{f}}$ is computed by multiplying the fermion (anti-fermion) wave-function from the right(left), as well as the polarisation vector for the Z-boson, and averaging over the spin and color space. In the approximation that $m_f \ll m_Z$ (m_Z denotes the Z-boson mass), one obtains

$$|\mathcal{M}_{Z\to f\bar{f}}|^2 = \frac{N_c}{3} \frac{g_2^2}{\cos^2\theta_W} m_Z^2 (c_V^2(f) + c_A^2(f)) \tag{7}$$

and N_c is the number of different color states the fermion f can occupy.

i) What are the fermion anti-fermion pairs relevant for the decay $Z \to f\bar{f}$? You should identify four distinct cases.

ii) Employing (3) in the limit $m_f \ll m_Z$ and (7), verify that the decay width $\Gamma_{f\bar{f}}$ of the Z-boson into a fermion anti-fermion pair is given by

$$\Gamma_{f\bar{f}} = \frac{N_c}{48\pi} \frac{g_2^2}{\cos^2\theta_W} m_Z (c_V^2 + c_A^2) \ . \tag{8}$$

iii) Evaluate c_V, c_A for the four cases and provide the explicit formula for $\Gamma_{f\bar{f}}$.

iv) Using $\sin^2\theta_w = 0{,}23$, $g_2 = \frac{e}{\sin\theta_w}$ (or more conveniently $g_2^2 \approx \frac{4\pi}{30}$) , and $m_Z = 91{,}187\,\mathrm{GeV}$, compute the numerical values for the partial widths $\Gamma_{f\bar{f}}$ as well as the total Z decay width

$$\Gamma_Z = \sum_f \Gamma_{f\bar{f}} \ . \tag{9}$$