Problem 8: Two-particle decay  For the decay $1 \rightarrow 2 + 3$, where particle 1 is assumed to be at rest, the decay rate is given by

$$\Gamma = \frac{S}{32\pi^3 m_1} \int |M|^2 \frac{\delta^4(p_1 - p_2 - p_3)}{\sqrt{p_2^2 + m_2^2} \sqrt{p_3^2 + m_3^2}} \ d^3p_2 \ d^3p_3 ,$$  

where $m_i$ is the mass of the $i$th particle, $p_i$ is its four-momentum, and $\vec{p}_i$ is its spatial momentum. $S$ is a statistical factor that corrects double-counting when there are identical particles in the final state: i.e. if particle 2 and 3 are identical then $S = 1/2!$. The dynamics of the decay process is contained in the amplitude $M(p_1, p_2, p_3)$, which we assume to be averaged over spin degrees of freedom.

(i) Express the 4-dimensional delta function into the temporal and the 3-dimensional spatial delta function. Employing that particle 1 is at rest, perform the integral over $\vec{p}_3$.

(ii) The amplitude originally depended on all three four-momenta. However, $p_1$ is constant for the integration, and $p_3$ has been taken care of in the previous step. Moreover, $M$ must be a Lorentz scalar, such that it can only depend on $\vec{p}_2^2$.

For the remaining integral change to spherical coordinates $(r, \theta, \phi)$ for $\vec{p}_2$ and perform the angular integration.

(iii) Simplify the argument of the remaining 1-dimensional delta function by a change of variable to

$$u = \sqrt{r^2 + m_2^2} + \sqrt{r^2 + m_3^2} .$$  

Now, evaluate the final integral and verify that

$$\Gamma = \frac{S |\vec{p}_2|}{8\pi m_1^2} |M(\vec{p}_2^2)|^2 ,$$  

where the formula is evaluated at the particular value

$$|\vec{p}_2| = \frac{1}{2m_1} \sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2}$$  

determined from the conservation laws.

Without ever knowing the functional form of $M$ we have been able to carry out the integrals for the 2-body decay. Formula (3) is sometimes referred to as golden rule for a 2-body decay.

Problem 9: Z width  We recall that the Standard Model Lagrangian contains the following interaction vertex

$$\frac{-i g_2}{2 \cos \theta_W} \gamma^\mu (c_V(f) - c_A(f)) \gamma^5$$  

between the Z boson and a fermion anti-fermion pair $f \bar{f}$. $c_V$ ($c_A$) denotes the vector (axial-vector) coupling of the fermion to the Z-boson and are given by

$$c_V(f) = T_3(f) - 2Q(f) \sin^2 \theta_W$$  
$$c_A(f) = T_3(f) .$$
Here, $T_3$ denotes the third component of the weak isospin and $Q$ the electric charge. These properties are provided in Tab. 1 for the Standard Model fermions. The amplitude $M_{Z \rightarrow f \bar{f}}$ is computed by multiplying the fermion (anti-fermion) wave-function from the right(left), as well as the polarisation vector for the $Z$-boson, and averaging over the spin and color space. In the approximation that $m_f \ll m_Z$ ($m_Z$ denotes the $Z$-boson mass), one obtains

$$|M_{Z \rightarrow f \bar{f}}|^2 = \frac{N_c}{3} \frac{g_2^2}{\cos^2 \theta_W} m_Z^2 (c_V^2(f) + c_A^2(f))$$  \hspace{1cm} (7)$$

and $N_c$ is the number of different color states the fermion $f$ can occupy.

i) What are the fermion anti-fermion pairs relevant for the decay $Z \rightarrow f \bar{f}$? You should identify four distinct cases.

ii) Employing (3) in the limit $m_f \ll m_Z$ and (7), verify that the decay width $\Gamma_{f\bar{f}}$ of the $Z$-boson into a fermion anti-fermion pair is given by

$$\Gamma_{f\bar{f}} = \frac{N_c}{48\pi \cos^2 \theta_W} m_Z^2 (c_V^2 + c_A^2) .$$  \hspace{1cm} (8)$$

iii) Evaluate $c_V$, $c_A$ for the four cases and provide the explicit formula for $\Gamma_{f\bar{f}}$.

iv) Using $\sin^2 \theta_W = 0.23$, $g_2 = \frac{e}{\sin \theta_W}$ (or more conveniently $g_2^2 \approx \frac{4\pi}{30}$), and $m_Z = 91, 187$ GeV, compute the numerical values for the partial widths $\Gamma_{f\bar{f}}$ as well as the total $Z$ decay width

$$\Gamma_Z = \sum_f \Gamma_{f\bar{f}} .$$  \hspace{1cm} (9)$$