Quantum Field Theory: Exercise Session 5

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Regularization, renormalization and the one-loop structure of $\phi^4$ theory

The Feynman rules in renormalized perturbation theory for the $\phi^4$ theory are

\[
\begin{align*}
\frac{i}{p^2 - m^2} &= -i\lambda, \\
\delta_z - \delta_m &= -i\delta\lambda.
\end{align*}
\]

(a) Write down the two-particle scattering amplitude $i\mathcal{M}$ in terms of Feynman diagrams to one loop order.

(b) Use the Feynman rules to write down explicitly the integral in momentum space corresponding to the diagram

\[
\begin{align*}
\frac{1}{ab} &= \int_0^1 \frac{dz}{[az + b(1-z)]^2}. \\
\int \frac{d^d q}{(q^2 + 2qr - \Omega^2)^\alpha} &= (-1)^{d/2} \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)} \frac{1}{(-r^2 - \Omega^2)^{\alpha - \frac{d}{2}}}. \\
\end{align*}
\]

(c) Now regularize the integral $V(p^2)$ using dimensional regularization as follows:

(i) Generalize the action for $\phi^4$ theory to $d$ spacetime dimensions, introducing an arbitrary mass parameter $\mu$ to allow the coupling $\lambda$ to keep mass dimension 0. Thus, write down the corresponding momentum integral $V(p^2)$ in $d$ dimensions.

(ii) Introduce a Feynman parameter $z$ to combine the factors in the denominator using

\[
\frac{1}{ab} = \int_0^1 \frac{dz}{[az + b(1-z)]^2}. \\
\]

(iii) Make the change of variables $k' = k+p(1-z)$ in order to perform the momentum integral by applying the relation

\[
\int \frac{d^d q}{(q^2 + 2qr - \Omega^2)^\alpha} = (-1)^{d/2} \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)} \frac{1}{(-r^2 - \Omega^2)^{\alpha - \frac{d}{2}}}. \\
\]
(iv) Take the limit $d \to 4$ using
\[
\Gamma(\epsilon) = \left[ \frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \right],
\]
with $\gamma$ the Euler-Mascheroni constant, and $\Gamma(n+1) = n!$ for $n$ a natural number. Be careful with dimensionful quantities when making your expansions. Thus, express $V(p^2)$ as the sum of a divergent term $\sim 1/\epsilon$ (where $\epsilon \equiv 4 - d$) and finite terms.

(d) For a two-particle to two-particle process

\[
\begin{array}{c}
\vdots \\
p_1 & k_1 \\
p_2 & k_2 \\
\end{array}
\]

the Mandelstam variables are given by $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$, $t = (p_3 - p_1)^2 = (p_4 - p_2)^2$, $u = (p_4 - p_1)^2 = (p_3 - p_2)^2$. Write down the entire amplitude $i\mathcal{M}$ in terms of the physical mass $m$ and coupling $\lambda$, the Mandelstam variables as $V(s), V(t), V(u)$, and the counterterms, $\delta_{\lambda}, \delta_m, \delta_Z$.

(e) By applying the renormalization conditions on the two-particle scattering amplitude, compute the shift $\delta_\lambda$ from the bare coupling constant to the physical coupling constant in the limit $d \to 4$.

(f) Combine your results to write down a finite expression for the two-particle scattering amplitude in terms of physically observable quantities.

(g) Compute the propagator to determine the remaining counterterms, $\delta_Z$ and $\delta_m$, working to one-loop order.