Perturbation theory for interacting $\phi^4$ scalar field theory

Consider a scalar field theory with quartic self-interaction, described by:

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4. \]  

In quantum field theory, all $n$-point correlation functions can be encoded in a single object called the generating functional, $Z[J]$, as:

\[ G_n(x_1, x_2, \ldots, x_n) = i^{-n} \left[ \frac{\delta^n Z[J]}{\delta J(x_1) \delta J(x_2) \cdots \delta J(x_n)} \right]_{J=0}. \]  

For the $\phi^4$ theory, the generating functional is given by:

\[ Z[J] = \frac{\exp \left[ \left( -i \frac{\delta}{\delta J(z)} \right)^4 \right] \left[ \frac{Z_0[J]}{Z_0[0]} \right]}{\exp \left[ \left( -i \frac{\delta}{\delta J(z)} \right)^4 \right] \left[ \frac{Z_0[J]}{Z_0[0]} \right]}_{J=0}, \]  

where $Z_0$ is the free generating functional:

\[ Z_0[J] = Z_0[0] \exp \left[ -\frac{1}{2} \int d^4x d^4y J(x) D_F(x-y) J(y) \right]. \]  

Assuming small interaction coupling, $\lambda \ll 1$, we can use perturbation theory. The Feynman rules for the $\phi^4$ theory read:

\[ \begin{align*}
\begin{array}{c}
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\bullet \quad \bullet \quad \bullet \quad \bullet \\
\end{array}
\end{array} & = D_F (x-y) \\
\begin{array}{c}
\begin{array}{c}
\bullet \quad \bullet \\
\end{array}
\end{array} & = \int d^4z \ D_F(x-z) i \bar{J}(z) \\
\begin{array}{c}
\begin{array}{c}
\bullet \quad \bullet \\
\end{array}
\end{array} & = \int d^4z \ \int d^4y \ i \bar{J}(z) \ D_F(x-y) i \bar{J}(y) \\
\begin{array}{c}
\begin{array}{c}
\bullet \quad \bullet \\
\end{array}
\end{array} & = \left( -i \frac{\lambda}{4!} \right) \int d^4z \\
\begin{array}{c}
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\bullet \quad \bullet \\
\end{array}
\end{array} & = \left( -i \frac{\lambda}{4!} \right) \int d^4z \\
\end{align*} \]
1 (a) Apply the functional derivative \( \frac{1}{i} \frac{\delta}{\delta J(x)} \) to \( Z_0[J] \), and draw the corresponding diagram. What does \( \frac{1}{i} \frac{\delta}{\delta J(x)} \) do to a diagram?

(b) By expanding \( Z[J] \) to first order in \( \lambda \), and applying \( \frac{\delta}{\delta J(x)} \) four times, show that to \( \mathcal{O}(\lambda) \):

\[
Z[J] = \frac{1}{\left[ 1 - i \frac{1}{3!} \int d^4x \left( 3(D_F(0))^2 + 6D_F(0) \left( \int d^4y D_F(x - y) J(y) \right) + \left( \int d^4y D_F(x - y) J(y) \right)^4 \right) \right] Z_0[0]} \]

You may choose whether to work explicitly or to use the Feynman diagrams.

(c) Show that the vacuum diagrams, which diverge, cancel thanks to the normalization.

2 The four-point function reads:

\[
\Gamma_4(x_1, x_2, x_3, x_4) = \left( \begin{array}{c}
\begin{array}{c}
1 \quad 3 \\
\hline
3 & +
\end{array}
\end{array} \right) + \left( \begin{array}{c}
\begin{array}{c}
1 \quad 4 \\
\hline
3 & +
\end{array}
\end{array} \right) + \left( \begin{array}{c}
\begin{array}{c}
1 \quad 2 \\
\hline
3 & +
\end{array}
\end{array} \right) + \left( \begin{array}{c}
\begin{array}{c}
1 \quad 2 \\
\hline
3 & 4
\end{array}
\end{array} \right) + \mathcal{O}(\lambda^2)
\]

What diagrams appear in \( G_4(x_1, x_2, x_3, x_4) \) at \( \mathcal{O}(\lambda^2) \)? Take the symmetry factors into account!

3 Only the connected Feynman diagrams in a correlation function contribute to the non-trivial (off-diagonal) part of the \( S \) matrix. Show to \( \mathcal{O}(\lambda) \) that the functional \( W[J] = -i \ln Z[J] \) generates only the connected diagrams of \( G_4(x_1, x_2, x_3, x_4) \).