Regularization, renormalization and the one-loop structure of $\phi^4$ theory

The Feynman rules in renormalized perturbation theory for the $\phi^4$ theory are:

\[ \frac{i}{p^2 - m^2} \]
\[ -i \lambda \]
\[ i (p^1 \delta_{2} - \delta_{m}) \]
\[ -i \delta_\lambda \]

(a) Write down the two-particle scattering amplitude, $iM$, in terms of Feynman diagrams to one loop order.

(b) Use the Feynman rules to write down explicitly the integral in momentum space, $V(p^2)$, corresponding to the diagram:

\[ (i\lambda)^2 \cdot i \cdot V(p^2) \]
(c) Now regularize the integral $V(p^2)$ using dimensional regularization as follows.

(i) Generalize the action for $\phi^4$ theory to $d$ spacetime dimensions, introducing an arbitrary mass parameter $\mu$ to allow the coupling $\lambda$ to keep mass dimension 0. Thus, write down the corresponding momentum integral $V(p^2)$, in $d$ dimensions.

(ii) Introduce a Feynman parameter, $z$, to combine the denominator factors using

$$\frac{1}{ab} = \int_0^1 \frac{dz}{[az + b(1-z)]^2}. \quad (1)$$

(iii) Make the change of variables $k' = k + p(1-z)$ in order to perform the momentum integral by applying the relation:

$$\int \frac{d^d q'}{(q'^2 + 2qr - \Omega^2)\alpha} = (-1)^{d/2} \pi^{d/2} \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)} \frac{1}{(-r^2 - \Omega^2)^{\alpha - \frac{d}{2}}}. \quad (2)$$

(iv) Take the limit $d \to 4$, using

$$\Gamma(\epsilon) = \left[ \frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \right], \quad (3)$$

with $\gamma$ the Euler-Mascheroni constant, and $\Gamma(n+1) = n!$ for $n$ a natural number. Be careful with dimensionful quantities when making your expansions! Thus, express $V(p^2)$ as the sum of a divergent term $\sim 1/\epsilon$ (where $\epsilon = 4 - d$) and finite terms.

(d) For a two-particle to two-particle process:

$$\begin{array}{c}
\text{p}_3 \quad \text{p}_4 \\
\text{p}_1 \quad \text{p}_2
\end{array}$$

the Mandelstam variables are given by $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$, $t = (p_3 - p_1)^2 = (p_4 - p_2)^2$, $u = (p_4 - p_1)^2 = (p_3 - p_2)^2$. Write down the entire amplitude $i\mathcal{M}$, in terms of the physical mass and coupling, $m, \lambda$, the Mandelstam variables as, $V(s), V(t), V(u)$, and the counterterms, $\delta_\lambda, \delta_m, \delta_Z$.

(e) By applying the renormalization conditions on the two-particle scattering amplitude, compute the shift $\delta_\lambda$ from the bare coupling constant to the physical coupling constant, in the limit $d \to 4$.

(f) Combine your results to write down a finite expression for the two-particle scattering amplitude, in terms of physically observable quantities.

(g) Compute the propagator to determine the remaining counter-terms, $\delta_Z$ and $\delta_m$, working to one-loop order.