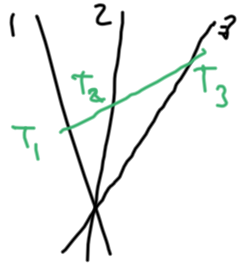
Relativistische Dynamik von Punktteilchen

$c=1$

zunächst noch Kinematik. Beispiel: Addition von Geschw.



$$T_3 = k_{32} \cdot T_2 = k_{32} \cdot (k_{21} \cdot T_1) \overset{!}{=} k_{31} \cdot T_1$$

$$\rightsquigarrow \quad k_{31} = k_{32} \cdot k_{21} \quad \rightsquigarrow \quad \frac{1+v_{31}}{1-v_{31}} = \frac{1+v_{32}}{1-v_{32}} \cdot \frac{1+v_{21}}{1-v_{21}}$$

oder

$$\rightsquigarrow \quad \theta_{31} = \theta_{32} + \theta_{21} \quad \rightsquigarrow$$

$$v_{31} = \operatorname{th}(\theta_{31}) = \operatorname{th}(\theta_{32}+\theta_{21}) \overset{\text{Add.-thm}}{=} \frac{\operatorname{th}(\theta_{32}) + \operatorname{th}(\theta_{21})}{1+\operatorname{th}(\theta_{32})\operatorname{th}(\theta_{21})} = \frac{v_{32}+v_{21}}{1+v_{32} \cdot v_{21}}$$

Lorentz-Trsf. in 1+3 Dimensionen

x-Boost:
$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

analog in y- & z- Richtg.

hinzu kommen Drehungen
$$\begin{pmatrix} t' \\ \vec{r}' \end{pmatrix} = \begin{pmatrix} 1 & 0\ 0\ 0 \\ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & D \end{pmatrix} \begin{pmatrix} t \\ \vec{r} \end{pmatrix}$$

$\Bigg\}$ erzeugen Lorentzgruppe

$SO(1,3) \ni \Lambda$

$\underline{x}' = \Lambda \cdot \underline{x}$ mit $\underline{x} = \begin{pmatrix} t\\ x\\ y\\ z\end{pmatrix}$ Vierspalte

Index-Schreibweise:

$x^\mu = (x^0, x^{i=1,2,3})$ $\qquad x = e_\mu x^\mu$ Vierervektor

$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$ Lorentz-Trsf.

Invarianz:

in 1+1 Dim.: $T'_+ T'_- = T_+ T_- \rightsquigarrow (t+r)(t-r) = t^2 - r^2 = \tau^2$ Lorentz-invariant

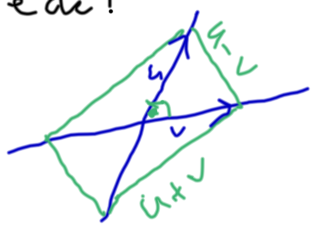
mit Drehinv. in 1+3 Dim.: $t^2 - \vec{r}^{\,2} = t^2 - x^2 - y^2 - z^2 = (t,x,y,z)\underbrace{\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}}_{\eta}\begin{pmatrix}t\\ x\\ y\\ z\end{pmatrix}$ inv.

Index-Schreibweise: $u^\mu \eta_{\mu\nu} u^\nu =: u\cdot u$ Bilinearform

Skalarprodukt: $u\cdot v = \frac{1}{4}(u+v)^2 - \frac{1}{4}(u-v)^2 = u^\mu \eta_{\mu\nu} u^\nu$
$= u^0v^0 - u^1v^1 - u^2v^2 - u^3v^3$

Licht: $u\cdot u = 0$

Lichtecke:



$$\left.\begin{aligned} 0 &= (u+v)^2 = u^2 + 2u\cdot v + v^2 \\ 0 &= (u-v)^2 = u^2 - 2u\cdot v + v^2 \end{aligned}\right\} \quad \begin{aligned} u^2 &= -v^2 \\ u\cdot v &= 0 \end{aligned}$$

$\rightsquigarrow u \perp v$. Kurve τ^2=konst. sind Hyperbeln

Index-Stellung relevant, weil

$$x^\mu \doteq \begin{pmatrix} t \\ \vec r \end{pmatrix} \rightsquigarrow x_\mu = \eta_{\mu\nu} x^\nu \doteq \begin{pmatrix} t \\ -\vec r \end{pmatrix}$$

kann η verstecken: $x \cdot y = x^\mu \eta_{\mu\nu} y^\nu = x^\mu y_\mu = x_\mu y^\mu$

wie transformiert x_μ?

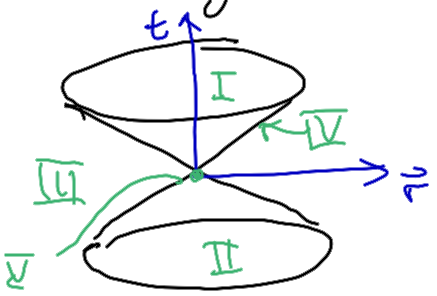
$$x'_\mu = \eta_{\mu\nu} x'^\nu = \eta_{\mu\nu} \Lambda^\nu{}_\rho x^\rho = \eta_{\mu\nu} \Lambda^\nu{}_\rho \eta^{\rho\sigma} x_\sigma =: \tilde\Lambda_\mu{}^\sigma x_\sigma$$

mit $\tilde\Lambda_\mu{}^\sigma = (\eta \Lambda \eta^{-1})_\mu{}^\sigma = (\Lambda^{-1T})_\mu{}^\sigma$

$$\eta_{\mu\nu} \eta^{\nu\rho} = \delta_\mu^\rho$$

weil $\Lambda^T \eta \Lambda = \eta \Leftrightarrow \Lambda^{T-1} = \eta \Lambda \eta^{-1}$ def. $SO(1,3)$

Lichtkegel



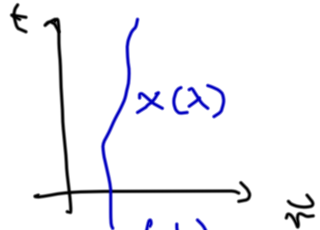
im Minkowskiraum $\mathbb{R}^{1,3}$

lässt Lorentz-Trsf. 5 Bereiche invariant

- I zeitartige Zukunft
- II —"— Vergangenheit $\Big\}$ $x^2 > 0$
- III raumartig — $x^2 < 0$
- IV lichtartig — $x^2 = 0$
- V Null — $x = 0$

· Weltlinie eines Punktteilchens

$$\lambda \longmapsto x^\mu(\lambda) \doteq \begin{pmatrix} x^0(\lambda)\\ x^1(\lambda)\\ x^2(\lambda)\\ x^3(\lambda)\end{pmatrix}$$



Newton: $\lambda = t = x^0 \rightsquigarrow x^\mu \doteq \begin{pmatrix} t \\ \vec r(t)\end{pmatrix} \rightsquigarrow \dot x^\mu \doteq \begin{pmatrix}1\\ \vec v\end{pmatrix}$

Eigenzeit: $\lambda = \tau$ mit $\tau_{12} = \int_1^2 ds = \int_1^2 dt\,\sqrt{\dot x\cdot\dot x} = \int_1^2 dt\,\overbrace{\sqrt{1-\vec v^{\,2}(t)}}^{\gamma^{-1}}$

↳ Lorentz-Skalar!

$$\gamma = \frac{1}{\sqrt{1-v^2}} = dt/d\tau \rightsquigarrow d\tau = \sqrt{1-v^2}\,dt$$

· Geschwindigkeit & Beschleunigung

teile durch $d\tau$ (Skalar) statt durch dt (Teil von dx^μ)

4-Geschw.: $u = \frac{dx}{d\tau} = \frac{dt}{d\tau}\frac{dx}{dt} \doteq \gamma\begin{pmatrix}1\\ \vec v\end{pmatrix} \rightsquigarrow u^2 = \gamma^2(1-\vec v^{\,2}) = 1$

4-Beschl.: $b = \frac{du}{d\tau} = \frac{d^2x}{d\tau^2} = \frac{dt}{d\tau}\frac{du}{dt} \doteq \gamma^2\begin{pmatrix}0\\ \vec a\end{pmatrix} + \gamma^4\,\vec v\cdot\vec a\begin{pmatrix}1\\ \vec v\end{pmatrix}$

$$\rightsquigarrow 0 = \frac{d}{d\tau}u^2 = 2u\cdot b \rightsquigarrow u^2>0,\ b^2<0,\ b\perp u$$

momentanes Ruhesystem: $\gamma=1, \vec v = 0 \rightsquigarrow u = \begin{pmatrix}1\\ \vec 0\end{pmatrix},\ b = \begin{pmatrix}0\\ \vec a\end{pmatrix}$

kleine Geschw.: $\gamma = \frac{1}{\sqrt{1-v^2}} = 1 + \frac{1}{2}v^2 + \frac{3}{8}v^4 + \dots$

· Dynamik   ersetze alle 3-Vektoren durch 4-Vektoren

4-Impuls: $p \doteq \begin{pmatrix} p^0 \\ \vec{p} \end{pmatrix} \doteq m u \doteq m \begin{pmatrix} \gamma \\ \gamma \vec{v} \end{pmatrix} = m \begin{pmatrix} 1 + \frac{1}{2} v^2 + \frac{3}{8} v^4 + \dots \\ \vec{v} + \frac{1}{2} v^2 \vec{v} + \dots \end{pmatrix}$

4-Kraft: $F \doteq \begin{pmatrix} F^0 \\ \vec{F} \end{pmatrix} \doteq m b = m \begin{pmatrix} \gamma^4 \vec{v} \cdot \vec{a} \\ \gamma^2 \vec{a} + \gamma^4 (\vec{v} \cdot \vec{a}) \vec{v} \end{pmatrix}$

$$= m \begin{pmatrix} \vec{v} \cdot \vec{a} + 2 v^2 \, \vec{v} \cdot \vec{a} + \dots \\ \vec{a} + v^2 \vec{a} + (\vec{v} \cdot \vec{a}) \vec{v} + \dots \end{pmatrix}$$

räumliche Komponenten: Newton + relativist. Korrekturen

zeitliche Komponenten sind Zugaben:

$E = c p^0 = \gamma m c^2 = \underbrace{m c^2}_{\text{Ruheenergie}} + \frac{1}{2} m v^2 + \frac{3}{8} m \, v^4/c^2 + \dots$ Energie

$P = c F^0 = \vec{v} \cdot \vec{F} + \dots$ Leistung

$\Big\} \; P = \frac{dE}{d\tau}$

Bewegungsgl.: $F^\mu = \frac{dp^\mu}{d\tau} = m \frac{d^2 x^\mu}{d\tau^2} = m b^\mu$

Energie-Impuls-Beziehung: $m^2 c^2 = p^2 = \left(E/c\right)^2 - \vec{p}^{\,2} \rightsquigarrow E = c\sqrt{m^2 c^2 + \vec{p}^{\,2}}$ Massenschale

masselose Teilchen $m = 0$ sind möglich!