P R E L I M I N A R Y   V E R S I O N

Introduction to Perturbative String Theory

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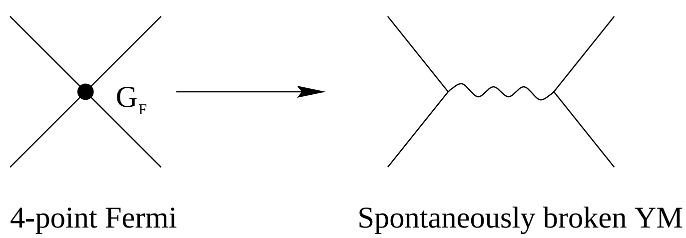
1 Motivation

Presumably the deepest problem of 20th century fundamental physics has been the resolution of the conflict between quantum and gravitational theory. Evidence for this clash are the incurable ultraviolet divergences emerging whenever gravitational interactions are being included in a quantum field theoretical framework. Superficially, this is to be expected due to the fact that (in four spacetime dimensions) the strength of gravitational interactions grows with energy since its coupling has negative (mass) dimensions, $[1/M_{pl}] = \mathrm{GeV}^{-1}$. Interactions of this kind are infrared-safe but non-renormalizable. Two possible ways out of such a situation come to mind:

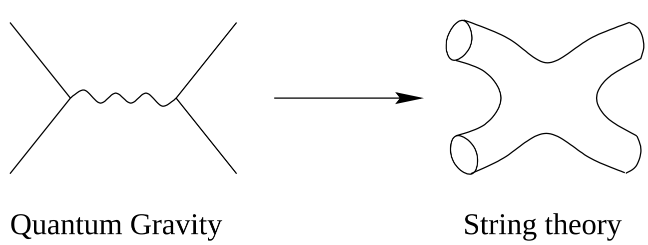
a) a nontrivial UV fixed point, meaning that our divergences are artifacts.

b) new physics to be uncovered at higher energies.

For lack of evidence pointing at solution a), we shall presume case b) to be realized in nature. Moreover, historically, an analogous situation was encountered in the theory of weak interactions. Those were first described with a local four-point coupling by Fermi. Although successful at energies well below 100 GeV, the Fermi theory suffered from UV divergences. The present theory of weak interactions, due to Glashow, Salam, and Weinberg, solves this problem uniquely by 'point-splitting' the interaction according to the Feynman graphs depicted. The Feynman diagrams of (tree-level) quantum gravity suggest a similar approach

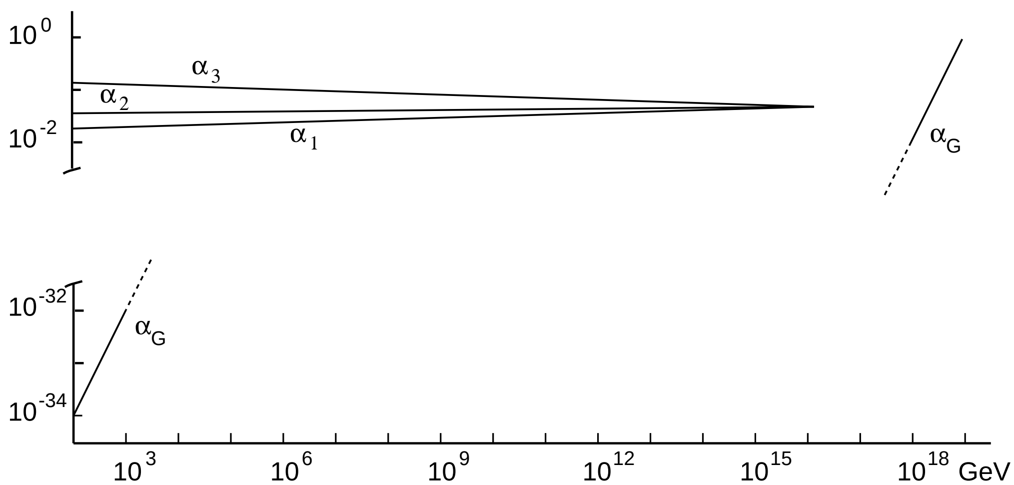


to alleviate the UV desaster. Further 'smearing' of the three-point couplings naturally (uniquely?) leads one to consider extended objects, i.e. strings in the first place. From the string diagram it becomes obvious that the notion of a



Lorentz-invariant interaction point has ceased to exist.

In the absence of experimental input our only guidance is mathematical consistency. Reassuringly, string theory unifies many of the theoretical concepts beyond the Standard Model. In fact, it necessarily includes gravity, GUTs, higher dimensions and supersymmetry, sometimes with a new twist. It also displays amazing mathematical structures and cohesiveness. As we see in the following figure, the gravitational coupling constant almost meets the other three near the Planck scale. Although it should be kept in mind that this extrapolation is not to



be trusted too much, the low-energy world certainly hints at some kind of grand unification including gravity. And it is fair to say that currently string theory is the only promising path leading towards this goal.

The attentive student may ask herself: why stop the smearing at one-dimensional extended objects, rather than allowing for membranes and so on? Before 1995, the answer would have involved better space-time but worsening world-volume properties with increasing dimension (strings strike an optimal balance in this regard). In the meantime, however, one has learned that (classical) string theory indeed contains all sorts of higher-dimensional objects (called ‘branes’) which become invisible in the perturbative weak-coupling regime. Since their (perturbative?) quantization is posing enormous difficulties, we shall content ourselves with (pre-1995) perturbative string theory in these lectures, leaving the more recent developments for the continuation by Dieter Lüst. Let me end these preliminaries by remarking that string theory is more than a theory of strings only, just like a quantum theory of strongly interacting particles may reveal non-pointlike structures.

2 The Bosonic String

2.1 The Classical Bosonic String

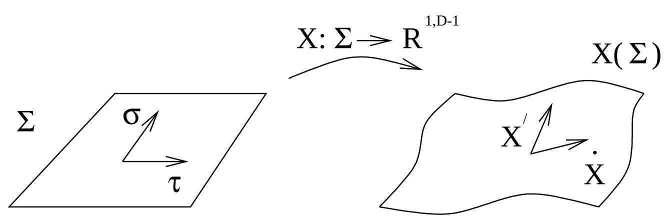
Our starting point is the classical discussion of the structureless relativistic string, called 'bosonic string', in a flat spacetime $\mathbb{R}^{1,D-1}$. It is described by a map X from a two-dimensional parameter space $\Sigma \subset \mathbb{R}^{1,1}$, into the target space(time) $\mathbb{R}^{1,D-1}$,

$$X : \ \Sigma \ \longrightarrow \ \mathbb{R}^{1,D-1} \quad .$$

The coordinates on Σ are denoted by $(\tau, \sigma) = (\xi^0, \xi^1) = (\xi^\alpha, \alpha = 0, 1)$, and its flat metric is $(\eta_{\alpha\beta}) = \mathrm{diag}(-1, 1)$. The image point of (τ, σ) has coordinates $X^\mu(\tau, \sigma)$, with $\mu = 0, 1, \ldots, D{-}1$, i.e.

$$X^\mu : \ (\tau, \sigma) \ \longmapsto \ X^\mu(\tau, \sigma) \quad .$$

We take the flat target space metric to be $(\eta_{\mu\nu}) = \mathrm{diag}(-1, \underbrace{1, \ldots, 1}_{D-1})$ and abbreviate $\eta_{\mu\nu} A^\mu B^\nu = A{\cdot}B$. The embedded surface $X(\Sigma) \subset \mathbb{R}^{1,D-1}$ swept out by the string moving through flat space and is called the 'worldsheet'. Each point p on



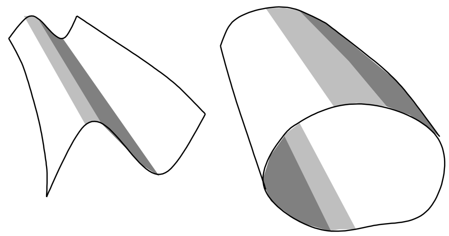
$X(\Sigma)$ carries a two-dimensional tangent space $T_p X(\Sigma)$. (The coordinates of) its two basis vectors read

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau} = \partial_0 X^\mu \qquad \text{and} \qquad X'^\mu = \frac{\partial X^\mu}{\partial \sigma} = \partial_1 X^\mu \quad . \tag{2.1}$$

There are two types of strings, distinguished by the worldsheet topology for the case of free string propagation. The closed string originates from the cylinder $\Sigma = [\tau_1, \tau_2] \times S^1$, or

$$X^\mu(\tau, \sigma{+}2\pi) \ = \ X^\mu(\tau, \sigma) \quad \forall \tau \quad , \tag{2.2}$$

so that the boundary of $X(\Sigma)$ is given by the initial and final string configuration. The open string possesses end points, at $\sigma = 0$ and $\sigma = \pi$, yielding the strip $\Sigma = [\tau_1, \tau_2] \times [0, \pi]$. The additional worldsheet boundaries require a choice of boundary conditions, which will be discussed in conjunction with the equations of motion.



The Nambu-Goto Action

The bosonic string motion is governed by the Nambu-Goto action

$$S_{\surd}[X] = \int_\Sigma \mathrm{d}^2\xi \; \mathcal{L}(\dot{X}, X') = -\frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2(\text{area}) = -\frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2\xi \,\sqrt{-\gamma} \qquad (2.3)$$

where $\gamma = \det\gamma_{\alpha\beta} < 0$ is the determinant of a metric induced on the parameter space by pulling back with X the flat target space metric $\eta_{\mu\nu}$,

$$\gamma_{\alpha\beta}[X] \;=\; ({}^*\eta)_{\alpha\beta} \;=\; \eta_{\mu\nu}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu \;=\; \begin{pmatrix} \dot{X}\cdot\dot{X} & \dot{X}\cdot X' \\ X'\cdot\dot{X} & X'\cdot X' \end{pmatrix} \quad . \qquad (2.4)$$

The factor $\frac{1}{2\pi\alpha'}$ of dimension GeV^2 is the tension of the string, as can be seen from the static open-string configuration $(X^0, X^1, X^2, \ldots, X^{D-1}) = (\tau T, \sigma\frac{L}{\pi}, 0, \ldots, 0)$ whose action (for $\tau \in [0,1]$) is $S_{\surd} = \frac{-LT}{2\pi\alpha'}$ so that $[\frac{1}{2\pi\alpha'}] = [\frac{S/T}{L}] =$ energy per length. The meaning of the parameter α' will be explained in section 3.2. The requirement that γ be negative is related to the causality of the string propagation; the tangent space to the worldsheet should be Lorentzian at each (interior) point.

Local symmetries and constraints. The Nambu-Goto action is invariant under reparametrizations of the worldsheet, infinitesimally $\delta_\varepsilon\xi^\alpha = -\varepsilon^\alpha(\xi)$ implying $\delta_\varepsilon X^\mu = \varepsilon^\gamma\partial_\gamma X^\mu$. Hence, there will be constraints on the phase space. The canonical momentum densities are

$$\Pi_\mu(\xi) \;=\; \frac{\partial\mathcal{L}}{\partial\dot{X}^\mu} \;=\; -\frac{1}{2\pi\alpha'}\frac{(\dot{X}\cdot X')X'_\mu - (X'^2)\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}} \quad . \qquad (2.5)$$

Exercise 2.1 *Show that the covariant form of* (2.5) *reads*

$$\Pi_\mu(\xi) \;=\; -\frac{1}{2\pi\alpha'}\sqrt{-\gamma}\,\gamma^{0\alpha}\,\eta_{\mu\nu}\,\partial_\alpha X^\nu \quad ,$$

and use this to derive the two constraints

$$\Pi\cdot X' \;=\; 0 \qquad \textit{and} \qquad \Pi\cdot\Pi + \frac{1}{4\pi^2\alpha'^2}X'^2 \;=\; 0 \quad .$$

Combining the two equations gives the Virasoro constraints

$$2\alpha' T_{\pm\pm}(\Pi, X') \;:=\; \frac{1}{2}(2\pi\alpha'\Pi^\mu \pm X'^\mu)^2 \;=\; 0 \quad . \tag{2.6}$$

Constraints which follow from the definition of the canonical momenta without using the equations of motion are called primary. A constraint ϕ_a is first class if its Poisson bracket $\{\phi_a, \phi_k\}$ with any constraint ϕ_k vanishes upon application of the constraints; otherwise it is called second class. The Virasoro constraints are primary and first class. A detailed discussion of constrained systems is given in [**?**]. Here we just note that first class constraints are associated with gauge invariance. The canonical Hamiltonian vanishes identically,

$$H \;=\; \int d\sigma\; (\dot{X}\cdot\Pi - \mathcal{L}) \;\equiv\; 0 \quad , \tag{2.7}$$

which seems to imply that we have no time evolution. Actually, the theory of singular systems demands that we have to add a linear combination of the constraints to H. Therefore, the dynamics is completely governed by the Virasoro constraints. This is more thoroughly discussed in [**?**].

Gauge fixing. We can use the invariance under worldsheet reparametrizations, $\xi^\alpha \longmapsto \xi'^\alpha = f^\alpha(\xi)$, to bring the induced metric on any coordinate patch of the parameter space to the conformally flat form

$$\gamma_{\alpha\beta} \;=\; \lambda(\xi)\, \eta_{\alpha\beta} \quad . \tag{2.8}$$

This is called the conformal gauge or orthogonal gauge, because it translates via (2.4) into

$$\dot{X}\cdot X' \;=\; 0 \qquad \text{and} \qquad \dot{X}^2 + X'^2 \;=\; 0 \quad . \tag{2.9}$$

In this gauge, the Nambu-Goto action (2.3), the canonical momentum density (2.5), and the Virasoro constraints (2.6) simplify to

$$\begin{aligned} S^{\rm gf}_{\surd}[X] &= \frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi\; (\dot{X}^2 - X'^2) \quad , \\ \Pi^{\rm gf}_\mu &= \frac{1}{2\pi\alpha'}\,\dot{X}_\mu \quad , \\ 2\alpha' T^{\rm gf}_{\pm\pm} &= \frac{1}{2}(\dot{X} \pm X')^2 \;=\; 0 \quad . \end{aligned} \tag{2.10}$$

The Polyakov Action

The Nambu-Goto action is non-polynomial und therefore difficult to quantize. As in relativistic point-particle mechanics, an alternative and classically equivalent description is obtained by introducing an auxiliary degree of freedom, namely the intrinsic worldsheet metric $h_{\alpha\beta}(\xi)$ as a field independent of $\gamma_{\alpha\beta}[X]$. This is the price to pay for getting an action quadratic in X, called the Polyakov action (as usual, $h := \det h_{\alpha\beta}$):

$$\begin{aligned} S_0[X,h] &= -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\xi\, \sqrt{-h}\, h^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,\eta_{\mu\nu} \\ &= -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\xi\, \sqrt{-h}\, h^{\alpha\beta}\,\gamma_{\alpha\beta}[X] \quad . \end{aligned} \tag{2.11}$$

Local symmetries and constraints. This action is again reparametrization invariant. An infinitesimal change of coordinates $\xi^\alpha \longmapsto \xi^\alpha - \varepsilon^\alpha(\xi)$ induces

$$\begin{aligned} \delta_\varepsilon X^\mu &= \varepsilon^\alpha \partial_\alpha X^\mu \quad , \\ -\delta_\varepsilon h^{\alpha\beta} &= \nabla^\alpha \varepsilon^\beta + \nabla^\beta \varepsilon^\alpha && (2.12) \\ &= -\varepsilon^\gamma \partial_\gamma h^{\alpha\beta} + \partial_\gamma \varepsilon^\beta h^{\alpha\gamma} + \partial_\gamma \varepsilon^\alpha h^{\gamma\beta} && (2.13) \end{aligned}$$

which implies $\delta_\varepsilon \sqrt{-h} = \partial_\gamma(\varepsilon^\gamma \sqrt{-h})$. In addition, the Polyakov action possesses a local Weyl (rescaling) invariance,

$$\begin{aligned} \delta_\Lambda X^\mu &= 0 \quad , \\ \delta_\Lambda h_{\alpha\beta} &= \Lambda(\xi)\, h_{\alpha\beta} \quad , \end{aligned} \tag{2.14}$$

which is not to be confused with a (conformal) coordinate transformation! No primary constraints appear, but the equation of motion for the auxiliary field $h_{\alpha\beta}$ turns out to be algebraic.

Exercise 2.2 *Show that*

$$\begin{aligned} 0 &= -\frac{4\pi\alpha'}{\sqrt{-h}}\,\frac{\delta S_0}{\delta h^{\alpha\beta}} \\ &= \partial_\alpha X\cdot\partial_\beta X - \frac{1}{2}h_{\alpha\beta}\left(h^{\gamma\delta}\,\partial_\gamma X\cdot\partial_\delta X\right) \;=\; \gamma_{\alpha\beta} - \frac{1}{2}h_{\alpha\beta}\,\mathrm{tr}(h^{-1}\gamma) \\ &=: \alpha'\, T_{\alpha\beta}[X,h] \end{aligned} \tag{2.15}$$

($T_{\alpha\beta}$ will be related to $T_{\pm\pm}$ in (2.6) *shortly).*

Comparison with Nambu-Goto action. From (2.15) we see that the auxiliary metric $h_{\alpha\beta}$ is classically proportional to the induced metric $\gamma_{\alpha\beta}$. If we evaluate the Polyakov action (2.11) on this solution,

$$S_0[X, h_{..}{=}\lambda\gamma_{..}] = \frac{-1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\xi\sqrt{-\lambda^2\gamma}\,\lambda^{-1}\gamma^{\alpha\beta}\gamma_{\alpha\beta} = \frac{-1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2\xi\sqrt{-\gamma} = S_{\surd}[X]\ ,$$

we obtain back the Nambu-Goto action (2.3), which demonstrates their equivalence on the level of the equations of motion.

Exercise 2.3 *Using the inequality* $(\mathrm{tr}M)^2 \geq 4\det M$ *for any real symmetric* $2{\times}2$ *matrix* M*, with equality for* $M \propto \mathbb{1}$*, show that*

$$|S_0[X,h]| \;\geq\; |S_{\surd}[X]| \qquad \forall h$$

with equality if and only if $h_{\alpha\beta} \propto \gamma_{\alpha\beta}$*.*

Let us stress that the Polyakov action is *not* the worldsheet area measured with the intrinsic metric. Both actions are equal if and only if the metrics $h_{\alpha\beta}$ and $\gamma_{\alpha\beta}[X]$ are conformally related, which is the case when the equation of motion for $h_{\alpha\beta}$ is satisfied. Note that the proportionality factor λ drops out due to Weyl invariance.

Conformal gauge fixing. Since the intrinsic metric $h_{\alpha\beta}$ is not dynamical but a pure gauge degree of freedom, we may fix it to a convenient value. We use the worldsheet reparametrizations (2.13) to go again to the conformal (orthogonal) gauge, now

$$h_{\alpha\beta} \;=\; \lambda(\xi)\,\eta_{\alpha\beta} \quad . \tag{2.16}$$

This is always possible by solving the so-called Beltrami equation for the transformation functions. The action then simplifies to

$$\begin{aligned} S_0^{\mathrm{gf}}[X] &= S_0[X, h_{..}{=}\lambda(\xi)\eta_{..}] \\ &= -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\xi\; \eta^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,\eta_{\mu\nu} \\ &= +\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\xi\;\left(\dot{X}^2 - X'^2\right) \;=\; S_{\surd}^{\mathrm{gf}}[X] \quad . \end{aligned} \tag{2.17}$$

Finally, the equation of motion for X (dropping boundary terms)

$$0 \;=\; \frac{\delta S_0[X,h]}{\delta X_\mu} \;\propto\; \nabla^\alpha\partial_\alpha X^\mu \;=\; \partial_\alpha\left(\sqrt{-h}\,h^{\alpha\beta}\,\partial_\beta X^\mu\right) \tag{2.18}$$

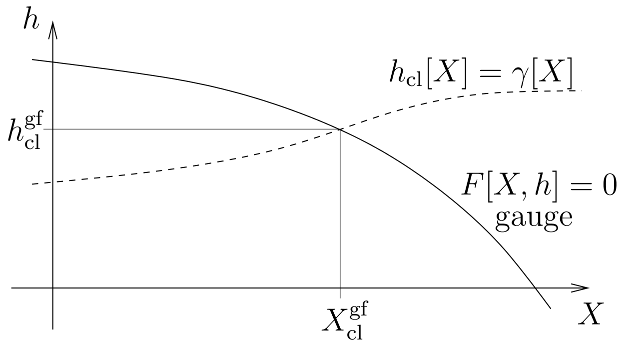
turns into the free wave equation

$$0 \;=\; \Box X^\mu \;=\; (\partial_\tau^2 - \partial_\sigma^2)X^\mu \quad . \tag{2.19}$$

We have arrived at a collection of free massless scalar fields $X^\mu(\xi)$ on the parameter space. The Weyl invariance might be employed to specialize to $\lambda = 1$, but this is neither necessary nor advisable, since that symmetry acquires a quantum anomaly as we will see in section 2.2. It is important to note that not all solutions of the wave equation (2.19) extremize S_0. We still must impose on the solutions of (2.19) the (gauge fixed) equations of motion of $h_{\alpha\beta}$,

$$0 \;=\; \alpha'\,T^{\mathrm{gf}}_{\alpha\beta} \;=\; \begin{pmatrix} \frac12(\dot{X}^2 + X'^2) & \dot{X}\cdot X' \\ \dot{X}\cdot X' & \frac12(\dot{X}^2 + X'^2) \end{pmatrix} \quad . \tag{2.20}$$

Here we recognize the Virasoro constraints (2.10), which in this context appear as first class but secondary. They generalize the familiar mass-shell constraint $0 = p^2 + m^2 \propto \dot{X}^2 + 1$ or $\propto \dot{X}^2$ for a massive or massless relativistic particle.



It is convenient to introduce world-sheet light-cone coordinates

$$\xi^{\pm} = \xi^0 \pm \xi^1 = \tau \pm \sigma \qquad \Longrightarrow \qquad \partial_{\pm} = \frac{1}{2}\left(\partial_0 \pm \partial_1\right) \quad , \tag{2.21}$$

in which a two-tensor $t_{\alpha\beta}$ acquires the new components

$$t_{\pm\pm} = \frac{1}{4}(t_{00} \pm t_{10} \pm t_{01} + t_{11}) \qquad \text{and} \qquad t_{\pm\mp} = \frac{1}{4}(t_{00} \pm t_{10} \mp t_{01} - t_{11}) \quad ,$$

so that the worldsheet metric reads

$$\eta_{\pm\pm} = 0 \qquad \text{and} \qquad \eta_{\pm\mp} = -\frac{1}{2} \quad .$$

In this basis the Virasoro constraints become

$$\alpha' \, T^{\mathrm{gf}}_{\pm\pm} \;=\; \partial_{\pm} X \cdot \partial_{\pm} X \;=\; 0 \qquad \text{and} \qquad T_{\pm\mp} \;\equiv\; 0. \tag{2.22}$$

The Polyakov formulation makes contact with two-dimensional general relativity. We may interpret (2.11) as the action of a collection of D massless real scalar fields (X^{μ}) coupled to gravity ($h_{\alpha\beta}$) in two dimensions. The Einstein-Hilbert part $\int \sqrt{-h}\, R$ is missing from this action because in two dimensions it is merely a constant (4π times the Euler number χ of Σ, a topological invariant) — there are no transversal gravitational fields here. From (2.15) one learns that $T_{\alpha\beta}$ can be identified with the energy-momentum tensor of the matter (X^{μ}). Since the Einstein tensor $R_{\alpha\beta} - \frac{1}{2} h_{\alpha\beta} R$ vanishes identically in two dimensions, the Einstein equation simply reads $T_{\alpha\beta} = 0$. The energy-momentum tensor is covariantly conserved,

$$\nabla^{\alpha} T_{\alpha\beta} \;=\; 0 \quad , \tag{2.23}$$

and it is traceless because

$$0 \;=\; \delta_\Lambda S_0 \;=\; \int_\Sigma \frac{\delta S_0}{\delta h^{\alpha\beta}}\,\delta_\Lambda h^{\alpha\beta} \;=\; \int_\Sigma \frac{\sqrt{-h}}{4\pi}\,T_{\alpha\beta}\,\Lambda h^{\alpha\beta} \;=\; \int_\Sigma \frac{\sqrt{-h}}{4\pi}\,T^\alpha{}_\alpha\,\Lambda \qquad (2.24)$$

due to Weyl invariance. In other words, the theory possesses dilatation and therefore conformal invariance. Let us mention that two dimensions is the maximal number where the local symmetries $(\varepsilon^\alpha, \Lambda)$ allow one to completely eliminate the gravitational field $(h_{\alpha\beta})$, gauge-fixing locally $h_{\alpha\beta} \mapsto \eta_{\alpha\beta}$. More generally, in d dimensions one has $\frac{1}{2}d(d{+}1)$ field degrees of freedom in the metric but only d degrees of reparametrization freedom, so $\frac{1}{2}d(d{-}1)$ metric degrees of freedom remain dynamical.

The conformal gauge (2.16) is incomplete; a class of residual gauge transformations survive (in the Polyakov as well as the Nambu-Goto case). These are reparametrizations which can be compensated by a Weyl rescaling and are called (pseudo)conformal coordinate transformations because they preserve (hyperbolic) angles. These transformations will play an important role when we discuss the global aspects of the conformal gauge.

Exercise 2.4 *Characterize coordinate transformation parameters* $\varepsilon^\alpha(\xi)$ *which respect the conformal gauge fixing, i.e. which produce* $\delta_\varepsilon h_{\alpha\beta}|_{h=\lambda\eta} = (\delta\lambda)\eta_{\alpha\beta}$. *First, show that this implies the property* $\partial_{(\alpha}\varepsilon_{\beta)} = (\partial\cdot\varepsilon)\eta_{\alpha\beta}$. *In worldsheet light-cone coordinates* (2.21) *the parameters* ε^α *combine to* $\varepsilon^\pm = \varepsilon^0 \pm \varepsilon^1$. *Second, show that in these coordinates the above property reads* $\partial_+\varepsilon^- = 0 = \partial_-\varepsilon^+$, *which means that the residual gauge transformations take the form*

$$\varepsilon^+ = \varepsilon^+(\xi^+) \qquad \textit{and} \qquad \varepsilon^- = \varepsilon^-(\xi^-) \quad .$$

Integrating we find that globally the (pseudo)conformal transformations are

$$\xi^+ \;\longmapsto\; f^+(\xi^+) \qquad \text{and} \qquad \xi^- \;\longmapsto\; f^-(\xi^-) \qquad (2.25)$$

with the functions $f^\pm$ only depending on the one coordinate exhibited. It follows that the new coordinates satisfy the wave equation with respect to the old ones, $\Box f^\alpha = 0$.

Transversal gauge fixing. A complete gauge fixing is possible when sacrificing manifest Lorentz covariance. We specialize the conformal gauge to the transversal gauge by performing a (pseudo)conformal transformation with $f^+{+}f^- = 2n_\mu X^\mu$ which is possible since both sides obey the wave equation. As a result, the new worldsheet time becomes

$$\tau \;=\; \tfrac{1}{2}(\xi^+ + \xi^-) \;=\; n_\mu X^\mu \quad , \qquad (2.26)$$

and no freedom remains because this also determines σ, up to a constant shift. It is convenient to take the vector n lightlike, choosing $n_\mu = \frac{\beta}{2\alpha' p^+}(1,0,\ldots,0,1)$ which gives

$$X^+ \ := \ X^0 + X^{D-1} \ = \ \tfrac{2}{\beta}\alpha' p^+ \tau \qquad \text{with} \quad \beta := \begin{cases} 1 & \text{for open strings} \\ 2 & \text{for closed strings} \end{cases} \tag{2.27}$$

and a constant p^+. Here, we have introduced spacetime light-cone coordinates $X^\pm = X^0 \pm X^{D-1}$ which define a splitting of $\{X^\mu\}$ into timelike/longitudinal components ($X^\pm$) and transversal ones (X^i, $i = 1,\ldots,D{-}2$). With the help of $\partial_\pm X^+ = \frac{1}{2}\dot{X}^+ = \frac{\alpha'}{\beta}p^+$ we can solve the Virasoro constraints

$$0 \ = \ \alpha' \, T^{\text{gf}}_{\pm\pm} \ = \ \partial_\pm X \cdot \partial_\pm X \ = \ -\partial_\pm X^+ \, \partial_\pm X^- + \sum_{i=1}^{D-2} \partial_\pm X^i \, \partial_\pm X^i \tag{2.28}$$

for X^- and find (summation convention for i understood)

$$\partial_\pm X^- \ = \ \frac{\beta}{\alpha' p^+} \, \partial_\pm X^i \, \partial_\pm X^i \quad . \tag{2.29}$$

Adding the zero mode q^- of X^- we see that the set of physical degrees of freedom is $\{q^-, p^+, X^i(\xi)\}$.

Finally, the Hamiltonian (2.7) in conformal gauge takes the form

$$H^{\text{gf}} \ = \ \frac{1}{4\pi\alpha'} \int \! \mathrm{d}\sigma \ (\dot{X}^2 + X'^2) \ = \ \frac{1}{2\pi\alpha'} \int \! \mathrm{d}\sigma \ \big((\partial_+ X)^2 + (\partial_- X)^2 \big) \quad . \tag{2.30}$$

We note that it is proportional to the average of $T^{\text{gf}}_{++} + T^{\text{gf}}_{--}$.

Global symmetries. The global symmetries of the Polyakov (or Nambu-Goto) action are the Poincaré symmetries in $\mathbb{R}^{1,D-1}$,

$$\delta X^\mu \ = \ a^\mu{}_\nu \, X^\nu + b^\mu \qquad \text{with} \quad a_{\mu\nu} = -a_{\nu\mu} \quad , \qquad \text{but} \quad \delta h_{\alpha\beta} \ = \ 0 \quad . \tag{2.31}$$

The associated Noether charge densities and charges are (in conformal gauge)

$$\Pi_\mu = \frac{1}{2\pi\alpha'} \, \dot{X}_\mu \qquad \Longrightarrow \qquad P_\mu = \int \! \mathrm{d}\sigma \ \Pi_\mu \ = \ \frac{1}{2\pi\alpha'} \int \! \mathrm{d}\sigma \ \dot{X}_\mu \quad , \tag{2.32}$$

$$\mathcal{J}_{\mu\nu} = \frac{1}{2\pi\alpha'} \, X_{[\mu} \dot{X}_{\nu]} \quad \Longrightarrow \quad J_{\mu\nu} = \int \! \mathrm{d}\sigma \ \mathcal{J}_{\mu\nu} \ = \ \frac{1}{2\pi\alpha'} \int \! \mathrm{d}\sigma \ X_{[\mu} \dot{X}_{\nu]} \quad . \tag{2.33}$$

The charge P^μ is nothing but the center-of-mass energy-momentum of the string, and $P^+ = p^+$ explains the normalization in (2.27). The total energy P^0 of the string is not to be confused with the Hamiltonian H (which vanishes).

Exercise 2.5 *Show that the charges above are conserved for closed strings, and find the boundary conditions under which they are conserved for open strings as well. What is their interpretation in terms of worldsheet momentum flow off the ends of the string?*

Solution of the Equation of Motion

In the conformal gauge (2.8), we identified the equations of motion as the two-dimensional wave equation (2.19). The general solution to this linear problem is well known and given by

$$X^\mu(\xi) \;=\; \frac{1}{2}\left(X_R^\mu(\xi^-) + X_L^\mu(\xi^+)\right) \quad . \tag{2.34}$$

Since X_R (X_L) depends only on the combination ξ^- (ξ^+) of worldsheet coordinates, it represents a right-moving (left-moving) wave along the string and is called a 'right-mover' ('left-mover'). The Virasoro constraints (2.22) then read $T_{\pm\mp} = 0$ and

$$\begin{aligned} \alpha' T_{--} \;&=\; \partial_- X \cdot \partial_- X \;=\; \tfrac{1}{4}(\partial X_R)^2 \;=\; 0 \quad , \\ \alpha' T_{++} \;&=\; \partial_+ X \cdot \partial_+ X \;=\; \tfrac{1}{4}(\partial X_L)^2 \;=\; 0 \quad , \end{aligned} \tag{2.35}$$

where from now on we drop the 'gf' label. Energy-momentum conservation (2.23) boils down to

$$\begin{aligned} \partial_+ T_{--} + \partial_- T_{+-} \;&=\; 0 \qquad \Longrightarrow \qquad T_{--} \;=\; T_{--}(\xi^-) \quad , \\ \partial_+ T_{-+} + \partial_- T_{++} \;&=\; 0 \qquad \Longrightarrow \qquad T_{++} \;=\; T_{++}(\xi^+) \quad , \end{aligned} \tag{2.36}$$

which reveals T_{--} (T_{++}) as a right-mover (left-mover).

Closed string. Recalling that the closed string satisfies periodicity conditions (2.2), we may expand right- and left-movers in Fourier series plus linear terms,

$$\begin{aligned} X_R^\mu(\tau{-}\sigma) \;&=\; q_R^\mu + (\tau{-}\sigma)\,\alpha' p_R^\mu + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\mathrm{i}}{n}\alpha_n^\mu\,\mathrm{e}^{-\mathrm{i}n(\tau-\sigma)} \quad , \\ X_L^\mu(\tau{+}\sigma) \;&=\; q_L^\mu + (\tau{+}\sigma)\,\alpha' p_L^\mu + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\mathrm{i}}{n}\widetilde{\alpha}_n^\mu\,\mathrm{e}^{-\mathrm{i}n(\tau+\sigma)} \quad , \end{aligned} \tag{2.37}$$

which yields

$$\begin{aligned} X^\mu_{\text{closed}} \;&=\; \tfrac{q_L^\mu+q_R^\mu}{2} + \alpha'\tfrac{p_L^\mu+p_R^\mu}{2}\,\tau + \alpha'\tfrac{p_L^\mu-p_R^\mu}{2}\,\sigma + \sqrt{\tfrac{\alpha'}{2}}\sum_{n\neq 0}\frac{\mathrm{i}}{n}(\alpha_n^\mu\,\mathrm{e}^{\mathrm{i}n\sigma} + \widetilde{\alpha}_n^\mu\,\mathrm{e}^{-\mathrm{i}n\sigma})\,\mathrm{e}^{-\mathrm{i}n\tau} \\ &=:\; q^\mu + \alpha'\,p^\mu\,\tau + \alpha'\,w^\mu\,\sigma + \sqrt{\tfrac{\alpha'}{2}}\sum_{n\neq 0}\frac{\mathrm{i}}{n}(\alpha_n^\mu\,\mathrm{e}^{\mathrm{i}n\sigma} + \widetilde{\alpha}_n^\mu\,\mathrm{e}^{-\mathrm{i}n\sigma})\,\mathrm{e}^{-\mathrm{i}n\tau} \end{aligned} \tag{2.38}$$

so that indeed (2.32) yields $P^\mu = p^\mu$. The coordinate zero mode q^μ is shared, and the σ-periodicity of X^μ enforces[1] $w^\mu = 0$, putting $p_R^\mu = p_L^\mu = p^\mu$. The reality condition $(X^\mu)^* = X^\mu$ tells us that q^μ and p^μ are real and

$$(\alpha_n^\mu)^* \;=\; \alpha_{-n}^\mu \qquad \text{and} \qquad (\widetilde{\alpha}_n^\mu)^* \;=\; \widetilde{\alpha}_{-n}^\mu \quad . \tag{2.39}$$

[1]Not so if the target space is multiply connected, since then we may have $X^\mu(\tau,\sigma{+}2\pi) = X^\mu(\tau,\sigma) + 2\pi\alpha' w^\mu$ with a 'winding' w^μ. For this reason we mostly carry w^μ along.

It is convenient to define

$$\alpha_0^\mu \;:=\; \sqrt{\tfrac{\alpha'}{2}}\, p_R^\mu \qquad \text{and} \qquad \widetilde{\alpha}_0^\mu \;:=\; \sqrt{\tfrac{\alpha'}{2}}\, p_L^\mu \tag{2.40}$$

because then the derivative of (2.37) can be represented as

$$\begin{aligned} \partial_- X_R^\mu \;&=\; \dot{X}_R^\mu \;=\; \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^\mu \, \mathrm{e}^{-in(\tau-\sigma)} \quad , \\ \partial_+ X_L^\mu \;&=\; \dot{X}_L^\mu \;=\; \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \widetilde{\alpha}_n^\mu \, \mathrm{e}^{-in(\tau+\sigma)} \quad . \end{aligned} \tag{2.41}$$

Open string. When deriving the equations of motion for the open string, an assumption has to be made about the variation of X^μ at the worldsheet boundaries $\sigma = 0$ and $\sigma = \pi$. (The boundaries in τ represent initial and final string configurations which are kept fixed or moved to temporal infinity.) If we insist in spacetime translation invariance, we must not restrict the string ends in any way. Extremizing the action in (2.18) then produces the wave equation in the conformal gauge only up to boundary terms which must vanish seperately,

$$n^\alpha\, \partial_\alpha X^\mu|_{\sigma=0,\pi} \;=\; X'^\mu|_{\sigma=0,\pi} \;=\; 0 \qquad\qquad \text{Neumann} \tag{2.42}$$

with n^α being a vector normal to the worldsheet boundary.

Exercise 2.6 *Using the constraints* (2.20) *and the Neumann boundary conditions* (2.42) *show that the ends of an open string move at the speed of light. (The ensueing degeneracy of the worldsheet metric at the boundary poses only a formal difficulty.)*

Alternatively, one can avoid the boundary terms by keeping the worldsheet boundaries fixed, e.g.

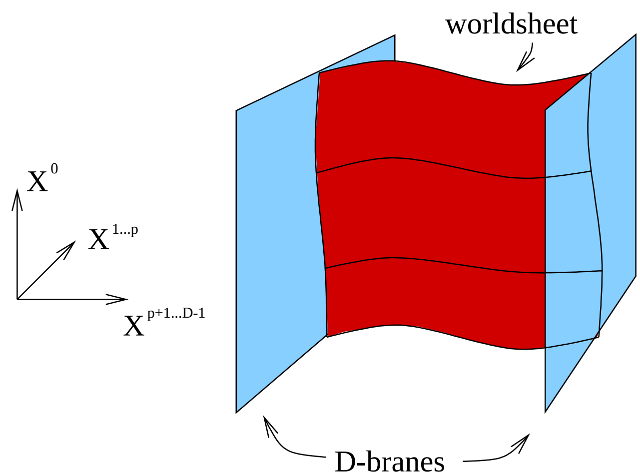
$$X^\mu|_{\sigma=0} \;=\; 0 \qquad \text{and} \qquad X^\mu|_{\sigma=\pi} \;=\; L^\mu \qquad\qquad \text{Dirichlet} \tag{2.43}$$

where L^μ is an arbitrary constant spacelike vector. The choice between Neumann and Dirichlet boundary conditions can be made seperately at each string end for each coordinate X^μ, which leads to the more general situation of

$$\begin{array}{rcl} \text{Neumann at } \sigma=0 & \qquad \text{for} \qquad & X^0,\ldots,X^p \quad , \\ \text{Dirichlet at } \sigma=0 & \qquad \text{for} \qquad & X^{p+1},\ldots,X^{D-1} \quad , \end{array}$$

and similarly (with a potentially different distribution) at $\sigma = \pi$. The geometrical interpretation is obvious: The left end of the open string is constraint to a p-dimensional hypersurface on which it can move freely (we usually take X^0 to be Neumann). Clearly, translation invariance is broken in the $D{-}p{+}1$ spatial directions transversal to the hypersurface. As will be shown in the lectures of

Lüst [?], such hypersurfaces are actually dynamical and represent (the classical limit of) solitonic objects which are intrinsic to string theory and are called Dp-branes. Their p+1 dimensional worldvolumes are defined as the locations where the worldlines of open string ends can live. For $p = 0$ we have a D-particle, for $p = 1$ we have a D-string, for $p = 2$ we have a D-membrane, ..., for $p = D-1$ the D-brane fills the whole space. The latter represents the traditional situation (full translational invariance), while the other extreme ($p = -1$) is localized even in time and therefore called a D-instanton. It is important to notice that the open string and the D-string are different objects.



Like the closed string periodicity, the open string boundary conditions suggest a Fourier mode expansion of the solution (2.34) to the wave equation. The standard trick to implement the boundary conditions employs the mirror charge idea: combine the solution X^μ_{closed} with its reflection at the boundary. For the σ=0 end, this yields (see also Exercise (2.7))

$$\begin{aligned} X^\mu_{\text{open}} &= X^\mu_{\text{closed}}(\tau,\sigma) \ \pm \ X^\mu_{\text{closed}}(\tau,-\sigma) \\ &= \frac{1}{2}\big(X^\mu_R(\tau-\sigma) \pm X^\mu_R(\tau+\sigma) + X^\mu_L(\tau+\sigma) \pm X^\mu_L(\tau-\sigma)\big) \\ &= \left\{ \begin{matrix} (q^\mu_L+q^\mu_R) + \alpha'(p^\mu_L+p^\mu_R)\,\tau \\ \alpha'(p^\mu_L+p^\mu_R)\,\sigma \end{matrix} \right\} + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\mathrm{i}}{n}(\alpha^\mu_n \pm \widetilde{\alpha}^\mu_n)\,\mathrm{e}^{-\mathrm{i}n\tau}\left\{ \begin{matrix} \cos n\sigma \\ \mathrm{i}\sin n\sigma \end{matrix} \right\} \\ &=: \left\{ \begin{matrix} q^\mu + 2\alpha' p^\mu \tau \\ 2\alpha' w^\mu \sigma \end{matrix} \right\} + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\mathrm{i}}{n}\beta^\mu_n\,\mathrm{e}^{-\mathrm{i}n\tau}\left\{ \begin{matrix} \cos n\sigma \\ \mathrm{i}\sin n\sigma \end{matrix} \right\} \quad , \end{aligned} \tag{2.44}$$

where we introduced

$$\beta^\mu_n := \alpha^\mu_n \pm \widetilde{\alpha}^\mu_n \qquad \text{including} \qquad \beta^\mu_0 = \sqrt{\tfrac{\alpha'}{2}}\,(p^\mu_L \pm p^\mu_R) = \sqrt{2\alpha'}\left\{ \begin{matrix} p^\mu \\ w^\mu \end{matrix} \right\} \quad . \tag{2.45}$$

Note that again $P^\mu = p^\mu$ but $q^\mu_{\text{closed}} = 2q^\mu_{\text{open}}$. The functions X^μ_R and X^μ_L are taken from the closed-string expansion (2.37) (and do not refer to (2.34) literally). The upper and lower signs correspond to the Neumann and Dirichlet cases, respectively. Note that in the latter case the center-of-mass data q^μ and p^μ are absent while a so-called 'winding term' linear in σ is now admitted due to the lack of periodicity and even necessary to agree with (2.43) upon choosing $L^\mu = 2\pi\alpha' w^\mu$. The right- and left-moving amplitudes α^μ_n and $\widetilde{\alpha}^\mu_n$ are no longer independent but get swapped upon reflection at the string ends. In effect, we have built the open-string vibrations by combining left- and right-movers into a standing wave.

Exercise 2.7 *The construction* (2.44) *automatically imposes the σ=0 boundary condition also at the σ=π end. Can you generalize it to the mixed (DN and ND) cases, and write down the corresponding mode expansions?*

Virasoro constraints. We have seen that an arbitrary closed string configuration is encoded in the set $\{q^\mu, p^\mu, \alpha^\mu_{n\neq0}, \widetilde{\alpha}^\mu_{n\neq0}\}$. However, these data are too general since they must be subjected to the Virasoro constraints. We now show how to implement those. Since the energy-momentum tensor is the current associated with general coordinate transformations, the Noether theorem tells us that the (weighted) Virasoro constraints

$$T^{\varepsilon^+}_{++}(\tau) \;=\; \oint \mathrm{d}\sigma\; \varepsilon^+(\xi^+)\, T_{++} \qquad \text{and} \qquad T^{\varepsilon^-}_{--}(\tau) \;=\; \oint \mathrm{d}\sigma\; f\varepsilon^-(\xi^-)\, T_{--} \tag{2.46}$$

generate (via Poisson brackets) the residual (pseudo)conformal coordinate transformations (2.25) of the closed-string worldsheet in the conformal gauge.

Exercise 2.8 *Show that the above charges are conserved, e.g. $\dot{T}^{\varepsilon^+}_{++} = 0$. Consequently, for the Cauchy problem the Virasoro constraints represent restrictions on the initial string configuration only.*

Without loss of generality, we put $\tau = 0$. A convenient basis for the periodic weight functions is $\varepsilon^\pm_n(\sigma) = \frac{1}{2\pi}\mathrm{e}^{\pm\mathrm{i}n\sigma}$. The associated charges are the Virasoro operators ($\to$ quantum theory)

$$\begin{aligned} L_n \;&:=\; T^{\varepsilon^-_n}_{--} \;=\; \oint \frac{\mathrm{d}\sigma}{2\pi}\,\mathrm{e}^{-\mathrm{i}n\sigma}\frac{1}{4\alpha'}\,(\partial X_R)^2 \;=\; \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{n-m}\cdot\alpha_m \\ \widetilde{L}_n \;&:=\; T^{\varepsilon^+_n}_{++} \;=\; \oint \frac{\mathrm{d}\sigma}{2\pi}\,\mathrm{e}^{+\mathrm{i}n\sigma}\frac{1}{4\alpha'}\,(\partial X_L)^2 \;=\; \frac{1}{2}\sum_{m\in\mathbb{Z}} \widetilde{\alpha}_{n-m}\cdot\widetilde{\alpha}_m \end{aligned} \tag{2.47}$$

for the closed string.

Exercise 2.9 *Verify equation* (2.47).

The L_n and $\widetilde{L}_n$ are therefore the Fourier components of the energy-momentum tensor,

$$T_{--}(\sigma) \;=\; \sum_{n\in\mathbb{Z}} L_n \,\mathrm{e}^{+\mathrm{i}n\sigma} \qquad \text{and} \qquad T_{++}(\sigma) \;=\; \sum_{n\in\mathbb{Z}} \widetilde{L}_n \,\mathrm{e}^{-\mathrm{i}n\sigma} \quad , \tag{2.48}$$

and are subject to the reality conditions

$$(L_n)^* \;=\; L_{-n} \qquad \text{and} \qquad (\widetilde{L}_n)^* \;=\; \widetilde{L}_{-n} \quad . \tag{2.49}$$

Hence, the closed-string Virasoro constraints (2.22) translate into

$$L_n \;=\; \widetilde{L}_n \;=\; 0 \qquad \forall\, n \in \mathbb{Z} \quad . \tag{2.50}$$

Of particular importance are the combinations $L_0 + \widetilde{L}_0$ and $L_0 - \widetilde{L}_0$ because they generate rigid translations in τ and σ. respectively. From

$$\begin{aligned} L_0 \;&=\; \tfrac{1}{2}\alpha_0^2 + \sum_{m>0} \alpha_{-m}\cdot\alpha_m \;=\; \frac{\alpha'}{4}p_R^2 + N_R \;=\; \frac{\alpha'}{4}(p+w)^2 + N_R \\ \widetilde{L}_0 \;&=\; \tfrac{1}{2}\widetilde{\alpha}_0^2 + \sum_{m>0} \widetilde{\alpha}_{-m}\cdot\widetilde{\alpha}_m \;=\; \frac{\alpha'}{4}p_L^2 + N_L \;=\; \frac{\alpha'}{4}(p-w)^2 + N_L \quad , \end{aligned} \tag{2.51}$$

where N_L and N_R abbreviate the left-moving respective right-moving vibrational contributions, we invoke that

$$0 \;=\; L_0 - \widetilde{L}_0 \;=\; \alpha' p{\cdot}w + N_R - N_L \qquad \overset{p\cdot w=0}{\Longrightarrow} \qquad N_R \;=\; N_L \;=:\; N \quad , \tag{2.52}$$

as a consequence of rotational invariance, as well as

$$0 \;=\; L_0 + \widetilde{L}_0 \;=\; \tfrac{\alpha'}{2}(p^2{+}w^2) + N_R + N_L \qquad \overset{w=0}{\Longrightarrow} \qquad 0 \;=\; \tfrac{\alpha'}{2}p^2 + 2N \quad , \tag{2.53}$$

which is nothing but the Hamiltonian (2.30) in terms of Fourier modes,

$$H \;=\; \frac{1}{2}\sum_{m\in\mathbb{Z}}(\alpha_{-m}{\cdot}\alpha_m + \widetilde{\alpha}_{-m}{\cdot}\widetilde{\alpha}_m) \;=\; \frac{\alpha'}{2}p^2 + \sum_{m>0}(\alpha_{-m}{\cdot}\alpha_m + \widetilde{\alpha}_{-m}{\cdot}\widetilde{\alpha}_m) \quad . \tag{2.54}$$

Let us define $N = \frac{1}{2}(N_R{+}N_L)$ also for $w{\neq}0$. Note that it is only through condition (2.52) at fixed w^μ that left-movers know about right-movers.

For the open-string case, left- and right-movers are related, and we have to take the appropriate linear combinations of T_{++} and T_{--} as well as of $\mathrm{e}^{-\mathrm{i}n\sigma}$ and $\mathrm{e}^{+\mathrm{i}n\sigma}$ in order to respect Neumann or Dirichlet boundary conditions.

Exercise 2.10 *Work out the details and obtain the following two equations.*

The result is a single set $\{L_n\}$ of Virasoro operators,

$$L_n \;=\; \frac{1}{2}\sum_{m\in\mathbb{Z}} \beta_{n-m}\cdot\beta_m \quad , \tag{2.55}$$

whose distinguished member reads

$$L_0 \;=\; \frac{1}{2}\beta_0^2 + \sum_{m>0}\beta_{-m}\cdot\beta_m \;=\; \alpha'(p^2+w^2)+N \quad , \tag{2.56}$$

for center-of-mass momentum p^μ parallel and 'winding' w^μ orthogonal to the brane as well as vibration intensity N. We have chosen a notation which makes the open-string constraints look like the right-moving constraints of the closed string. Accordingly, we shall from now on denote β_n^μ by α_n^μ, too, and write

$$\alpha_0^\mu \;=\; \tfrac{\sqrt{2\alpha'}}{\beta}(p^\mu+w^\mu) \qquad \text{with} \quad p\cdot w=0 \quad \text{for open strings} \quad . \tag{2.57}$$

Let us consider a situation where the target space is partially compactified, and adapt our string coordinates such that the momentum components split as $(p^\mu) = (p^m_{\text{ext}}, p^I_{\text{int}})$ with m and I labelling the 'large' and compactified dimensions, respectively. Viewed with low resolution, a string then appears like a point particle in a lower-dimensional spacetime, carrying energy-momentum p^m_{ext} and, therefore, mass $M=\sqrt{-p^2_{\text{ext}}}$. The L_0 constraint thus relates this mass to the vibrational intensity (as well as the winding and compact momentum),

$$\alpha' M^2 \;=\; -\alpha' p^2_{\text{ext}} \;=\; \beta^2 N + \alpha'(p^2_{\text{int}}+w^2) \qquad \text{with} \quad \beta^2 = \begin{cases} 1 & \text{for open strings} \\ 4 & \text{for closed strings} \end{cases} \quad . \tag{2.58}$$

2.2 The Quantized Bosonic String

If string theory is to describe fundamental physics it has to be subjected to quantization. In these lectures, we shall use the so-called first-quantized framework, i.e. develop the quantum *mechanics* of strings (string *field* theory is much harder). Like for a point-particle, there exist a number of quantization schemes for the string, namely

- Path integral quantization
  (which will not be discussed in these lectures, see [**?**])
- Canonical quantization
  - covariant (i.e. in the conformal gauge)
  - light-cone (i.e. in the transversal gauge)
- BRST quantization

Covariant Canonical Quantization

Point-particle mechanics may be considered as a (1+0) dimensional field theory (on the worldline). Likewise, string mechanics can be treated as a (1+1) dimensional field theory, with the string coordinates X^μ being viewed as scalar fields on the parameter space Σ. Since we have learned from (2.17) that this field theory is non-interacting in the conformal gauge, our task seems easy: quantize a free-field theory with some constraints! (Presently we consider only worldsheets of free string propagation; string interactions will be implemented in section 3.1 by generalizing the worldsheet topology, which however does not affect the local X dynamics.) With canonical momenta $\Pi^\mu = \frac{1}{2\pi\alpha'}\dot{X}^\mu$ the standard equal-time canonical commutation relations read

$$\begin{aligned} \mathrm{i}\,\eta^{\mu\nu}\delta(\sigma-\sigma') &= \Big[X^\mu(\tau,\sigma)\,,\,\Pi^\nu(\tau,\sigma')\Big] \\ &= \frac{1}{2\pi\alpha'}\Big[X^\mu(\tau,\sigma)\,,\,\dot{X}^\nu(\tau,\sigma')\Big] \quad . \end{aligned} \tag{2.59}$$

For the closed string, we can insert the mode expansion (2.38) and obtain the equivalent relations

$$\begin{aligned} [q^\mu\,,\,p^\nu] &= \mathrm{i}\,\eta^{\mu\nu} \quad , \\ [\alpha^\mu_m\,,\,\alpha^\nu_n] &= m\,\eta^{\mu\nu}\delta_{m+n,0} \quad , \\ [\widetilde{\alpha}^\mu_m\,,\,\widetilde{\alpha}^\nu_n] &= m\,\eta^{\mu\nu}\delta_{m+n,0} \quad , \end{aligned} \tag{2.60}$$

with all other commutators being zero. Since w^μ does not enter, we treat it as a constant parameter for the time being.[2] Note that even q^0 turns into an operator, a price to pay in any covariant first-quantization scheme. In fact, the vanishing of the covariant Hamiltonian (2.7) is intimately related. As usual, the 'field' amplitudes are promoted to quantum operators, calling for a Fock space to act on. For each $m{>}0$, the set $\{\frac{1}{\sqrt{m}}\alpha^\mu_m, \frac{1}{\sqrt{m}}\alpha^\mu_{-m}|\mu=0,\ldots,D{-}1\}$ gives us D copies of a harmonic oscillator. We recognize the (right-moving) number operator N_R for the string modes as the Hamilton operator (up to a constant) for this infinite collection of harmonic oscillators, each of which contributes an amount m to the energy,

$$N_R \;=\; \sum_{m>0}\alpha_{-m}\cdot\alpha_m \;=\; \sum_{m>0} m\left(\tfrac{1}{\sqrt{m}}\alpha_{-m}\cdot\tfrac{1}{\sqrt{m}}\alpha_m\right) \quad . \tag{2.61}$$

The commutators (2.60) tell us that an application of $\alpha^\mu_{n>0}$ lowers the eigenvalue of N_R by n while the action of $\alpha^\mu_{n<0}$ raises it by n. In order to bound the spectrum of N_R (and of N_L) from below we define a vacuum state $|0\rangle$ by

$$\alpha^\mu_n|0\rangle \;=\; 0 \qquad \text{and} \qquad \widetilde{\alpha}^\mu_n|0\rangle \;=\; 0 \qquad\qquad \text{for} \quad n\geq 0 \quad , \tag{2.62}$$

which at $n{=}0$ includes translation invariance, $p^\mu|0\rangle{=}0$. Normal ordering $:\ldots:$ then rearranges operator products in the order $\alpha_<, q, p, \alpha_>$, with obvious notation.

[2] More accurately, it is the momentum conjugate to $\frac{1}{2}(q_L{-}q_R)$.

Exercise 2.11 *For w^μ=0, split $X^\mu_{\rm closed} = q^\mu + \alpha' p^\mu \tau + X^\mu_< + X^\mu_>$ in* (2.38) *and consider the normal-ordered exponential* $:\mathrm{e}^{\mathrm{i}k\cdot X_{\rm closed}}:$. *Show that, for $\tau \to -\infty(1-\mathrm{i}0)$,*

$$:\mathrm{e}^{\mathrm{i}k\cdot X_{\rm closed}}: |0\rangle \;\longrightarrow\; \mathrm{e}^{\mathrm{i}k\cdot q}\,|0\rangle \;=:\; |k\rangle$$

and verify that it is a momentum eigenstate, i.e.

$$p^\mu |k\rangle \;=\; k^\mu |k\rangle \quad . \tag{2.63}$$

The states $|k\rangle$ are still killed by α^μ_n and $\widetilde{\alpha}^\mu_n$ for $n > 0$, but they furnish a representation of the center-of-mass Heisenberg algebra $[q^\mu, p^\nu] = \mathrm{i}\eta^{\mu\nu}$. Hence, the Fock space $\mathcal{F}$ is equipped with a scalar product satisfying

$$\langle k|k'\rangle \;=\; 0 \qquad \text{for} \qquad k \neq k' \quad . \tag{2.64}$$

As usual, $|k\rangle$ is normalized to one in compact directions, but noncompact spacetime directions require wave-packet formation for properly normalized states. In the following, we suppress this technicality and proceed as if spacetime were compact, i.e. with $\langle k|k\rangle = 1$. The Fock space $\mathcal{F}$ is constructed in the usual manner by applying raising operators $\alpha_<$ to the oscillator vacuum $|k\rangle$ for any fixed momentum vector k. With respect to our scalar product, we have the hermiticity properties

$$\begin{aligned} (\alpha^\mu_n)^\dagger &= \alpha^\mu_{-n} \quad , \quad (p^\mu)^\dagger = p^\mu \quad , \\ (\widetilde{\alpha}^\mu_n)^\dagger &= \widetilde{\alpha}^\mu_{-n} \quad , \quad (q^\mu)^\dagger = q^\mu \quad . \end{aligned} \tag{2.65}$$

For the open string, the mode expansion (2.38) inside the canonical commutation relations produces the same commutators as in (2.60), after renaming $\beta^\mu_n \to \alpha^\mu_n$.

Negative norm states. Due to the Minkowski signature of the metric, we encounter a standard difficulty in the covariant quantization of vector fields, namely the occurrence of negative norm-squared states in the Fock space:

$$\|\varepsilon_\mu \alpha^\mu_{-1}|k\rangle\|^2 \;=\; \varepsilon^*_\mu \varepsilon_\nu \left\langle k \left| \alpha^\mu_1 \alpha^\nu_{-1} \right| k \right\rangle \;=\; \varepsilon^*_\mu \varepsilon_\nu\, \eta^{\mu\nu}\, \langle k|k\rangle \;=\; \varepsilon^* \cdot \varepsilon \quad , \tag{2.66}$$

which can take any sign depending on the vector ε being spacelike, timelike, or lightlike. Lorentz covariance and boundedness of N_R do not allow us to switch the roles of α^0_n and α^0_{-n}, and so the probabilistic interpretation of quantum mechanics is jeopardized.

The solution is well known, for example from the quantization of the Maxwell field. First, the Virasoro constraints are yet to be implemented. We shall see that this must be done in a weak version and defines a subspace of so-called 'physical states' $|\text{phys}\rangle \in \mathcal{F}_{\rm phys} \subset \mathcal{F}$. Second, one has to prove that $\mathcal{F}_{\rm phys}$ is positive semidefinite. Even then, it remains to make sure that the negative norm-squared states remain decoupled when turning on interactions.

The Virasoro algebra. The Virasoro constraints now involve composite operators and therefore may suffer from an ordering ambiguity. To fix notation, we simply define the operators

$$L_n \;=\; \frac{1}{2}\sum_{m\in\mathbb{Z}} :\alpha_{n-m}\cdot\alpha_m: \qquad \text{and} \qquad \widetilde{L}_n \;=\; \frac{1}{2}\sum_{m\in\mathbb{Z}} :\widetilde{\alpha}_{n-m}\cdot\widetilde{\alpha}_m: \quad . \tag{2.67}$$

Equation (2.60) implies that reordering merely produces c-number shifts for L_0 and $\widetilde{L}_0$ alone. Hence, the two corresponding quantum constraints are so far determined only up to a constant, which will be fixed in a while. Clearly, the Virasoro operators are hermitian,

$$(L_n)^\dagger \;=\; L_{-n} \qquad \text{and} \qquad (\widetilde{L}_n)^\dagger \;=\; \widetilde{L}_{-n} \quad . \tag{2.68}$$

In order to properly implement the operator constraints, we need to know their algebra. This calls for the computation of commutators of type $[\alpha\cdot\alpha,\alpha\cdot\alpha]$. While the 'single-contraction' terms reproduce $L\sim\alpha{\cdot}\alpha$ like in a classical Poisson-bracket calculation, the 'double-contraction' terms are quantum mechanical and occur only for $[L_n,L_{-n}]$ and $[\widetilde{L}_n,\widetilde{L}_{-n}]$ as c-number contributions. Their direct calculation requires a regularization of the infinite sums involved; however, the c-number ansatz plus the Jacobi identity and the evaluation of $\langle 0|[L_n,L_{-n}]|0\rangle$ for $n{=}1,2$ allows one to fix these so-called central terms uniquely. The result is a commuting pair of algebras known as 'Virasoro algebras',

$$[L_m,L_n] \;=\; (m{-}n)\,L_{m+n} + \frac{c}{12}m(m{+}1)(m{-}1)\,\delta_{n+m,0} \tag{2.69}$$

$$[\widetilde{L}_m,\widetilde{L}_n] \;=\; (m{-}n)\,\widetilde{L}_{m+n} + \frac{\widetilde{c}}{12}m(m{+}1)(m{-}1)\,\delta_{n+m,0} \tag{2.70}$$

$$[L_m,\widetilde{L}_n] \;=\; 0 \tag{2.71}$$

where c ($\widetilde{c}$) is known as the central charge or conformal anomaly. For the case at hand one finds

$$c \;=\; \widetilde{c} \;=\; \eta^\mu{}_\mu \;=\; D \quad . \tag{2.72}$$

The open string, of course, supports only a single Virasoro algebra.

Exercise 2.12 *Derive the Virasoro algebra* (2.69) *with* $c{=}D$ *following the stategy sketched above.*

Remarks. Let us make four useful remarks concerning the Virasoro algebra (2.69). First, a redefinition $L_n\to L_n-\alpha\delta_{n,0}$ just shifts the central term by $2\alpha m\,\delta_{n+m,0}$ which is a 'trivial' deformation of (2.69). Second, the commutator $[L_0,L_n]=-nL_n$ reveals that L_0 measures the $\mathbb{Z}$ grading of the algebra. The eigenvalue $-n$ is called 'level'. Third, the Virasoro algebra can be generated from

$\{L_{-2}, L_{-1}, L_1, L_2\}$ alone. Therewith, an infinite number of equations (see below) often reduces to a few. Fourth, (2.69) contains plenty of subalgebras. Besides the two Borel subalgebras involving only $\{L_{n>0}\}$ or $\{L_{n<0}\}$, the most important (and largest finite-dimensional) one is spanned by $\{L_{-1}, L_0, L_1\}$. This set of generators avoids the central term, annihilates the vacuum $|0\rangle$, and produces the algebra of $\mathrm{SL}(2,\mathbb{R})$. The subgroup of conformal transformations generated by this algebra is the group of *real* fractional linear transformations

$$z \ \mapsto\ \frac{az+b}{cz+d} \qquad \text{with} \quad a,b,c,d \in \mathbb{R} \quad \text{and} \quad ad-bc=1$$

whose infinitesimal version reads

$$\delta z \ =\ \alpha + \beta z + \gamma z^2 \qquad \text{with} \quad \alpha,\beta,\gamma \in \mathbb{R} \quad .$$

For the closed string we have to combine this with the left-moving copy based on $\{\widetilde{L}_{-1}, \widetilde{L}_0, \widetilde{L}_1\}$ and obtain the group $\mathrm{SL}(2,\mathbb{C})$ of *complex* fractional linear transformations. Both groups play an important role in the calculation of scattering amplitudes as we will see in section 3.1.

Physical state conditions. Classically, we have $L_n = 0 \quad \forall n$. The natural quantum analog, $L_n|\mathrm{phys}\rangle = 0$, is, first, incomplete since the ordering ambiguity in the L_0 condition requires a relaxation to $(L_0{-}a)|\mathrm{phys}\rangle = 0$ with a (yet to be determined) real constant a, and second, inconsistent since it implies

$$0 \ =\ [L_m, L_{-m}]|\mathrm{phys}\rangle \ =\ \frac{c}{12} m(m{+}1)(m{-}1)|\mathrm{phys}\rangle \qquad \Longrightarrow \qquad c=0 \quad . \tag{2.73}$$

One therefore imposes weaker conditions, namely

$$\langle \mathrm{phys}' |(L_n - a\delta_{n,0})|\,\mathrm{phys}\rangle \ =\ 0 \ =\ \langle \mathrm{phys}' |(\widetilde{L}_n - \widetilde{a}\delta_{n,0})|\,\mathrm{phys}\rangle \qquad \forall\, n \quad . \tag{2.74}$$

Taking into account the hermiticity (2.68) it suffices to demand

$$\begin{aligned} (L_0 - a)|\mathrm{phys}\rangle \ &=\ 0 \qquad \text{and} \qquad L_n|\mathrm{phys}\rangle \ =\ 0 \qquad \forall\, n>0 \quad , \\ (\widetilde{L}_0 - \widetilde{a})|\mathrm{phys}\rangle \ &=\ 0 \qquad \text{and} \qquad \widetilde{L}_n|\mathrm{phys}\rangle \ =\ 0 \qquad \forall\, n>0 \quad . \end{aligned} \tag{2.75}$$

The tilded equations are absent in the open-string case. For the closed string, the only relation between left- and right-moving oscillators (at fixed w^μ) arises as in (2.52),

$$(N_L - N_R)|\mathrm{phys}\rangle \ =\ (\alpha' p{\cdot}w + a - \widetilde{a})|\mathrm{phys}\rangle \quad . \tag{2.76}$$

Anticipating a later result, we now specialize to $\widetilde{a} = a$, so that the 'level-matching condition' at $p{\cdot}w{=}0$ reads $N_L = N_R = N$ on physical states.

The definition (2.75) of $\mathcal{F}_{\mathrm{phys}}$ is consistent but suffers from a redundancy. States of the form

$$|\mathrm{spur}\rangle \ :=\ \sum_{n>0} L_{-n}|\mathrm{any}_n\rangle \ +\ \sum_{n>0} \widetilde{L}_{-n}|\widetilde{\mathrm{any}}_n\rangle \tag{2.77}$$

are called 'spurious' and have the property that they are orthogonal to all physical states,

$$\langle \text{spur}|\text{phys}\rangle \;=\; \sum_{n>0} \langle \text{any}_n\,|L_n|\,\text{phys}\rangle + \sum_{n>0} \langle \widetilde{\text{any}}_n\,|\widetilde{L}_n|\,\text{phys}\rangle \;=\; 0 \quad . \tag{2.78}$$

For certain values of a and $c(=D)$ there exist a subspace $\mathcal{F}_{\text{null}} \subset \mathcal{F}_{\text{phys}}$ of states which are simultaneously spurious and physical, the so-called 'null' states

$$|\text{null}\rangle \;:=\; |\text{spur phys}\rangle \qquad \Longrightarrow \qquad \langle \text{null}|\text{phys}\rangle \;=\; 0 \;=\; \langle \text{null}|\text{null}\rangle \quad . \tag{2.79}$$

It follows that null states may be added to physical states without a consequence for the latter's scalar products. We shall see that this is precisely a gauge freedom of the spacetime field theory built on the quantum-mechanical states. Naturally, null states define an equivalence relation

$$|\text{phys}\rangle \;\sim\; |\text{phys}\rangle + |\text{null}\rangle \tag{2.80}$$

on $\mathcal{F}_{\text{phys}}$. Clearly, physical amplitudes depend only on the equivalence classes $[|\text{phys}\rangle]$ of the states involved. The physical Hilbert space is then the coset space of equivalence classes,

$$\mathcal{H} \;=\; \{[|\text{phys}\rangle]\} \;=\; \mathcal{F}_{\text{phys}}/\mathcal{F}_{\text{null}} \quad . \tag{2.81}$$

Since p^μ commutes with the Virasoro operators we can confine our investigation to spacetime momentum eigenstates $|\text{phys}, k\rangle$. The L_0 condition then yields the quantum version of (2.58), namely the mass-shell condition

$$\alpha'\, M^2 \;=\; -\alpha'\, k_{\text{ext}}^2 \;=\; \beta^2\,(N-a) \;+\; \alpha'(k_{\text{int}}^2 + w^2) \qquad \text{on} \quad |\text{phys}, k\rangle \tag{2.82}$$

for open and closed strings. Because compactified momenta k_{int}^I are quantized and the spectrum of N consists of the non-negative integers (called 'mass levels'), the quantity $\alpha' M^2$ is quantized (for fixed w^2 and a), and a semi-infinite tower of massive states emerges! We see that quantization has replaced the continuous excitation spectrum of the classical string with a discrete one, thus enabling a sensible particle interpretation.

Null state construction. From the physical-state conditions

$$\begin{aligned} (L_0 - a)|\text{phys}, k\rangle \;&=\; \big(\tfrac{\alpha'}{\beta^2}(k^2{+}w^2) + N - a\big)|\text{phys}, k\rangle \;=\; 0 \\ \text{and} \qquad L_{n>0}|\text{phys}, k\rangle \;&=\; \big(\tfrac{\sqrt{2\alpha'}}{\beta}(k{+}w)\cdot \alpha_{n>0} + \ldots\big)|\text{phys}, k\rangle \;=\; 0 \end{aligned} \tag{2.83}$$

we can infer that for suitably shifted momenta k_ℓ^μ with $\alpha' k_\ell^2 = \alpha' k^2 - \beta^2 \ell$ the (off-shell) states $|\chi_\ell\rangle := |\text{phys}, k\rangle|_{k\to k_\ell}$ fulfil

$$L_0|\chi_\ell\rangle \;=\; (a{-}\ell)|\chi_\ell\rangle \qquad \text{and still} \qquad L_{n>0}|\chi_\ell\rangle \;=\; 0 \quad . \tag{2.84}$$

Using $[L_0, L_{-n}] = nL_{-n}$ it follows that any linear combination of the spurious states

$$L_{-n_1}L_{-n_2}\cdots L_{-n_r}|\chi_\ell\rangle \qquad \text{with} \qquad n_i>0 \qquad \text{and} \qquad \textstyle\sum_i n_i=\ell \tag{2.85}$$

satisfies again the L_0 condition but generically not the others. Only if it is annihilated by L_1 and L_2 (and thus by all $L_{n>0}$), such a linear combination will form a null state of level ℓ built on the physical state $|\text{phys}, k\rangle$. In fact, all open-string null states can be obtained in this fashion. Therefore, working out the L_1 and L_2 conditions for successive levels $\ell = 1, 2, \ldots$ with heavy use of (2.69) yields the full open-string null space $\mathcal{F}_{\text{null}} = \bigoplus_\ell \mathcal{F}_\ell$. This computation also produces a certain relation $D = D(\ell, a)$ at each level for $\mathcal{F}_\ell$ to be nonempty. For generic values of (a, D) null states are absent, i.e. $\mathcal{F}_{\text{null}} = 0$.

In particular, at a=1 it is easy to see that (independent of D) each physical state $|\text{phys}, k\rangle$ gives rise to the level-one null state

$$\begin{aligned} |\text{null}, k_1\rangle &= L_{-1}|\chi_1\rangle \qquad \text{with} \quad \alpha' k_1^2 = \alpha' k^2 - \beta^2 \quad , \\ \text{since} \quad L_1|\text{null}, k_1\rangle &= 2L_0|\chi_1\rangle = 2(a{-}1)|\chi_1\rangle \\ \text{and} \quad L_2|\text{null}, k_1\rangle &= 3L_1|\chi_1\rangle = 0 \quad . \end{aligned} \tag{2.86}$$

In fact, for generic D (but a=1) this is all, i.e. the open-string null space reads

$$\mathcal{F}_{\text{null}}^{(a=1,D)} = \mathcal{F}_1 = \big\{ L_{-1}|\chi_1\rangle \;\big|\; (L_0, L_1, L_2)|\chi_1\rangle = 0 \big\} \quad . \tag{2.87}$$

For certain particular values of D, higher-level null states occur. For example, a level-two null state demands that $D = \frac{4-2a}{3-2a}(21{-}8a)$, a curve which intersects a=1 at D=26. Hence, at the point $(a, D) = (1, 26)$ additional null states appear, so that $\mathcal{F}_{\text{null}}^{(1,26)} = \mathcal{F}_1 + \mathcal{F}_2$. This observation will be crucial to obtain agreement with light-cone quantization. By the way, the level-three null state curve reads $D = \frac{5-a}{2-a}(10{-}3a)$. All higher-level null state curves avoid the range $1 < D < 25$ but touch it from both sides. The closed-string results obtain from tensoring left- and right-movers; both copies of the Virasoro algebra then produce null states.

Exercise 2.13 *Use the Virasoro algebra* (2.69) *to show that for a=1 a nonempty*

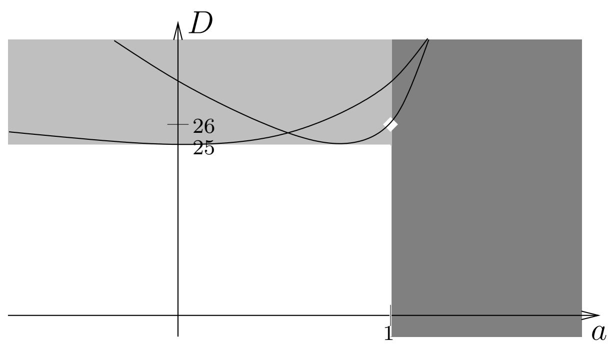
$$\mathcal{F}_2 = \big\{ (L_{-2} + \gamma L_{-1}L_{-1})|\chi_2\rangle \;\big|\; (L_0{+}1, L_1, L_2)|\chi_2\rangle = 0 \big\}$$

fixes γ=$\frac{3}{2}$ and demands D=26 for the open string (and also for the closed string).

No-ghost theorem. It is not an easy task to find out for which, if any, values of D and a the physical Fock space $\mathcal{F}_{\text{phys}}$ is positive semi-definite. A partial answer is provided by the no-ghost theorem of Goddard, Thorn and Brower [**?**] which was later extended by Kač [**?**, **?**]. For both the open and the closed string, the theorem states the following:

- $a > 1$: $\mathcal{F}_{\text{phys}}$ contains negative norm-squared states for any D.
- $a < 1$: $\mathcal{F}_{\text{phys}}$ is positive semi-definite for $1 \leq D \leq 25$.
- $a = 1$: $\mathcal{F}_{\text{phys}}$ is positive semi-definite for $1 \leq D \leq 25$ and $D = 26$.

Negative norm-squared physical states exist also in the regions ($a{\leq}1, D{>}25$) and ($a{\leq}1, D{<}1$) but perhaps not everywhere outside the special point ($a{=}1, D{=}26$); the analysis is not conclusive there.



If one follows in the (a, D) plane a curve which crosses the boundary of the 'physical region', some physical state will change the sign of its norm-squared at the intersection with the boundary. Thus, the interior of the 'physical region' is free of null states but they appear everywhere on its boundary.

Light-Cone Quantization

One may avoid the subtleties of determining the subspace of physical states in the covariant quantization scheme by employing the light-cone gauge (2.27) instead, which in terms of vibration modes reads[3]

$$\alpha^+_{n\neq 0} \;=\; \widetilde{\alpha}^+_{n\neq 0} \;=\; 0 \qquad \text{but} \qquad \alpha^+_0 \;=\; \widetilde{\alpha}^+_0 \;=\; \tfrac{\sqrt{2\alpha'}}{\beta}\, p^+ \;\neq\; 0 \quad . \tag{2.88}$$

The solution (2.29) of the classical constraints $L_n{=}0{=}\widetilde{L}_n$ expresses the modes in the minus direction in terms of the transversal ones ($i = 1, \ldots, D{-}2$),

$$\alpha^-_n \;=\; \frac{1}{\alpha^+_0} \sum_{m\in\mathbb{Z}} \alpha^i_{n-m}\, \alpha^i_m \;=\; \frac{\beta}{\sqrt{2\alpha'}p^+} \sum_{m\in\mathbb{Z}} \alpha^i_{n-m}\, \alpha^i_m \tag{2.89}$$

and analogously for $\widetilde{\alpha}^-_n$. We keep (q^-, p^+) classical and quantize $\{q^i, p^i, \alpha^i_n, \widetilde{\alpha}^i_n\}$ as the remaining independent degrees of freedom, obtaining the composites

$$\alpha^-_n \;=\; \frac{1}{\alpha^+_0} \Big(\sum_{m\in\mathbb{Z}} :\alpha^i_{n-m}\, \alpha^i_m: \; - 2a\, \delta_{n,0} \Big) \tag{2.90}$$

[3] We assume that $w^+ = 0$, i.e. no winding in the 'light-cone time' direction.

and similarly for $\widetilde{\alpha}_n^-$, where we have again allowed for an unknown normal-ordering constant a for p^-. For n=0 this relation is a mass-shell condition in disguise (remember that $\alpha_0^\mu = \frac{\sqrt{2\alpha'}}{\beta}(p^\mu + w^\mu)$):

$$\begin{aligned}
\alpha' M^2 &= \alpha'\left(p^+p^- - p_{\text{ext}}^{\perp\,2}\right) \\
&= \tfrac{\beta^2}{2}\,\alpha_0^+\alpha_0^- - \alpha' p^+ w^- - \alpha' p_{\text{ext}}^{\perp\,2} \\
&= \tfrac{\beta^2}{2}\Big(\sum_{m\in\mathbb{Z}} :\alpha_{-m}^i\,\alpha_m^i: \, - 2a\Big) - \alpha' p^+ w^- - \alpha' p_{\text{ext}}^{\perp\,2} \\
&= \beta^2\Big(\sum_{m>0} \alpha_{-m}^i\,\alpha_m^i - a\Big) + \tfrac{\beta^2}{2}\,\alpha_0^i\,\alpha_0^i - \alpha' p^+ w^- - \alpha' p_{\text{ext}}^{\perp\,2} \\
&= \beta^2\left(N^\perp - a\right) + \alpha'(p_{\text{int}}^2 + w^2)
\end{aligned} \tag{2.91}$$

and a corresponding left-moving identity hold as operator equations. This mass-shell condition takes the same form as (2.82) in the covariant quantization scheme, except that only transverse oscillators contribute to $N^\perp$, whose spectrum is $\mathbb{N}_0$.

In the transversal Fock space $\mathcal{F}^\perp$, spanned by the action of transversal creation operators $\alpha_<^i$ on $|k\rangle$, all states obviously have positive norm. However, there is no free lunch because Lorentz covariance is no longer manifest. Hence, it becomes necessary to explicitly check the realization of the Lorentz algebra on $\mathcal{F}^\perp$, as provided by the quantum version of the Lorentz charges (2.33).

With the help of (2.37), the latters' mode expansion reads

$$J^{\mu\nu} \;=\; q^{[\mu}p^{\nu]} - \sum_{n=1}^{\infty} \frac{\mathrm{i}}{n}\Big(\alpha_{-n}^{[\mu}\,\alpha_n^{\nu]} + \widetilde{\alpha}_{-n}^{[\mu}\,\widetilde{\alpha}_n^{\nu]}\Big) \quad . \tag{2.92}$$

In light-cone coordinates we can arrange these generators as $\{J^{ij}, J^{i+}, J^{i-}, J^{+-}\}$. The J^{ij} generate an SO($D-2$) subgroup which is the manifest symmetry of rotations in the transverse directions. Hence, those are safe and can be discarded from further study. Furthermore, the light-cone choice of $\alpha_{n\neq 0}^+ = 0 = \widetilde{\alpha}_{n\neq 0}^+$ renders J^{i+} and J^{+-} oscillator-free and thus harmless. Therefore, only the commutators $[J^{i-}, J^{j-}]$, which should (and classically do) vanish, may be anomalous and have to be checked. Due to (2.90), schematically $J^{i-} \sim \alpha^i\alpha^- \sim \alpha^i :\alpha^k\alpha^k:$, which makes this calculation quite non-trivial. Details are found on pp. 97 of [**?**], where the result

$$\begin{aligned}
[J^{i-}, J^{j-}] &= -\tfrac{1}{(p^+)^2}\sum_{m=1}^{\infty} \Delta_m \left(\alpha_{-m}^{[i}\,\alpha_m^{j]} + \widetilde{\alpha}_{-m}^{[i}\,\widetilde{\alpha}_m^{j]}\right) \\
\text{with}\quad \Delta_m &= \tfrac{26-D}{12}\,m \;+\; \left(\tfrac{D-26}{12} + 2\,(1-a)\right)\tfrac{1}{m}
\end{aligned} \tag{2.93}$$

indeed spoils Lorentz covariance unless D=26 and a=1. It is certainly reassuring that the magical value for (a, D) has reappeared in a completely different computational context. We shall see that, for other points in the (a, D) plane, covariant quantization and light-cone quantization yield different collections of physical states. In conclusion, the bosonic string is quantum-mechanically healthy only in a 26-dimensional spacetime! This illustrates that the presence of a string determines certain features of the spacetime it is moving in (and vice versa, as familiar from general relativity). It does not spell disaster for our four-dimensional world in the large because our starting point of $\mathbb{R}^{1,D-1}$ as spacetime (chosen for pedagogical and technical reasons) can be generalized to realistic spacetimes with some (22) dimensions being rolled up at small scales.

Spectrum of the Bosonic String

Open string. In light-cone quantization, the spectrum of states for the open string is simply generated by acting with the transverse creation operators, $\alpha^i_<$, $i{=}1,\ldots,D{-}2$, on the ground state $|k\rangle$ (not to be confused with the vacuum!). Let us put $a{=}1$ in the mass-shell condition (2.91) with $\beta{=}1$ and freeze the internal and winding contributions at their minimum, i.e. $p^2_{\text{int}}{+}w^2 = 0$. Then, the ground state $|k\rangle$ itself ($N^\perp{=}0$) fulfils

$$\alpha' M^2 |k\rangle \;=\; (N^\perp - 1)|k\rangle \;=\; -|k\rangle \quad , \tag{2.94}$$

hence it is a scalar excitation with negative mass-squared, i.e. a tachyon! This seems a disastrous result for a hopefully physical theory; yet, it only signals an instability of flat spacetime in the bosonic string theory, which is driven to a new 'vacuum' by a process of 'tachyon condensation'. Moreover, we shall be able to get rid of tachyons in more sophisticated string models later.

The first excited states ($N^\perp{=}1$)

$$|\varepsilon^\perp, k\rangle \;:=\; \varepsilon^\perp_i\, \alpha^i_{-1}|k\rangle \tag{2.95}$$

span a $D{-}2$ dimensional space parametrized by a polarization vector $\varepsilon^\perp$. These states are massless, since

$$\alpha' M^2 |\varepsilon^\perp, k\rangle \;=\; (N^\perp{-}1)\, \varepsilon^\perp_i \alpha^i_{-1}|k\rangle \;=\; 0 \quad , \tag{2.96}$$

and form a vector representation of the transversal rotation group $\mathrm{SO}(D{-}2)$.

On the next mass level ($N^\perp{=}2$) one finds

$$|\beta^\perp, \varepsilon^\perp, k\rangle \;:=\; (\beta^\perp_{ij}\, \alpha^i_{-1}\alpha^j_{-1} + \varepsilon^\perp_i\, \alpha^i_{-2})|k\rangle \tag{2.97}$$

with $\alpha' M^2 = +1$, yielding as irreps of $\mathrm{SO}(D{-}2)$ a traceless symmetric two-tensor representing a spin-two excitation, a scalar (the trace part of $\beta^\perp$), and a vector.

It is now obvious how to go on to higher mass levels. In fact, this investigation provides a check on Lorentz covariance: If the full Lorentz group of $\mathrm{SO}(1, D{-}1)$ is realized in the light-cone quantization, the string states will fall into representations of the little group, which is $\mathrm{SO}(D{-}1)$ for massive states and $\mathrm{SO}(D{-}2)$

for massless ones. Thus, the SO($D-2$) irreps found at a given mass level $M^2>0$ should organize themselves into SO($D-1$) multiplets. Indeed, at the first massive level we observe that the three SO($D-2$) irreps combine to a traceless symmetric two-tensor of SO($D-1$)! Turning the argument around, the fact that level $N^{\perp}=1$ supports only $D-2$ states (and no additional scalar) is compatible with Lorentz covariance only if the little group in this case is the massless one, which happens just for $a=1$.

It is instructive to repeat the spectrum analysis in covariant quantization, keeping a and D general at the beginning. Here, the task is, first, to restrict the general state on a given mass level by the physical state conditions $L_{1,2}|\mathrm{phys}, k\rangle = 0$ and, second, to identify the null states among the physical ones. Clearly, the ground state $|k\rangle$ is physical and non-null; it represents a scalar with $\alpha' M^2 = -a$. At level $N=1$ we have $\alpha' M^2 = 1-a$, and the most general state reads

$$|\varepsilon, k\rangle \;:=\; \varepsilon_\mu\, \alpha^\mu_{-1}|k\rangle \quad . \tag{2.98}$$

Let us work out the physical state conditions:

$$\begin{aligned} 0 \;&=\; L_1|\varepsilon,k\rangle \;=\; L_1\,\varepsilon\cdot\alpha_{-1}\,|k\rangle \;=\; [L_1\,,\,\varepsilon\cdot\alpha_{-1}]|k\rangle \\ &=\; [\alpha_0\cdot\alpha_1\,,\,\varepsilon\cdot\alpha_{-1}]|k\rangle \;=\; \varepsilon\cdot\alpha_0\,|k\rangle \;=\; \sqrt{2\alpha'}\,\varepsilon\cdot k\,|k\rangle \quad , \end{aligned} \tag{2.99}$$

$$\begin{aligned} 0 \;&=\; L_2|\varepsilon,k\rangle \;=\; L_2\,\varepsilon\cdot\alpha_{-1}\,|k\rangle \;=\; [L_2\,,\,\varepsilon\cdot\alpha_{-1}]|k\rangle \\ &=\; [\tfrac12\,\alpha_1\cdot\alpha_1\,,\,\varepsilon\cdot\alpha_{-1}]|k\rangle \;=\; \varepsilon\cdot\alpha_1\,|k\rangle \;=\; 0 \quad , \end{aligned} \tag{2.100}$$

where we have used (2.60)–(2.67). The result implies that $|\varepsilon,k\rangle$ is physical provided $\varepsilon\cdot k = 0$, i.e. the 'polarization vector' ε should be orthogonal to the momentum vector. Among these physical states we still have to detect possible null states given by (2.77), in order to be left with the inequivalent physical states. The most general spurious state at level one is

$$|\mathrm{spur}\rangle \;=\; L_{-1}|k\rangle \;=\; \alpha_0\cdot\alpha_{-1}|k\rangle \;=\; \sqrt{2\alpha'}\,k\cdot\alpha_{-1}|k\rangle \;=\; \sqrt{2\alpha'}\,|\varepsilon{=}k,k\rangle \tag{2.101}$$

which is physical (and therefore null) precisely if $0 = \varepsilon(k)\cdot k = k^2$. Hence, the null states arise only if $M^2{=}0$ which means $a{=}1$.

For a fuller discussion let us look at norm of the states $|\varepsilon,k\rangle$ with $\varepsilon\cdot k = 0$. A quick computation yields $\|\,|\varepsilon,k\rangle\,\|^2 = \varepsilon^*\cdot\varepsilon$. This opens three possibilities:

- $a>1 \;\longrightarrow\; M^2<0 \;\longrightarrow\; k^2$ is spacelike. We may choose a frame where $k=(0,|M|,0,\ldots,0)$ and can still take $\varepsilon=(*,0,\ldots,0)$, so that negative norm-squared states are present.

- $a<1 \;\longrightarrow\; M^2>0 \;\longrightarrow\; k^2$ is timelike. We may choose the rest frame $k=(M,0,\ldots,0)$, which forces $\varepsilon=(0,*,\ldots,*)$, so that there are $D-1$ independent states with positive norm-squared.

- $a = 1 \longrightarrow M^2 = 0 \longrightarrow k^2$ is lightlike. We may choose a frame where $k = (\kappa, \kappa, 0, \ldots, 0)$, which allows for $D{-}2$ states $\varepsilon = (0, 0, *, \ldots, *)$ with positive norm-squared plus one state $\varepsilon \sim k$ of zero norm.

The first case is clearly eliminated: The Virasoro constraints do not suffice to remove all negative norm-squared states. The second case describes a massive vector, while the third case yields a massless vector, suggesting the presence of gauge symmetry in the spacetime dynamics.

Exercise 2.14 *Repeat this analysis at level N=2 for a=1, but D arbitrary. The mass-shell condition for*

$$|\beta, \varepsilon, k\rangle \;:=\; (\beta_{\mu\nu}\, \alpha^{\mu}_{-1} \alpha^{\nu}_{-1} + \varepsilon_{\mu}\, \alpha^{\mu}_{-2})|k\rangle$$

reads $L_0|k\rangle = -|k\rangle$, hence $\alpha' k^2 = -1$. Proceed with the following steps:

a) Compare the states in light-cone quantization with those in covariant quantization.

b) Determine the physical states $|\mathrm{phys}\rangle$ by deducing from $L_{1,2}|\beta, \varepsilon, k\rangle = 0$ conditions on ε and β.

c) Write down the most general spurious state $|\mathrm{spur}\rangle$, building on $\lambda \cdot \alpha_{-1}|k\rangle$ and on $|k\rangle$.

d) Determine the null states $|\mathrm{null}\rangle$ by deriving from $L_{1,2}|\mathrm{spur}\rangle = 0$ conditions on the parameters of $|\mathrm{spur}\rangle$.

e) Discuss the degrees of freedom.

f) Check that the state $[10\, \alpha_{-1}{\cdot}\alpha_{-1} + (D{+}4)(\alpha_0{\cdot}\alpha_{-1})^2 + 2(D{-}1)\, \alpha_0{\cdot}\alpha_{-2}]\, |k\rangle$ is physical and calculate its norm.

Only for a=1 and D=26 does the spectrum of inequivalent physical states exactly agree with that in light-cone quantization. The massless vector excitation is tentatively identified with a gauge boson (photon or gluon); the relevant gauge group will appear later. Since at level N with $\alpha' M^2 = N{-}1$ we always have a state described by a symmetric tensor of rank N, we find that the maximal spin at each level is $J^{\max} = N = \alpha' M^2 + 1$.

level N	(mass)^2 $\alpha' M^2$	states and their SO(24) representations	little group	representation content with respect to the little group
0	-1	$\lvert k\rangle$ $\bullet$ $\mathbf{1}$	SO(25)	$\bullet$ $\mathbf{1}$
1	0	$\alpha^i_{-1}\lvert k\rangle$ $\square$ $\mathbf{24}$	SO(24)	$\square$ $\mathbf{24}$
2	1	$\alpha^i_{-2}\lvert k\rangle$ $\quad$ $\alpha^i_{-1}\alpha^j_{-1}\lvert k\rangle$ $\square$ $\quad$ $\square\!\square + \bullet$ $\mathbf{24}$ $\quad$ $\mathbf{299}+\mathbf{1}$	SO(25)	$\square\!\square$ $\mathbf{324}$

Closed string. Since in the case of the closed string (β=2) we can excite both left- and right-moving degrees of freedom, its states are simply tensor products of the open string states, except for the center-of-mass data (q^μ, p^μ) common to left- and right-movers. Again we put $p^2_{\mathrm{int}}+w^2=0$ for simplicity. We have already seen that Lorentz covariance enforces $a=\widetilde{a}=1$. The ground state is again a scalar tachyon $|k\rangle$ with mass $\alpha' M^2=-4$. The most general first excited state is massless and reads

$$|e^\perp, k\rangle \;:=\; e_{ij}\,\alpha^i_{-1}\widetilde{\alpha}^j_{-1}|k\rangle \quad . \tag{2.102}$$

We can decompose this state space into irreps of SO(D−2),

$$e_{ij} \;=\; \left[\frac{1}{2}(e_{ij}+e_{ji}) - \frac{\operatorname{tr} e}{D-2}\,\delta_{ij}\right] + \left[\frac{1}{2}(e_{ij}-e_{ji})\right] + \left[\frac{\operatorname{tr} e}{D-2}\,\delta_{ij}\right] \quad , \tag{2.103}$$

which describe a traceless symmetric two-tensor, an antisymmetric two-tensor, and a scalar. As in the case of the open string, this matches up with the result of covariant quantization only for $a=\widetilde{a}=1$ and $D=26$. Precisely then not only are negative norm-squared states absent but also exist sufficiently many null states in order to kill all longitudinal modes. At D=26 the three SO(24) representations are the **299**, the **276**, and the **1**. The corresponding excitations are interpreted as the graviton, the Kalb-Ramond field, and the dilaton, respectively. The maximal spin at each level is $J^{\max}=2N=\frac{1}{2}\alpha' M^2+2$.

level N	(mass)^2 $\alpha' M^2$	states and their SO(24) representations	little group	representation content with respect to the little group
0	-4	$\lvert k\rangle$ $\underset{\mathbf{1}}{\bullet}$	SO(25)	$\underset{\mathbf{1}}{\bullet}$
1	0	$\alpha^i_{-1}\widetilde{\alpha}^j_{-1}\lvert k\rangle$ $\underset{\mathbf{24}}{\square} \times \underset{\mathbf{24}}{\square}$	SO(24)	$\underset{\mathbf{299}}{\square\!\square} + \underset{\mathbf{276}}{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + \underset{\mathbf{1}}{\bullet}$
2	4	$\alpha^i_{-2}\widetilde{\alpha}^j_{-2}\lvert k\rangle$ $\underset{\mathbf{24}}{\square} \times \underset{\mathbf{24}}{\square}$ $\alpha^i_{-2}\widetilde{\alpha}^j_{-1}\widetilde{\alpha}^k_{-1}\lvert k\rangle$ $\underset{\mathbf{24}}{\square} \times \left(\underset{\mathbf{299}}{\square\!\square} + \underset{\mathbf{1}}{\bullet} \right)$ $\alpha^i_{-1}\alpha^j_{-1}\widetilde{\alpha}^k_{-2}\lvert k\rangle$ $\left(\underset{\mathbf{299}}{\square\!\square} + \underset{\mathbf{1}}{\bullet} \right) \times \underset{\mathbf{24}}{\square}$ $\alpha^i_{-1}\alpha^j_{-1}\widetilde{\alpha}^k_{-1}\widetilde{\alpha}^l_{-1}\lvert k\rangle$ $\left(\underset{\mathbf{299}}{\square\!\square} + \underset{\mathbf{1}}{\bullet} \right) \times \left(\underset{\mathbf{299}}{\square\!\square} + \underset{\mathbf{1}}{\bullet} \right)$	SO(25)	$\underset{\mathbf{324}}{\square\!\square} \times \underset{\mathbf{324}}{\square\!\square}$ $= \underset{\mathbf{20150}}{\square\!\square\!\square\!\square} + \underset{\mathbf{32175}}{\begin{smallmatrix}\square\!\square\\\square\!\square\end{smallmatrix}}$ $+ \underset{\mathbf{52026}}{\begin{smallmatrix}\square\!\square\!\square\\\square\end{smallmatrix}} + \underset{\mathbf{324}}{\square\!\square}$ $+ \underset{\mathbf{300}}{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + \underset{\mathbf{1}}{\bullet}$

Number of states. Using the equivalence of covariant and light-cone quantization at a=1 and D=26, we can easily estimate the total number of independent states at each mass level ℓ for a fixed k^μ. This number is nothing but the dimension d_ℓ of the eigenspace of the number operator $N^\perp$ at level ℓ, where

$$N^\perp|\text{phys}\rangle \;=\; \ell\,|\text{phys}\rangle \quad . \tag{2.104}$$

Instead of computing the sequence of the d_ℓ by direct counting, it is more elegant to evaluate its generating function. For the open string one finds ($\alpha' M^2 = \ell - 1$)

$$
\begin{aligned}
G(q) &= \sum_{\ell=0}^{\infty} d_\ell \, q^{\ell-1} \;=\; \sum_{\text{basis}} \langle \text{phys}| q^{N^\perp - 1} |\text{phys}\rangle \;=\; \mathrm{tr}_{\mathcal{F}^\perp} \, q^{N^\perp - 1} \\
&= q^{-1} \, \mathrm{tr}_{\mathcal{F}^\perp} \, q^{\sum_{i=1}^{24} \sum_{n=1}^{\infty} \alpha^i_{-n} \alpha^i_n} \;=\; q^{-1} \prod_{i=1}^{24} \prod_{n=1}^{\infty} \mathrm{tr}_{n,i} \, q^{\alpha^i_{-n} \alpha^i_n} \\
&= q^{-1} \prod_{i=1}^{24} \prod_{n=1}^{\infty} \sum_{m=0}^{\infty} \langle m | q^{\alpha^i_{-n} \alpha^i_n} | m \rangle \;=\; q^{-1} \prod_{i=1}^{24} \prod_{n=1}^{\infty} \sum_{m=0}^{\infty} (q^n)^m \\
&= q^{-1} \left[\prod_{n=1}^{\infty} \frac{1}{1-q^n} \right]^{24} \;=\; \eta(\tau)^{-24} \quad , \qquad \text{where} \quad q := \mathrm{e}^{2\pi \mathrm{i} \tau}
\end{aligned}
\tag{2.105}
$$

and the complete trace is taken over the basis states $|\{m_{n,i}\}\rangle$ with $m_{n,i} \in \mathbb{N}_0$ and $\ell = \sum_{n,i} n \, m_{n,i}$. In conformal field theory, $G(q) = \mathrm{tr} \, q^{L_0 - c/24}$ is also known as a Virasoro character. Expanding the result again in powers of q one confirms that

$$
\big(d_\ell\big) \;=\; \big(p_{24}(\ell)\big) \;=\; \big(1, 24, 324, 3200, 25650, 176256, 1073720, \ldots\big) \quad , \tag{2.106}
$$

which denotes the number of 24-colored partitions of the integer ℓ. The function η in the last line is the Dedekind function known from complex function and number theory. Using its asymptotic expansion for $q \to 1$ we find the number of states for large $\ell \sim \alpha' M^2$ to be growing like

$$
d_\ell \;\sim\; \ell^{-27/4} \, \mathrm{e}^{4\pi\sqrt{\ell}} \;\sim\; M^{-27/2} \, \mathrm{e}^{M/M_0} \qquad \text{with} \quad M_0 = \tfrac{1}{4\pi\sqrt{\alpha'}} \quad . \tag{2.107}
$$

The closed-string level density explodes similarly.

2.3 BRST Quantization

General Aspects.

BRST quantization (named after Becchi, Rouet, Stora, and Tyutin) is a general method that is taylored to quantize systems with constraints, for instance systems with a local gauge symmetry. Consider a theory of fields $\phi_i(\xi)$ with a gauge group G. The gauge transformations δ_λ are parametrized by $\lambda^a(\xi)$ and are generated by hermitian charges T_a spanning a Lie algebra

$$
[T_a, T_b] \;=\; \mathrm{i} f^c{}_{ab} \, T_c \qquad \text{with} \quad a, b, c = 1, \ldots, \dim G \tag{2.108}
$$

and $f^c{}_{ab}$ being the (real) structure constants of G. We use a general notation for the gauge-fixing conditions (labelled by an index A),

$$
F^A[\phi_i, \xi] \;=\; 0 \quad . \tag{2.109}
$$

Recall the Faddeev–Popov trick in the path-integral quantization of gauge theory,

$$\int \frac{[\mathrm{d}\phi_i]}{\mathrm{Vol}(G)}\,\mathrm{e}^{\mathrm{i}S[\phi_i]} \qquad\longrightarrow\qquad \int [\mathrm{d}\phi_i][\mathrm{d}c^a][\mathrm{d}b_A][\mathrm{d}B_A]\,\mathrm{e}^{\mathrm{i}S[\phi_i]+\mathrm{i}S_{\mathrm{fix}}+\mathrm{i}S_{\mathrm{gh}}} \tag{2.110}$$

with the gauge-fixing term

$$S_{\mathrm{fix}} \;=\; -\int \mathrm{d}\xi\, B_A(\xi)\, F^A[\phi_i,\xi] \tag{2.111}$$

and the ghost action

$$S_{\mathrm{gh}} \;=\; \int \mathrm{d}\xi\, b_A(\xi)\,\delta_c F^A[\phi_i,\xi] \;=\; \iint \mathrm{d}\xi\mathrm{d}\xi'\, b_A(\xi)\,\frac{\delta F^A[\phi_i,\xi]}{\delta\phi_i(\xi')}\,(\delta_c\phi)_i(\xi') \quad , \tag{2.112}$$

where δ_c denotes a gauge transformation with an *anticommuting* parameter function $\lambda^a(\xi)=c^a(\xi)$. Moreover, the path integral involves c^a and b_A as new *anticommuting fields*, called 'ghosts' and 'antighosts', respectively. Integrating them out brings back the Faddeev–Popov determinant associated with the chosen gauge-fixing F^A. The auxiliary field B_A is commuting and implements the gauge fixing via $\int[\mathrm{d}B_A]\mathrm{e}^{\mathrm{i}S_{\mathrm{fix}}}=\prod_\xi \delta(F^A[\phi_i,\xi])$. Due to the presence of both statistics, one has a $\mathbb{Z}_2$ grading in the enlarged field space, but there exists an even finer $\mathbb{Z}$ grading, the ghost number U, with the assignments of $U{=}{+}1$ for ghosts, $U{=}{-}1$ for antighosts, and $U{=}0$ otherwise.

Fermionic symmetry. We now extend the local fermionic transformations δ_c to include the ghost, antighost, and auxiliary fields by defining

$$\begin{aligned} \mathfrak{s}\,\phi_i &= (\delta_c\phi)_i \quad , & \mathfrak{s}c^a &= -\tfrac12 f^a{}_{bc}\,c^b c^c \quad , \\ \mathfrak{s}B_A &= 0 \quad , & \mathfrak{s}b_A &= B_A \quad . \end{aligned} \tag{2.113}$$

The new transformations are tuned in such a way that the 'BRST transformation' $\mathfrak{s}$ is nilpotent, i.e. $\mathfrak{s}\mathfrak{s}=0$. As derivations they also satisfy a graded Leibniz rule.

Exercise 2.15 *Check the nilpotency of* $\mathfrak{s}$ *on* ϕ_i *and on* c^a.

It is quite useful to realize that (suppressing ξ dependence)

$$S_{\mathrm{fix}}+S_{\mathrm{gh}} \;=\; -\int(\mathfrak{s}b_A)\,F^A[\phi_i]+\int b_A\,\mathfrak{s}F^A[\phi_i] \;=\; -\mathfrak{s}\int b_A\,F^A \quad , \tag{2.114}$$

because the nilpotency of $\mathfrak{s}$ then guarantees that not only $S[\phi_i]$ but also $S_{\mathrm{fix}}{+}S_{\mathrm{gh}}$ is BRST invariant. Hence, we have found a local fermionic invariance of the gauge-fixed action as a remnant of the original gauge symmetry. BRST invariance is an invaluable tool for quantization when interactions are included.

Quantum-mechanically, BRST transformations are effected via $\mathfrak{s}W = [Q, W\}$ where $[.\,,.\}$ denotes the graded commutator and Q is known as the BRST operator. The Hilbert space of quantum states $|\psi\rangle$ is extended by the ghost/antighost degrees of freedom in such a way that Q is hermitian, $Q^\dagger = Q$. Note also that Q raises the ghost number by one unit. Consider now a small change of the gauge-fixing condition, $F^A \to F^A + \delta F^A$. Such a shift should not affect physical quantities like transition amplitudes of physical states $|\text{phys}\rangle$. In other words,

$$\begin{aligned} 0 \;&=\; \delta\,\langle\text{phys}\,|\,\text{phys}'\rangle \;=\; \mathrm{i}\,\langle\text{phys}\,|\,\delta(S_{\text{fix}}+S_{\text{gh}})\,|\,\text{phys}'\rangle \\ &=\; -\mathrm{i}\,\big\langle\text{phys}\,\big|\,\mathfrak{s}\textstyle\int b_A\,\delta F^A\big|\,\text{phys}'\big\rangle \;=\; -\mathrm{i}\,\big\langle\text{phys}\,\big|\,\{Q,\textstyle\int b_A\,\delta F^A\}\big|\,\text{phys}'\big\rangle \end{aligned} \tag{2.115}$$

for any δF^A, hence we conclude that

$$Q\,|\text{phys}\rangle \;=\; 0 \quad , \tag{2.116}$$

i.e. all physical states in the enlarged Fock space must be BRST invariant. The converse is also true. Abbreviating $I \equiv \int b_A \delta F^A$, we just saw that $\{Q, I\}$ generates a shift of the gauge-fixing condition. Yet, as a conserved charge, Q itself must not change under this shift, meaning that

$$0 \;=\; [Q,\{Q,I\}] \;=\; Q^2 I - QIQ + QIQ - IQ^2 \;=\; [Q^2, I] \qquad \forall\,\delta F^A \quad , \tag{2.117}$$

so that

$$Q^2 \;=\; 0 \tag{2.118}$$

as expected from the (classical) nilpotency of $\mathfrak{s}$.

Exercise 2.16 *$Q^\dagger = Q$ suggests that Q can be diagonalized. $Q^2 = 0$ then means that all eigenvalues are zero, hence $Q \equiv 0$! What is wrong with this argument?*

Consequently, $Q|\text{any}\rangle$ is physical but also orthogonal to any $|\text{phys}\rangle$,

$$\langle\text{phys}|\big(Q|\text{any}\rangle\big) \;=\; \big(\langle\text{phys}|Q\big)|\text{any}\rangle \;=\; 0 \quad . \tag{2.119}$$

Hence, $Q|\text{any}\rangle = |\text{null}\rangle$, and the physical Hilbert space (2.81) of cosets in the ghost-extended Fock space can be identified with the *cohomology* of the BRST operator,

$$\mathcal{H} \;=\; \frac{\{|\text{phys}\rangle\}}{\{|\text{null}\rangle\}} \;=\; \frac{\ker Q}{\operatorname{im} Q} \;=\; \frac{\{Q\text{-closed states}\}}{\{Q\text{-exact states}\}} \;=\; H^*(Q) \quad . \tag{2.120}$$

The nilpotent algebra (2.118) can be represented irreducibly only in two ways. Either $Q \sim \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ on a doublet of states, or $Q = 0$ on a singlet. Obviously, $\mathcal{H}$ is the direct sum of all singlets.

BRST Quantization of the Bosonic String.

Let us apply the BRST formalism to the case $\{\phi_i\} = \{X^\mu, h_{\alpha\beta}\}$ of the bosonic string. Recall that the Polyakov action (2.11) enjoys invariance under general coordinate transformations (2.12) and local Weyl transformations (2.14), which are parametrized by ε^α and Λ, respectively. Accordingly, we have to introduce reparametrization ghosts c^α and a Weyl ghost w. It is convenient to perform a linear combination of δ_ε and δ_Λ,

$$\left.\begin{array}{rcl} -\widetilde{\delta}_\varepsilon h^{\alpha\beta} &=& \nabla^{(\alpha}\varepsilon^{\beta)} - (\nabla\cdot\varepsilon)\, h^{\alpha\beta} \;=:\; (P_1\varepsilon)^{\alpha\beta} \\ -\widetilde{\delta}_\Lambda h^{\alpha\beta} &=& (\Lambda + \nabla\cdot\varepsilon)\, h^{\alpha\beta} \end{array}\right\} \Longrightarrow \quad \mathfrak{s} h^{\alpha\beta} \;=\; -(P_1 c)^{\alpha\beta} - w h^{\alpha\beta} \tag{2.121}$$

which separates the traceless from the trace part and defines P_1 as the map from vectors ε to traceless symmetric two-tensors h. The string coordinates X^μ transform as before. We have seen that these symmetries suffice to completely gauge-fix the worldsheet metric $h_{\alpha\beta}$, if not globally to the flat metric $\eta_{\alpha\beta}$ so at least to some fixed reference metric $\widehat{h}_{\alpha\beta}$. Using $F^{\alpha\beta} = -\frac{1}{4\pi}\sqrt{-h}\,(h^{\alpha\beta}-\widehat{h}^{\alpha\beta})$, our gauge-fixing term (2.111) then reads

$$S_{\text{fix}} \;=\; -\int B_{\alpha\beta}\, F^{\alpha\beta}[h] \;=\; \tfrac{1}{4\pi}\int B_{\alpha\beta}\sqrt{-h}\,\big(h^{\alpha\beta} - \widehat{h}^{\alpha\beta}\big) \quad , \tag{2.122}$$

and the ghost action (2.112) takes the form

$$S_{\text{gh}} \;=\; \int b_{\alpha\beta}\, \mathfrak{s}F^{\alpha\beta} \;=\; \tfrac{1}{4\pi}\int b_{\alpha\beta}\sqrt{-h}\,\big((P_1 c)^{\alpha\beta} + w\,\widehat{h}^{\alpha\beta}\big) \tag{2.123}$$

with the help of

$$\mathfrak{s}\sqrt{-h} \;=\; -\tfrac{1}{2}\sqrt{-h}\,h_{\alpha\beta}\,\mathfrak{s}h^{\alpha\beta} \;=\; \sqrt{-h}\,w \quad . \tag{2.124}$$

We denote the total action by $S = S_0 + S_{\text{fix}} + S_{\text{gh}}$. Since the Weyl ghost w and the auxiliary field $B_{\alpha\beta}$ are not dynamical we can integrate them out and obtain $\prod_\xi \delta(\widehat{h}\cdot b)\,\delta(h-\widehat{h})$ in the path-integral measure. Thus, the antighost b gets projected to its traceless part, and the worldsheet metric is replaced by the reference metric $\widehat{h}$. However, we must not forget the h equations of motion which provide the (ghost-extended) Virasoro constraints

$$0 \;=\; -\frac{4\pi}{\sqrt{-h}}\frac{\delta S}{\delta h^{\alpha\beta}}\bigg|_{h=\widehat{h}} \;=\; T^X_{\alpha\beta} + T^{\text{gh}}_{\alpha\beta} - B_{\alpha\beta} \quad , \tag{2.125}$$

where T^X was already obtained in (2.15) while T^{gh} stems from the h dependence (also of P_1) in (2.123). The BRST transformations (2.113) specialize to

$$\begin{aligned} \mathfrak{s}X^\mu &= c^\alpha\,\partial_\alpha X^\mu \quad , \\ \mathfrak{s}c^\alpha &= c^\beta\,\nabla_\beta\, c^\alpha \quad , \\ \mathfrak{s}b_{\alpha\beta} &= B_{\alpha\beta} \;=\; T^X_{\alpha\beta} + T^{\text{gh}}_{\alpha\beta} \;=:\; T^{\text{tot}}_{\alpha\beta} \quad . \end{aligned} \tag{2.126}$$

We now specialize to the conformal gauge, $\widehat{h}_{\alpha\beta}(\xi) = \lambda(\xi)\,\eta_{\alpha\beta}$, and convert to light-cone coordinates, yielding $c^\alpha \to c^\pm$ and $b_{\alpha\beta} \to b_{\pm\pm}$. Then, the total action simplifies to

$$S^{\text{gf}}[X,b,c] \;=\; \frac{1}{\pi\alpha'}\int \mathrm{d}^2\xi\;\partial_+X\cdot\partial_-X + \frac{1}{\pi}\int \mathrm{d}^2\xi\;(b_{++}\partial_-c^+ + b_{--}\partial_+c^-)\;. \tag{2.127}$$

The BRST transformation can be split, $\mathfrak{s} = \mathfrak{s}_R + \mathfrak{s}_L$, with

$$\begin{aligned} \mathfrak{s}_R\,X^\mu &= c^-\partial_-X^\mu \quad, & \mathfrak{s}_L\,X^\mu &= c^+\partial_+X^\mu \quad,\\ \mathfrak{s}_R\,c^- &= c^-\partial_-c^- \quad, & \mathfrak{s}_L\,c^+ &= c^+\partial_+c^+ \quad,\\ \mathfrak{s}_R\,b_{--} &= T^{\text{tot}}_{--} \quad, & \mathfrak{s}_L\,b_{++} &= T^{\text{tot}}_{++} \quad, \end{aligned} \tag{2.128}$$

where the gauge-fixed total energy-momentum tensor is given by

$$\begin{aligned} T^{\text{tot}}_{--} &= \tfrac{1}{\alpha'}\,\partial_-X\cdot\partial_-X \;+\; c^-\partial_-b_{--} - 2\,b_{--}\partial_-c^- \quad,\\ T^{\text{tot}}_{++} &= \tfrac{1}{\alpha'}\,\partial_+X\cdot\partial_+X \;+\; c^+\partial_+b_{++} - 2\,b_{++}\partial_+c^+ \quad, \end{aligned} \tag{2.129}$$

which, in contrast to the action (2.127), is not symmetric under the interchange $c \leftrightarrow b$. From the free-field action one immediately extracts the equations of motion for the ghosts and antighosts, which directly imply that

$$\begin{aligned} c^- &= c^-(\xi^-) \quad, & b_{--} &= b_{--}(\xi^-) \quad,\\ c^+ &= c^+(\xi^+) \quad, & b_{++} &= b_{++}(\xi^+) \quad, \end{aligned} \tag{2.130}$$

so that BRST transformations and energy-momentum tensor indeed decompose on-shell into left- and right-moving parts. In addition, periodicity conditions for the closed string and boundary conditions for the open string emerge:

$$\begin{aligned} &\text{closed string:} \quad & b_{\pm\pm}(\sigma{+}2\pi) &= b_{\pm\pm}(\sigma)\;, \quad & c^\pm(\sigma{+}2\pi) &= c^\pm(\sigma) \quad;\\ &\text{open string:} \quad & b_{--}|_{\sigma=0,\pi} &= b_{++}|_{\sigma=0,\pi}\;, \quad & c^-|_{\sigma=0,\pi} &= c^+|_{\sigma=0,\pi} \quad. \end{aligned} \tag{2.131}$$

Like for the X^μ, such conditions are implemented by Fourier-expanding,

$$\begin{aligned} c^-(\tau,\sigma) &= \sum_{n\in\mathbb{Z}} c_n\,\mathrm{e}^{-\mathrm{i}n(\tau-\sigma)} \quad, & b_{--}(\tau,\sigma) &= \mathrm{i}\sum_{n\in\mathbb{Z}} b_n\,\mathrm{e}^{-\mathrm{i}n(\tau-\sigma)} \quad,\\ c^+(\tau,\sigma) &= \sum_{n\in\mathbb{Z}} \widetilde{c}_n\,\mathrm{e}^{-\mathrm{i}n(\tau+\sigma)} \quad, & b_{++}(\tau,\sigma) &= \mathrm{i}\sum_{n\in\mathbb{Z}} \widetilde{b}_n\,\mathrm{e}^{-\mathrm{i}n(\tau+\sigma)} \quad, \end{aligned} \tag{2.132}$$

with $\widetilde{c}_n = c_n$ and $\widetilde{b}_n = b_n$ for open strings but independent left- and right-moving modes for closed strings. Note that $c^\pm$ are real fields while the $b_{\pm\pm}$ are imaginary.

Ghost quantization. The ghost system is almost trivial to quantize because it represents two pairs of mutually conjugate first-order free fields, (b_{--},c^-) and (b_{++},c^+). Since they anticommute, one has the standard equal-time anticommutation relations

$$\{b_{++}(\tau,\sigma)\,,\,c^+(\sigma',\tau)\} \;=\; 2\pi\mathrm{i}\,\delta(\sigma{-}\sigma') \;=\; \{b_{--}(\tau,\sigma)\,,\,c^-(\sigma',\tau)\} \quad , \tag{2.133}$$

with all other anticommutators vanishing. Inserting the mode expansions (2.132) one obtains

$$\begin{aligned} \{b_m,c_n\} \;&=\; \delta_{m+n,0} \quad , & \{\widetilde{b}_m,\widetilde{c}_n\} \;&=\; \delta_{m+n,0} \quad , \\ \{b_m,b_n\} \;&=\; \{c_m,c_n\} \;=\; 0 \quad , & \{\widetilde{b}_m,\widetilde{b}_n\} \;&=\; \{\widetilde{c}_m,\widetilde{c}_n\} \;=\; 0 \quad . \end{aligned} \tag{2.134}$$

For each value of $m{\in}\mathbb{Z}$ we recognize a couple of fermionic harmonic oscillators, $\{b_m,c_{-m}\}=1=\{\widetilde{b}_m,\widetilde{c}_{-m}\}$, whose Fock spaces are two-dimensional but not real since

$$(c_n)^\dagger \;=\; c_{-n} \qquad \text{and} \qquad (b_n)^\dagger \;=\; b_{-n} \qquad \text{etc.} \quad . \tag{2.135}$$

Like in (2.47), the Fourier modes of $T^{\mathrm{gh}}_{\pm\pm}$ yield the ghost contributions to the hermitian Virasoro operators,

$$L^{\mathrm{gh}}_n \;=\sum_{m\in\mathbb{Z}}(m{+}n)\,{:}\,b_{n-m}\,c_m{:} \qquad \text{and} \qquad \widetilde{L}^{\mathrm{gh}}_n \;=\sum_{m\in\mathbb{Z}}(m{+}n)\,{:}\,\widetilde{b}_{n-m}\,\widetilde{c}_m{:} \tag{2.136}$$

with a normal-ordering prescription yet to be specified. In oder to bound

$$N^{\mathrm{gh}}_R \;:=\; L^{\mathrm{gh}}_0 \;=\; \sum_{m>0} m\,{:}\,(b_{-m}\,c_m + c_{-m}\,b_m){:} \tag{2.137}$$

(and $\widetilde{L}^{\mathrm{gh}}_0$) from below, one must define the ghost vacuum as follows,

$$c_n|0\rangle \;=\; 0 \quad \text{for} \quad n\ge\lambda \qquad \text{and} \qquad b_n|0\rangle \;=\; 0 \quad \text{for} \quad n\ge 1{-}\lambda \tag{2.138}$$

with some integer λ, and likewise for the left-movers. Mapping $\lambda\to 1{-}\lambda$ corresponds to interchanging $c_n\leftrightarrow b_n$.

By straightforward but lengthy calculation one asserts that the ghost Virasoro operators (2.136) generate their own pair of Virasoro algebras (2.69), with central charges $c^{\mathrm{gh}} = \widetilde{c}^{\mathrm{gh}} = -26$! The requirement that one obtains the canonical form (2.69) (and not a trivial deformation) determines the value of λ in (2.138):

$$\begin{aligned} 0 \;&=\; 2\,L^{\mathrm{gh}}_0|0\rangle \;=\; [L^{\mathrm{gh}}_1,L^{\mathrm{gh}}_{-1}]|0\rangle \;=\; L^{\mathrm{gh}}_1L^{\mathrm{gh}}_{-1}|0\rangle \\ &=\; (\lambda{+}1)b_{1-\lambda}c_\lambda\,(\lambda{-}2)b_{-\lambda}c_{\lambda-1}|0\rangle \;=\; (\lambda{+}1)(\lambda{-}2)|0\rangle \quad . \end{aligned} \tag{2.139}$$

We take $\lambda=2$ and achieve $SL(2)$ invariance of the vacuum since $L^{\mathrm{gh}}_{n\ge-1}|0\rangle=0$, just like for L^X_n. The $SL(2)$ invariant vacuum is, however, not a highest-weight state of the ghost algebra. Since c_1 acts as creation operators on $|0\rangle$, we can lower the L_0^{gh} eigenvalue by one unit and arrive at its 'ground state':

$$|\Omega\rangle \ := \ c_1|0\rangle \qquad \Longrightarrow \qquad L_0^{\mathrm{gh}}|\Omega\rangle \ = \ -|\Omega\rangle \quad . \tag{2.140}$$

The $b \leftrightarrow c$ asymmetry also leads to a peculiarity of the (indefinite) scalar product in the ghost Fock space. From (2.138) and (2.135) we learn that $\{c_{-1}, c_0, c_1\}$ are nonzero on $|0\rangle$ as well as on $\langle 0|$, while $\{b_{-1}, b_0, b_1\}$ kill both $|0\rangle$ and $\langle 0|$. Consider now the vacuum expectation value of an operator $\mathcal{O}$:

$$\begin{aligned} \langle 0|\mathcal{O}|0\rangle \ &= \ \langle 0|\{b_m, c_{-m}\}\,\mathcal{O}|0\rangle \\ &= \ \langle 0|\, b_m\, c_{-m}\,\mathcal{O} + c_{-m}\,\mathcal{O}\, b_m|0\rangle \ = \ 0 \qquad \text{for} \quad m=-1,0,+1 \end{aligned} \tag{2.141}$$

if $[b_m, \mathcal{O}\} = 0$, i.e. if $\mathcal{O}$ does not contain the product $c_{-1}c_0c_1$. Thus, $\langle 0|0\rangle = 0$, and the simplest non-vanishing expectation value is

$$\langle 0|\, c_{-1}\, c_0\, c_1|0\rangle \ = \ 1 \quad . \tag{2.142}$$

This leads to an off-diagonal pairing,

$$|0\rangle \ \longleftrightarrow \ c_{-1}c_0c_1|0\rangle \qquad \text{and} \qquad |\Omega\rangle \equiv c_1|0\rangle \ \longleftrightarrow \ c_0c_1|0\rangle \equiv c_0|0\rangle \quad . \tag{2.143}$$

BRST operator. Demanding that the BRST transformations (2.128) on all fields ϕ are generated via $\mathfrak{s}\phi = [Q, \phi\}$ one can reconstruct the (closed-string) BRST operator

$$Q \ = \ \oint \frac{\mathrm{d}\sigma}{2\pi}\ (J_- + J_+) \tag{2.144}$$

as an integral over the BRST current with components

$$\begin{aligned} J_- \ &= \ c^-\, T^X_{--} + \tfrac{1}{2} :c^-\, T^{\mathrm{gh}}_{--}: \ = \ \tfrac{1}{\alpha'}\, c^- :\partial_- X \cdot \partial_- X: \ + \ :b_{--}\, c^- \partial_- c^-: \ , \\ J_+ \ &= \ c^+\, T^X_{++} + \tfrac{1}{2} :c^+\, T^{\mathrm{gh}}_{++}: \ = \ \tfrac{1}{\alpha'}\, c^+ :\partial_+ X \cdot \partial_+ X: \ + \ :b_{++}\, c^+ \partial_+ c^+: \ , \end{aligned} \tag{2.145}$$

where the coefficient of $\frac{1}{2}$ is essential. In terms of oscillators this reads

$$\begin{aligned} Q \ &= \ \sum_{n\in\mathbb{Z}} \Big(c_{-n}\, L^X_n \ + \ \tfrac{1}{2} :c_{-n}\, L^{\mathrm{gh}}_n: \Big) \ + \ \text{left-movers} \\ &= \ \sum_{n,m\in\mathbb{Z}} \Big(:c_{-n}\, \alpha_{n-m} \cdot \alpha_m: \ - \ \tfrac{1}{2}(m-n) :c_{-m}\, c_{-n}\, b_{m+n}: \Big) \ + \ \text{left-movers} \quad . \end{aligned} \tag{2.146}$$

Similarly, the ghost number operator U is given by

$$U \ = \ \oint \frac{\mathrm{d}\sigma}{2\pi}\ (j_- + j_+) \ = \ \sum_{n\in\mathbb{Z}} \Big(:c_{-n}\, b_n: + :\widetilde{c}_{-n}\, \widetilde{b}_n: \Big) \tag{2.147}$$

through the ghost-number current components

$$j_- \ = :c^- b_{--}: \qquad \text{and} \qquad j_+ \ = :c^+ b_{++}: \quad , \tag{2.148}$$

where conventionally our vacuum $|0\rangle$ has ghost number zero.

Exercise 2.17 *Use* (2.138) *(with* λ=2*) to verify that* $U^\dagger = -U + 3$*. What can you conclude for a state* $|\psi\rangle$ *with ghost number* u*, computing* $0 = \langle\psi|U|0\rangle = \ldots$ *?*

Using the equations of motion (2.130) and the conservation of $T^X_{\pm\pm}$, we see that these currents are conserved, $\partial_- J_+ = \partial_+ J_- = 0$ and $\partial_- j_+ = \partial_+ j_- = 0$, so that both Q and U are conserved quantities. The last line of (2.128) implies that

$$L_n^{\rm tot} \;=\; \{Q, b_n\} \quad , \tag{2.149}$$

which demonstrates that the $L_n^{\rm tot}$ are BRST invariant,

$$\left[Q, L_n^{\rm tot}\right] \;=\; [Q, \{Q, b_n\}] \;=\; \tfrac{1}{2}\left[\{Q,Q\}, b_n\right] \;=\; 0 \quad , \tag{2.150}$$

as long as Q is nilpotent. In this case a graded Jacobi identity may be employed to derive a Virasoro algebra,

$$\begin{aligned} [L_m^{\rm tot}, L_n^{\rm tot}] \;&=\; [L_m^{\rm tot}, \{Q, b_n\}] \;=\; \{Q, [L_m^{\rm tot}, b_n]\} - \{[Q, L_m^{\rm tot}], b_n\} \\ &=\; \{Q, (m{-}n) b_{m+n}\} - 0 \;=\; (m{-}n) L_{m+n}^{\rm tot} \quad , \end{aligned} \tag{2.151}$$

with total central charge $c^{\rm tot} = 0$! Quite generally, adding the generators of two independent Virasoro algebras, one obtains a new Virasoro algebra, and the central charges simply add. For our case of $L_n^{\rm tot} = L_n^X + L_n^{\rm gh}$, this yields $c^{\rm tot} = D - 26$. Hence, $Q^2 = 0$ implies $D = 26$ again. We learn that outside the critical dimension the BRST framework breaks down because Q is no longer nilpotent.

Exercise 2.18 *Prove the converse of the above, i.e.* $c = 0 \implies Q^2 = 0$ *(dropping the label* 'tot'*). First, use the graded Jacobi identity in* (2.151) *and the commutator* $[L_m, b_n] = (m{-}n) b_{m+n}$ *as well as* (2.149) *to show that* $\{[Q, L_m], b_n\} = 0$*. Apply ghost number grading to argue that this enforces* $[Q, L_m] = 0$*. Second, recall* (2.150) *(graded Jacobi again!) to obtain* $[Q^2, b_n] = 0$*. Finally, ghost number grading once more.*

As a general recipe, one can associate a BRST operator Q with any Lie algebra (2.108) of constraints. Just introduce a canonical ghost/antighost pair (c^a, b_a) for any generator T_a, with

$$\{b_a, c^b\} \;=\; {\rm i}\, \delta_a{}^b \qquad \text{and} \qquad \{b_a, b_b\} \;=\; 0 \;=\; \{c^a, c^b\} \quad , \tag{2.152}$$

and define

$$Q \;:=\; c^a\, T_a \;-\; \tfrac{1}{2} f^c{}_{ab}\, c^a c^b\, b_c \quad . \tag{2.153}$$

The nilpotency of Q directly follows from the Jacobi identity for T_a. Indeed, our BRST operator (2.146) may be found in this fashion with $T_a \to L_n$. Note, however, that no central charge terms are allowed in this construction, another reason why they must disappear in (2.151).

BRST cohomology. According to (2.120) the physical Hilbert space should result from computing the (so-called absolute) cohomology of the BRST operator,

$$Q|\mathrm{phys}\rangle \;=\; 0 \qquad \text{modulo} \quad |\mathrm{null}\rangle \;=\; Q|\mathrm{any}\rangle \quad . \tag{2.154}$$

Since the closed-string cohomology is obtained from tensoring two open-string cohomologies, let us focus on the open string case for simplicity. Our task is simplified by the observation that the cohomology is graded by the simultaneous eigenvalues $\{k^\mu, h, u\}$ for the set $\{p^\mu, L_0^{\mathrm{tot}}, U\}$ of mutually commuting charges,

$$H^*(Q) \;=\; \bigoplus_{k,u} H^{k,u}(Q) \quad , \tag{2.155}$$

where the L_0^{tot} eigenvalue is already fixed by the following little argument:

$$L_0^{\mathrm{tot}}|h\rangle = h|h\rangle \quad \Longrightarrow \quad |h\rangle = \tfrac{1}{h}L_0^{\mathrm{tot}}|h\rangle = \tfrac{1}{h}\{Q,b_0\}|h\rangle = Q\big(\tfrac{1}{h}b_0|h\rangle\big) + \tfrac{1}{h}b_0\,Q|h\rangle$$

is applicable when $h \neq 0$ and shows that a BRST-closed state, $Q|h\rangle$=0, is automatically BRST-exact, $|h\rangle$=$Q|*\rangle$. Thus, no cohomology remains when $h \neq 0$, and we may restrict ourselves to the h=0 eigenspace,

$$\{Q,b_0\}|\mathrm{phys},k\rangle \;=\; L_0^{\mathrm{tot}}|\mathrm{phys},k\rangle \;=\; \big(L_0^X + L_0^{\mathrm{gh}}\big)|\mathrm{phys},k\rangle \;=\; 0 \quad . \tag{2.156}$$

As expected, one reads off a mass-shell relation (for $k_{\mathrm{int}}^2 + w^2 = 0$),

$$\alpha' M^2 \;=\; -\alpha' k^2 \;=\; \beta^2(N^X + N^{\mathrm{gh}}) \tag{2.157}$$

which agrees with the earlier result (2.82) (for Neumann boundary conditions) when $N^{\mathrm{gh}} = -a = -1$. Since (2.156) follows already from

$$b_0\,|\mathrm{phys}\rangle \;=\; 0 \tag{2.158}$$

it is consistent to impose the latter as a supplementary condition on the cohomology, thereby restricting it to the so-called *relative* BRST cohomology $H^*_{\mathrm{rel}} := H^*(Q|b_0{=}0)$. In fact, this restriction is necessary to remove a doubling of physical states in the absolute BRST cohomology, which stems from representing the algebra $\{b_0, c_0\} = 1$ on physical states.

The open-string BRST analysis produces the following result [**?**]:

- The exceptional cohomology $H_{\mathrm{rel}}^{k=0,u}$ is represented by the D+2 states $|0\rangle$, $\alpha_{-1}^\mu c_1|0\rangle$, and $c_{-1}c_1|0\rangle$ at $u = 0, 1, 2$, respectively.

- For $k^\mu \neq 0$ on the mass shell, $H_{\mathrm{rel}}^{k,u}$ is nonzero *only* at $u = 1$.

- $H_{\mathrm{rel}}^{k,u=1}$ has a representative of the form $|\mathrm{phys}\rangle = |\mathrm{phys}\rangle_X \otimes c_1|0\rangle_{\mathrm{gh}}$ where $|\mathrm{phys}\rangle_X$ satisfies the physical state conditions (2.75) with a=1.

Indeed, (2.146) shows that

$$Q\big(\,|\text{phys}\rangle_X \otimes c_1|0\rangle_{\text{gh}}\big) \;=\; \Big(c_0\,\big(L_0^X - 1\big) + \sum_{n>0} c_{-n}\,L_n^X\Big)\big(\,|\text{phys}\rangle_X \otimes c_1|0\rangle_{\text{gh}}\big) \tag{2.159}$$

where the 'tachyon shift' $a{=}1$ is seen to arise via $c_0 L_0^{\text{gh}}\,c_1|0\rangle_{\text{gh}} = -c_0 c_1|0\rangle_{\text{gh}}$ from the 'energy' difference of $|0\rangle$ and $|\Omega\rangle$, on which the physical state representatives are built. Moreover, the famous null states at $D{=}26$ appear as BRST-exact states,

$$\begin{aligned} L_{-1}^X|\chi_1\rangle \;&=\; Q\,b_{-1}|\chi_1\rangle \quad , \\ \big(L_{-2}^X + \tfrac{3}{2}L_{-1}^X L_{-1}^X\big)|\chi_2\rangle \;&=\; Q\big(b_{-2} + \tfrac{3}{2}L_{-1}^X b_{-1}\big)|\chi_2\rangle \quad . \end{aligned} \tag{2.160}$$

Finally, let us take a look at the open-string spectrum in this framework. Abbreviate $|k\rangle := |k\rangle_X \otimes c_1|0\rangle_{\text{gh}}$. At level zero,

$$Q\,|k\rangle \;=\; \big(\alpha' k^2 - 1\big)c_0\,|k\rangle \;=\; 0 \tag{2.161}$$

only determines the tachyon mass. Q-exact states cannot exist. At level one, the general state is

$$|\varepsilon,\beta,\gamma,k\rangle \;:=\; \big(\varepsilon\cdot\alpha_{-1} + \beta\,b_{-1} + \gamma\,c_{-1}\big)|k\rangle$$

with a norm-squared of $\varepsilon^*{\cdot}\varepsilon + \beta^*\gamma + \gamma^*\beta$, spanning a $D{+}2$ dimensional space with two timelike directions. Like ε represents a massless vector field, β and γ are associated with the two Faddeev-Popov ghosts of quantum gauge theory.

Exercise 2.19 *Use*

$$Q = c_{-1}(\alpha_0{\cdot}\alpha_1) + c_0(\tfrac{1}{2}\alpha_0^2 + \alpha_{-1}{\cdot}\alpha_1) + c_1(\alpha_{-1}{\cdot}\alpha_0) - c_{-1}c_0 b_1 - b_{-1}c_0 c_1 + \textit{irrelevant}$$

on $|\varepsilon,\beta,\gamma,k\rangle$ *to prove that physical states are subject to* $k^2 = 0$*,* $\varepsilon\cdot k = 0$*, and* $\beta = 0$*. The physical state space is* D *dimensional including two null directions.*

Exercise 2.20 *Work out the general null state*

$$Q\,\big(\varepsilon'\cdot\alpha_{-1} + \beta'\,b_{-1} + \gamma'\,c_{-1}\big)|k\rangle \;=\; \ldots$$

to show that the BRST-exact terms have $\gamma =$ *anything and* $\varepsilon \propto k$*. Modding out the null states thus yields the positive definite* $D{-}2$ *dimensional Hilbert space of transversal photons.*

This mechanism is general. The BRST cohomology removes from the Fock space quartets of two pairs of unphysical and null states. Schematically,

$$Q: \qquad b \;\longrightarrow\; \alpha^k \;\longrightarrow\; 0 \qquad \text{and} \qquad \alpha^{\bar k} \;\longrightarrow\; c \;\longrightarrow\; 0 \tag{2.162}$$

where $\alpha^{\bar k}$ and α^k denote oscillators in an unphysical and null direction, respectively. Of course, the positive definiteness of the resulting cohomology space is not guaranteed and has to be checked by other means. Not surprisingly, the no-ghost theorem may also be proved quite elegantly in the ghost-extended Fock space.

3 String Interactions

3.1 First-Quantized Description

So far we have been discussing the free string. In this section we introduce interactions into the theory of strings. To formulate interactions between quantum objects, a multi-object description seems necessary, in particular for relativistic systems. Indeed, the multi-particle Fock space is the appropriate arena for interacting relativistic particles. Such a quantum field theoretical ("second-quantized") framework does exist for strings as well, but it is beset with enormous technical difficulties and in a satisfactory stage presently only for the open bosonic string. Therefore, we refrain from introducing multi-string states here, but employ a single-string ("first-quantized") formulation for the emission and absorption of strings. Despite a certain conceptual awkwardness this method is relatively simple and easy to apply, once the so-called vertex operators are in place.

Vertex Operators

Multi-string state space. As for point particles, the multi-string Fock space

$$\mathcal{F} \;=\; \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \dots \tag{3.1}$$

decomposes into sectors of definite string number, with n-string product states labelled as $|A_1,\dots,A_n\rangle \in \mathcal{F}_n$. In particular, we have the zero-string state (vacuum), one-string states and two-string product states

$$|\;\;\rangle \in \mathcal{F}_0 \quad , \qquad |A\rangle \in \mathcal{F}_1 \quad , \qquad |A,B\rangle = |A\rangle \otimes |B\rangle \in \mathcal{F}_2 \quad , \tag{3.2}$$

respectively. The number of strings in a state can be altered by applying a string field operator,

$$\Psi_A \;:\quad \mathcal{F}_n \;\to\; \mathcal{F}_{n-1} \oplus \mathcal{F}_{n+1} \quad , \qquad \text{in particular} \quad \Psi_A|\;\;\rangle \;=\; |A\rangle \quad . \tag{3.3}$$

Each n-string sector $\mathcal{F}_n$ contains a ground state. In $\mathcal{F}_1$ this is simply the vacuum $|0\rangle = |0\rangle_X \otimes |0\rangle_{\text{gh}}$ of the oscillator and momentum modes of a single string, not to be confused with the zero-string vacuum $|\;\;\rangle$!

Operator-state correspondence. Any one-string state $|A\rangle$ can be obtained by applying a particular operator V_A to the ground state,

$$V_A\,|0\rangle \;=\; |A\rangle \quad . \tag{3.4}$$

This operator is called the "vertex operator" of the state $|A\rangle$ and establishes a one-to-one correspondence of operators and states in $\mathcal{F}_1$. For example, the open-string tachyon state (with $X \equiv X_{\text{open}}$ and $\alpha' k^2 = 1$) is

$$|T(k)\rangle \;:=\; |k\rangle_X \otimes c_1|0\rangle_{\text{gh}} \;=\; \lim_{\tau\to-\infty(1-\mathrm{i}0)} \mathrm{e}^{\mathrm{i}\tau} :c\,\mathrm{e}^{\mathrm{i}k\cdot X}:|0\rangle \;=\; V_T(k)\,|0\rangle \quad . \tag{3.5}$$

More generally, at mass level N (with $\alpha' k^2 = 1-N$) the correspondence takes the form

$$(\varepsilon_\mu \alpha^\mu_{-N} + \ldots + \beta_{\mu_1 \ldots \mu_N} \alpha^{\mu_1}_{-1} \cdots \alpha^{\mu_N}_{-1}) |k\rangle_X \otimes c_1 |0\rangle_{\mathrm{gh}} \quad \Longleftrightarrow \quad : c\, \mathcal{P}_N(\dot{X}, \ddot{X}, \ldots)\, \mathrm{e}^{\mathrm{i}k\cdot X} : \tag{3.6}$$

where $\mathcal{P}_N$ denotes a polynomial homogeneous of degree N in the number of τ derivatives.

Exercise 3.1 *Find the polynomials for the mass levels N=1 and N=2.*

The worldsheet of the free open string has the topology of a strip, with a global 'time' coordinate τ. The large-τ limit in (3.5) signifies that the our ket states are created by the action of a vertex operator in the far past. The conjugation

$$\mathcal{I} \;:\quad \mathcal{F}_1 \;\to\; \mathcal{F}_1^* \qquad \text{via} \qquad |A\rangle \;\mapsto\; \langle A| \tag{3.7}$$

based on our scalar product then relates the corresponding bra state to the hermitian conjugated vertex operator acting in the far future,

$$\mathcal{I} \;:\quad V_A(k) \;\mapsto\; V_A^\dagger(k) \;=\; V_A(-k)\, c_0 \quad , \tag{3.8}$$

since the exponential in (3.5) changes sign. If we agree upon reading all momenta as incoming, the dagger may be dropped:

$$|A\rangle \;=\; V_A(k;\tau{=}{-\infty})\,|0\rangle \qquad \text{and} \qquad \langle A| \;=\; \langle 0|\, V_A(k;\tau{=}\infty)\, c_0 \quad . \tag{3.9}$$

Note that in $\mathcal{F}_1$ physical kets carry ghost number $u=1$ while physical bras have $u=2$, to produce $u=3$ for the scalar product.

String fusion. While successive application of string field operators creates multi-string states, e.g.

$$\Psi_B\, \Psi_A\, |\;\;\rangle \;=\; \Psi_B\, |A\rangle \;=\; f(A,B)\, |\;\;\rangle \;+\; |A,B\rangle \quad , \tag{3.10}$$

the iteration of vertex operators produces fused one-string states:

$$V_B\, V_A\, |0\rangle \;=\; V_B\, |A\rangle \;=:\; c_0\, |A{+}B\rangle \;\in\; \mathcal{F}_1 \quad . \tag{3.11}$$

The relation between the two operations defines the 3-string vertex

$$\Upsilon \;:\quad \mathcal{F}_1 \otimes \mathcal{F}_1 \;\to\; \mathcal{F}_1^* \qquad \text{via} \qquad |A,B\rangle \;\mapsto\; \langle A{+}B| \quad , \tag{3.12}$$

which extends to

$$\Upsilon \;:\quad \mathcal{F}_1 \otimes \mathcal{F}_1 \otimes \mathcal{F}_1 \;\to\; \mathbb{C} \qquad \text{via} \qquad |A,B,C\rangle \;\mapsto\; \langle A{+}B|C\rangle \quad . \tag{3.13}$$

Likewise, the conjugation provides a 2-string vertex (with u=1),

$$\mathcal{I} \;:\quad \mathcal{F}_1 \otimes \mathcal{F}_1 \;\to\; \mathbb{C} \qquad \text{via} \qquad |A,B\rangle \;\mapsto\; \langle A|B\rangle \quad . \tag{3.14}$$

Sometimes the notation $\mathcal{I} = \langle\langle V_2|$ and $\Upsilon = \langle\langle\langle V_3|$ is used. Fusion introduces a geometric 3-string interaction in $\mathcal{F}_1$. It gives the vertex operators an algebraic structure with structure constants $f_{ABC} = \Upsilon|A,B,C\rangle$.

Open-String Tree-Level Correlations.

Multi-String Process. Another form of the 2- and 3-string vertices is

$$\begin{aligned} \mathcal{I}\,|A,B\rangle \;&=\; \langle 0|\,V_A(\infty)\,c_0\,V_B(-\infty)\,|0\rangle \qquad \text{and} \\ \Upsilon\,|A,B,C\rangle \;&=\; \langle 0|\,V_A(\infty)\,V_B(\infty)\,V_C(-\infty)\,|0\rangle \quad . \end{aligned} \tag{3.15}$$

Higher-order interactions, such as a process of elastic 2-string scattering, require sewing 3-string vertices with internal string propagators. To this end, two-string fusion must be considered at finite values of τ, which is achieved by evolving the physical vertex operators via

$$V_A(\tau,\sigma) \;=\; \mathrm{e}^{\mathrm{i}(\tau-\tau_0)H}\,V_A(\tau_0,\sigma)\,\mathrm{e}^{-\mathrm{i}(\tau-\tau_0)H} \qquad \text{with} \quad H \;=\; L_0^{\text{tot}} \;=\; L_0^X-1 \tag{3.16}$$

away from their asymptotic reference times $\tau_0=\pm\infty$. Practically, this amounts simply to dropping the limit taken in (3.5). As for point particles, the computation of amplitudes for string processes requires an integration over the interaction times. Open-string interactions occur only at the worldsheet boundaries, whence we must evaluate its vertex operators at $\sigma=0$ or at $\sigma=\pi$. We shall see that for tree diagrams we can restrict ourselves to $\sigma=0$. The string propagation between the interaction events is already described by the implicit factors of $\mathrm{e}^{-\mathrm{i}\Delta\tau(L_0^X-1)}$, which become explicit when passing from the Heisenberg to the Schrödinger picture. (External legs are being amputated.) Indeed, $\mathrm{i}\int_0^\infty \mathrm{d}\tau\;\mathrm{e}^{-\mathrm{i}\tau(L_0^X-1)}=(L_0^X-1)^{-1}$ is the relevant Greens function.

There is also the issue of ghost number counting. It is plausible that inverting the 2-string vertex in (3.15) enforces an extra antighost insertion for every internal propagator. Employing the machinery of string field theory, one may demonstrate that these insertions finally amount to removing the ghost factors of all but three vertex operators in the correlation function. (Note that an r-string tree diagram features precisely $r{-}3$ internal propagators.) Hence, the building blocks for string scattering amplitudes are the correlation functions of r vertex operators,

$$\langle 0|\,cV_1^X(\tau_1)\,cV_2^X(\tau_2)\,V_3^X(\tau_3)\,V_4^X(\tau_4)\ldots V_{r-1}^X(\tau_{r-1})\,cV_r^X(\tau_r)\,|0\rangle \quad , \tag{3.17}$$

where the times are ordered as $\tau_1>\tau_2>\cdots>\tau_r$ and

$$V_i^X(\tau_i) \;=\; \mathrm{e}^{(1-N_i)\mathrm{i}\tau}\;{:}\,\mathcal{P}_{N_i}(\partial_\tau^* X)\,\mathrm{e}^{\mathrm{i}k_i\cdot X}\,{:}\,(\tau_i,0) \quad . \tag{3.18}$$

Wick rotation and Koba-Nielsen factor. For technical reasons in the computation of the amplitudes it is convenient to extend the range of τ to the complex plane, perform a Wick rotation

$$\tau \;=\; -\mathrm{i}\,t \qquad \text{and define} \qquad y \;=\; \mathrm{e}^{\mathrm{i}\tau} \;=\; \mathrm{e}^{t}\;\in[0,\infty] \quad . \tag{3.19}$$

In this variable, (2.44) for Neumann boundary conditions at $\sigma{=}0$ yields

$${:}\,\mathrm{e}^{\mathrm{i}k\cdot X}\,{:}\,(y) \;=\; \mathrm{e}^{\sqrt{2\alpha'}\sum_{n>0}\frac{1}{n}k\cdot\alpha_{-n}y^n}\;\mathrm{e}^{\mathrm{i}k\cdot q}\,y^{2\alpha' k\cdot p}\,\mathrm{e}^{-\sqrt{2\alpha'}\sum_{n>0}\frac{1}{n}k\cdot\alpha_n y^{-n}} \quad . \tag{3.20}$$

Standard coherent-state techniques produce (for $y_1 > y_2$)

$$
\begin{aligned}
& {}_X\langle 0|\ :\mathrm{e}^{\mathrm{i}k_1\cdot X}:(y_1)\ \ :\mathrm{e}^{\mathrm{i}k_2\cdot X}:(y_2)\,|0\rangle_X \\
&= {}_X\langle -k_1|\, y_1^{2\alpha' k_1\cdot p}\, \mathrm{e}^{\mathrm{i}k_2\cdot q}\, \mathrm{e}^{-\sqrt{2\alpha'}\sum_{n>0}\frac{1}{n}k_1\cdot\alpha_n y_1^{-n}}\, \mathrm{e}^{\sqrt{2\alpha'}\sum_{m>0}\frac{1}{m}k_2\cdot\alpha_{-m}y_2^{m}}\,|0\rangle_X \\
&= {}_X\langle -k_1|k_2\rangle_X\ y_1^{2\alpha' k_1\cdot k_2}\, \mathrm{e}^{-2\alpha' k_1\cdot k_2\sum_{n>0}\frac{1}{n}(\frac{y_2}{y_1})^n} \\
&= {}_X\langle -k_1|k_2\rangle_X\ \mathrm{e}^{2\alpha' k_1\cdot k_2\ln(y_1-y_2)}\ =\ {}_X\langle 0|k_1{+}k_2\rangle_X\ (y_1-y_2)^{2\alpha' k_1\cdot k_2} \quad .
\end{aligned}
\tag{3.21}
$$

More generally, for the r-tachyon correlator ($\alpha' k_i^2 = 1$) one finds

$$
{}_X\langle 0|\prod_{i=1}^{r} :\mathrm{e}^{\mathrm{i}k_i\cdot X}:(y_i)\,|0\rangle_X \ =\ \delta^{(26)}(k_1{+}\ldots{+}k_r)\ \prod_{i<j}(y_i-y_j)^{2\alpha' k_i\cdot k_j} \tag{3.22}
$$

with $y_i > y_{i+1}$. This expression occurs in all vertex-operator correlation functions and is known as the Koba-Nielsen factor. Its most remarkable property is an invariance under simultaneous cyclic permutations of the y_i and k_i, because pushing the first vertex operator all the way to the right produces a factor of $\prod_2^r(-1)^{2\alpha' k_1\cdot k_i} = (-1)^{-2\alpha' k_1^2} = 1$. For higher-level states, one must in addition take into account contributions from the corresponding polynomials in $\mathrm{e}^{-*\mathrm{i}\tau}\partial_\tau^* X \sim \partial_y^* X$. Finally, we need the ghost correlator,

$$
{}_{\mathrm{gh}}\langle 0|\,c(y_1)\,c(y_2)\,c(y_r)\,|0\rangle_{\mathrm{gh}} \ =\ y_1^{-1}y_2^{-1}y_r^{-1}\,(y_1-y_2)\,(y_1-y_r)\,(y_2-y_r) \quad , \tag{3.23}
$$

by virtue of $c(y) = \sum_n c_n y^{-n}$ and ${}_{\mathrm{gh}}\langle 0|c_{-1}c_0c_1|0\rangle_{\mathrm{gh}} = 1$ as the only nonvanishing contribution (up to permutations). Therefore, the full primitive r-tachyon amplitude is given by the product of (3.22) and (3.23), to be integrated with $\mathrm{i}\int\mathrm{d}\tau_i\, y_i \ldots = \int \mathrm{d}y_i \ldots$ for all i. After summation over all external leg permutations involving identical particles one obtains the complete scattering amplitude.

Exercise 3.2 *Verify the relations* (3.22) *and* (3.23).

Möbius invariance. Recall that the open-string Fock space carries an action of the sl(2,$\mathbb{R}$) subalgebra of the Virasoro algebra. The corresponding group action, the so-called Möbius transformation, leaves the ground state $|0\rangle$ invariant and transforms the vertex operators tensorially, while shifting their argument as

$$
y \ \mapsto\ \frac{ay+b}{cy+d} \qquad \text{with} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \quad . \tag{3.24}
$$

This action extends from the $\sigma{=}0$ boundary to the whole parameter space, by replacing y with

$$
z \ =\ \mathrm{e}^{\mathrm{i}(\tau+\sigma)} \ =\ \mathrm{e}^{t+\mathrm{i}\sigma} \quad , \tag{3.25}
$$

which lives in the upper half plane. Note that that $\sigma{=}\pi$ boundary is included as the negative real axis ($y < 0$). In this variable, constant-t slices are semi-circles around the origin $z{=}0 \leftrightarrow t{=}{-}\infty$. Via $z \mapsto \frac{z-\mathrm{i}}{z+\mathrm{i}}$ the upper half plane is mapped conformally to the unit disk, showing that the worldsheet boundary has a single component and hinting at the cyclic invariance of the amplitude. The Möbius group (or projective group) SL(2,$\mathbb{R}$) comprises all singularity-free conformal maps of this tree-level open-string worldsheet onto itself. It captures the residual gauge invariance in the conformal gauge, which is still to be fixed.

Since the correlation functions are SL(2,$\mathbb{R}$) scalars, they are inert under the three-parameter family of Möbius transformations. Hence, the integration over all y_i will generate the volume of SL(2,$\mathbb{R}$) as an infinite factor, unless we restrict the integration to one representative per Möbius orbit. This is achieved by fixing the location of three vertex operators and dropping the corresponding τ integrations. The traditional choice is

$$y_1 \;=\; \infty \quad , \qquad y_2 \;=\; 1 \quad , \qquad y_r \;=\; 0 \quad , \tag{3.26}$$

in tune with our choice for the remaining ghost locations. Note that the remaining factor of $y_1 y_2 y_r$ is cancelled by a part of (3.23). Putting all together, we are to perform an integration $\prod_{i=3}^{r-1}\int_0^1 \mathrm{d}y_i$ with $1 > y_3 > y_4 > \ldots > y_{r-1} > 0$ of

$$y_{12}y_{1r}y_{2r}\prod_{i<j} y_{ij}^{2\alpha' k_i\cdot k_j} \;=\; \prod_{\ell}\big[(1{-}y_\ell)^{2\alpha' k_2\cdot k_\ell}\, y_\ell^{2\alpha' k_\ell\cdot k_r}\big]\prod_{m<n} y_{mn}^{2\alpha' k_m\cdot k_n} \quad , \tag{3.27}$$

where ℓ, m and n run from 3 to $r{-}1$ only, and we introduced the abbreviation $y_{ij} = y_i{-}y_j$. It is noteworthy that the potential divergence

$$y_1^{2+2\alpha'\sum_{j=2}^{r} k_1\cdot k_j} \;=\; y_1^{2-2\alpha' k_1^2} \;=\; y_1^0 \tag{3.28}$$

is tamed by the momentum conservation $\sum_i k_i = 0$ and the mass-shell condition $\alpha' k_i^2 = 1$.

Leg permutations. Although it is no longer obvious, the integrated expression (3.27) is still invariant under cyclic permutations of the momenta. The sum over permutations of the external legs of identical particles therefore reduces to a sum over *cyclicly inequivalent* orderings $\{\pi\}$ only,

$$T^{\mathrm{tree}}(12\ldots r) \;=\; \sum_{\{\pi\}} A^{\mathrm{tree}}(\pi_1\pi_2\ldots\pi_r) \quad , \tag{3.29}$$

where A denotes the primitive amplitude for a fixed leg ordering and T is the T-matrix element. We have abbreviated the external momentum arguments by the position labels. Amplitudes containing higher-level states are treated likewise and yield additional momentum factors as well as shifts of the exponents due to the modified mass-shell conditions. Finally, for the complete T-matrix element we should weigh the r-string tree-level amplitude with a factor of g^{r-2} with g being the strength of the open-string coupling. In the following we suppress the momentum delta functions.

3.2 Tree Amplitudes

Tachyons and Photons

Three-point. The simplest case is $r=3$, where the kinematics enforces $k_1{\cdot}k_2 = m_1^2+m_2^2-m_3^2$ plus cyclic permutations. For three tachyons, $\alpha' k_i\cdot k_j=-1$ and the full correlator reduces to unity, hence

$$A^{\rm tree}_{TTT}(123) \;=\; g \qquad\longrightarrow\qquad T^{\rm tree}_{TTT} \;=\; A^{\rm tree}_{TTT}(123)+A^{\rm tree}_{TTT}(213) \;=\; 2g \quad . \tag{3.30}$$

Exercise 3.3 *Compute $A^{\rm tree}_{TTT}$ for a generic choice of y_1, y_2 and y_3.*

The case of three photons is more interesting. With the photon vertex operator

$$V^X_\gamma(\varepsilon,k;y) \;=\; y\;{:}\varepsilon\cdot\partial_y X\;{\rm e}^{{\rm i}k\cdot X}{:}\,(y) \qquad \text{for}\quad \varepsilon\cdot k = k\cdot k = 0 \quad , \tag{3.31}$$

one straightforwardly finds the cyclicly symmetric expression

$$A^{\rm tree}_{\gamma\gamma\gamma} \;=\; {\rm i}g(2\alpha')^2\left(\varepsilon_1{\cdot}k_2\,\varepsilon_2{\cdot}\varepsilon_3+\varepsilon_2{\cdot}k_3\,\varepsilon_3{\cdot}\varepsilon_1+\varepsilon_3{\cdot}k_1\,\varepsilon_1{\cdot}\varepsilon_2+2\alpha'\,\varepsilon_1{\cdot}k_2\,\varepsilon_2{\cdot}k_3\,\varepsilon_3{\cdot}k_1\right) \quad . \tag{3.32}$$

Using momentum conservation and transversality of the polarization, we see that $A^{\rm tree}_{\gamma\gamma\gamma}$ is totally *anti*symmetric under external leg permutations. Photons being bosons, the permutation sum makes $T^{\rm tree}_{\gamma\gamma\gamma}$ vanish, in agreement with QED. The nonabelian generalization to the (non-vanishing) 3-gluon coupling, however, requires the implementation of the color charges into the open string, which will be given shortly. Of course, there are also mixed amplitudes $A^{\rm tree}_{TT\gamma}$ and $A^{\rm tree}_{T\gamma\gamma}$ at this mass level.

Exercise 3.4 *Amplitudes including photons are economically computed by means of the following trick: Write the vertex operator as*

$$V^X_\gamma(\varepsilon,k;y) \;=\; y\;{:}{\rm e}^{{\rm i}k\cdot X+\varepsilon\cdot\partial_y X}{:}\,(y)\,\Big|_{\textit{linear in }\varepsilon} \tag{3.33}$$

and verify that

$${}_X\langle 0|\prod_{i=1}^r {:}{\rm e}^{{\rm i}k_i\cdot X+\varepsilon_i\cdot\partial_y X}{:}\,(y_i)\,|0\rangle_X \;\sim\; \prod_{i<j}{\rm e}^{2\alpha'\left(k_i\cdot k_j\ln y_{ij}-{\rm i}\frac{\varepsilon_i\cdot k_j-k_i\cdot\varepsilon_j}{y_{ij}}-\frac{\varepsilon_i\cdot\varepsilon_j}{y_{ij}^2}\right)} \quad . \tag{3.34}$$

Apply this formula to confirm (3.32).

Four-point: Veneziano amplitude. Kinematical phase space first occurs for $r=4$. Traditionally, one introduces the Mandelstam variables

$$\begin{aligned} s &= -(k_1{+}k_2)^2 = -(k_3{+}k_4)^2 = m_1^2+m_2^2-2k_1{\cdot}k_2 = m_3^2+m_4^2-2k_3{\cdot}k_4 \quad ,\\ t &= -(k_2{+}k_3)^2 = -(k_1{+}k_4)^2 = m_2^2+m_3^2-2k_2{\cdot}k_3 = m_1^2+m_4^2-2k_1{\cdot}k_4 \quad ,\\ u &= -(k_1{+}k_3)^2 = -(k_2{+}k_4)^2 = m_1^2+m_3^2-2k_1{\cdot}k_3 = m_2^2+m_4^2-2k_2{\cdot}k_4 \quad , \end{aligned} \tag{3.35}$$

which are related by $s+t+u=\sum_i m_i^2$. We also define the "Regge trajectory"

$$\alpha(x) \;=\; \alpha'\,x/\beta^2 \;+\; \alpha_0 \qquad \text{with} \quad \alpha_0 = 1 \quad . \tag{3.36}$$

The computation of the (primitive) four-tachyon amplitude is a historical highlight. Specializing (3.27) and writing $y_3 \equiv x$, we obtain ($\alpha' m_i^2 = -1$)

$$\begin{aligned} A^{\text{tree}}_{TTTT}(1234) \;&=\; g^2 \int_0^1 \mathrm{d}x\; (1{-}x)^{2\alpha' k_2\cdot k_3}\; x^{2\alpha' k_3\cdot k_4} \\ &=\; g^2 \int_0^1 \mathrm{d}x\; x^{-\alpha(s)-1}\; (1{-}x)^{-\alpha(t)-1} \\ &=\; g^2\, \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \;=\; g^2\, \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(1+\alpha(u))} \quad , \end{aligned} \tag{3.37}$$

where the ratio of Euler gamma functions is also known as the Euler beta function. Although the integral diverges for $\alpha' s \geq -1$ or $\alpha' t \geq -1$, its analytic continuation features only poles at $\alpha' s = n{-}1$ and $\alpha' t = n{-}1$ for $n \in \mathbb{N}_0$.

Exercise 3.5 *Compute A^{tree}_{TTTT} for a generic choice of $y_1, \ldots, y_4$.*

Channel duality. This so-called Veneziano amplitude has wondrous properties: First, it is symmetric under the $s \leftrightarrow t$ interchange, which is equivalent to the cyclic permutation

$$(k_1, k_2, k_3, k_4) \;\mapsto\; (k_2, k_3, k_4, k_1) \quad . \tag{3.38}$$

This property is known as duality (in the s and t scattering channels). It implies that, in contrast to point-particle scattering, one does not have to add an s-channel string diagram and a t-channel string diagram but needs only *one* of these since either one captures both! In fact, this is intuitively clear from the representation of the worldsheet as a disk with four marked boundary points and fits in with the observation that the string joining and splitting events in spacetime are not Lorentz-invariant. Quite generally, each string diagram combines many Feynman graphs and degenerates to their sum in the point-particle limit $\alpha' \to 0$.

Second and related, the poles due to internal string exchange occur identically in the s and t channels: The poles of the gamma functions in the numerator of (3.37) are located at nonnegative integer values of $\alpha(s)$ and $\alpha(t)$,

$$\begin{aligned} A^{\text{tree}}_{TTTT}(s,t) \;&=\; -g^2 \sum_{n=0}^{\infty} \frac{(\alpha(t){+}1)(\alpha(t){+}2)\cdots(\alpha(t){+}n)}{n!}\; \frac{1}{\alpha(s)-n} \\ &=\; -g^2 \sum_{n=0}^{\infty} \frac{(\alpha(s){+}1)(\alpha(s){+}2)\cdots(\alpha(s){+}n)}{n!}\; \frac{1}{\alpha(t)-n} \quad , \end{aligned} \tag{3.39}$$

which perfectly agrees with the open-string mass spectrum reemerging when an internal propagator goes on mass-shell, since

$$
{}_X\langle\mathrm{phys},k|\tfrac{1}{L_0^X-1}|\mathrm{phys},k\rangle_X \quad \text{blows up at} \quad \alpha' k^2 = 1-n \quad \text{for} \quad n\in\mathbb{N}_0 \quad . \tag{3.40}
$$

It is remarkable that channel duality requires an infinite number of physical poles (or none at all) and hence cannot be realized with a finite particle spectrum.

High-energy limits. Third, the high-energy behavior of the Veneziano amplitude is more benign than that of any point-particle scattering. For the process $1+2\to3+4$ one has

$$
s \;=\; E^2 \quad , \qquad t \;=\; (4m^2-E^2)\sin^2\tfrac{\theta}{2} \quad , \qquad u \;=\; (4m^2-E^2)\cos^2\tfrac{\theta}{2} \tag{3.41}
$$

for the energy E and the scattering angle θ in the 1-2-center-of-mass frame, which for large s simplifies to $-\frac{2t}{s}=1-\cos\theta$. The so-called *Regge limit* corresponds to large s at fixed negative t, which is high-energy elastic forward scattering. Regulating by a small imaginary value for s and using Stirling's formula, one gets the Regge behavior

$$
A^{\mathrm{tree}}_{TTTT}(s{\to}\infty,t) \;\sim\; \Gamma(-\alpha(t))\,s^{\alpha(t)} \;\sim\; s^{\alpha' t+1} \quad . \tag{3.42}
$$

For $\alpha' t<-1$ the amplitude has a power-law fall-off at large s. In point-particle scattering, the t-channel exchange of a spin-j particle contributes a term $\sim s^j$ to the amplitude. Therefore, we may interpret (3.42) as the t-channel exchange of a ficticious particle with t-dependent effective spin $j=\alpha(t)$, summing up infinitely many t-channel exchanges of arbitrary integer spins at high center-of-mass energy s. The string excitations are situated on the Regge trajectory $j=\alpha(t)$ at the integer values $j=n$, i.e. $\alpha' t=n{-}1$. The so-called Regge slope α' determines the string tension via $T=\frac{1}{2\pi\alpha'}$. Alternatively, one may consider the *hard scattering limit* $s\to\infty$ at fixed θ (or t/s), which yields

$$
A^{\mathrm{tree}}_{TTTT}(s{\to}\infty,\theta) \;\sim\; \mathrm{e}^{-\alpha'(s\ln s+t\ln t+u\ln u)} \;\sim\; |f(\theta)|^{-\alpha(s)} \tag{3.43}
$$

with a specific function f, i.e. exponential fall-off for fixed-angle scattering at high energies. Deep-inelastic scattering probes the structure of the scattered objects. A soft behavior like in (3.43) suggests a smooth object of size $\sqrt{\alpha'}$.

Complete amplitude. The final sum over cyclicly inequivalent permutations,

$$
\begin{aligned}
T^{\mathrm{tree}}_{TTTT} \;&=\; A^{\mathrm{tree}}_{TTTT}(1234)+A^{\mathrm{tree}}_{TTTT}(1243)+A^{\mathrm{tree}}_{TTTT}(1324) \\
&\quad +A^{\mathrm{tree}}_{TTTT}(1342)+A^{\mathrm{tree}}_{TTTT}(1423)+A^{\mathrm{tree}}_{TTTT}(1432) \\
&=\; 2g^2\left(A^{\mathrm{tree}}_{TTTT}(s,t)\;+\;A^{\mathrm{tree}}_{TTTT}(t,u)\;+\;A^{\mathrm{tree}}_{TTTT}(u,s)\right) \quad ,
\end{aligned} \tag{3.44}
$$

produces a totally symmetric expression with identical poles in the s, t and u channels. The high-energy behavior of T^{tree}_{TTTT} is inherited from that of A^{tree}_{TTTT}.

It should be clear now how to evaluate four-point amplitudes including non-tachyonic legs and how to proceed to $r > 4$. The resulting expressions become increalingly complicated but share the nice properties of the Veneziano amplitude (with appropriate obvious adjustments).

Gluons

String theory could not claim to be a unified theory of all interactions if it did not contain nonabelian gauge bosons (gluons) among its excitations. Since gluons are self-interacting massless vector particles carrying color degrees of freedom, we must look for a generalization of the open string which admits such a structure at least for its massless excitations.

Chan-Paton charges. String theory is a very rigid framework which usually does not admit any tinkering without loosing consistency. Fortunately, the distinguished end points of open strings allow for an attachment of so-called Chan-Paton degrees of freedom, which then reside only on the worldsheet boundary and do not propagate.[4] Having trivial worldsheet dynamics, these do not enter the energy-momentum tensor and thus do not affect our previous considerations, but they alter the interactions and hence the spacetime dynamics.

Charging each end of the string with n such degrees of freedom enhances the Fock space of string states by a tensor factor of $\mathbb{C}^{n^2}$, and we may label open-string basis states (with suggestive notation) as

$$\alpha^*_< \, b^*_< \, c^*_< \, |k; ij\rangle \qquad \text{for} \quad i, j = 1, \ldots, n \quad , \tag{3.45}$$

where i and j count the 'colors' of the left and right endpoints, respectively. Since the Chan-Paton labels have no time evolution they remain constant along the worldsheet boundary, except for the interaction points, where vertex operators V^{ij} are inserted. Hence, in open-string fusion the right label of the left string must match with the left label of the right string. Employing a $\mathbb{C}$-complete set of n^2 hermitian matrices $\lambda^a = (\lambda^a_{ij})$, normalized to $\operatorname{tr} \lambda^a \lambda^b = \delta^{ab}$, we reorganize the basis into

$$|*, k; a\rangle \;=\; \sum_{i,j=1}^{n} |*, k; ij\rangle \; \lambda^a_{ij} \qquad \text{for} \quad a = 1, \ldots, n^2 \quad . \tag{3.46}$$

Color factors and symmetry. A vertex operator V^a creating an open-string state in a color state a just couples the boundaries i and j to a via λ^a_{ij}. For the

[4] This modification seems ad hoc but Chan-Paton charges are natural as D-brane labels.

r-string correlator we find

$$\begin{aligned}
\langle 0|\, V_1^{a_1}\, V_2^{a_2} \cdots V_r^{a_r}\, |0\rangle &= \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots,j_r}} \lambda^{a_1}_{i_1 j_1} \lambda^{a_2}_{i_2 j_2} \cdots \lambda^{a_r}_{i_r j_r}\ \langle 0|\, V_1^{i_1 j_1}\, V_2^{i_2 j_2} \cdots V_r^{i_r j_r}\, |0\rangle \\
&= \sum_{i_1,\ldots,i_r} \lambda^{a_1}_{i_1 i_2} \lambda^{a_2}_{i_2 i_3} \cdots \lambda^{a_r}_{i_r i_1}\ \langle 0|\, V_1\, V_2 \cdots V_r\, |0\rangle \qquad (3.47) \\
&= \mathrm{tr}\,(\lambda^{a_1}\, \lambda^{a_2} \cdots \lambda^{a_r})\ \langle 0|\, V_1\, V_2 \cdots V_r\, |0\rangle \quad ,
\end{aligned}$$

because V^{ij} differs from the colorless V only by providing the Kronecker deltas $\delta_{j_s i_{s+1}}$ implementing the sewing conditions. Thus, the complete amplitude now reads

$$T^{\mathrm{tree}}(12\ldots r) \;=\; \sum_{\{\pi\}} \mathrm{tr}\,(\lambda^{a_{\pi_1}}\lambda^{a_{\pi_2}} \cdots \lambda^{a_{\pi_r}})\ A^{\mathrm{tree}}(\pi_1\pi_2\ldots\pi_r) \quad , \qquad (3.48)$$

where π runs over the cyclicly inequivalent permutations of identical external particles.

We note that the matrices λ^a form an algebra

$$\lambda^a\, \lambda^b \;=\; \tfrac{\mathrm{i}}{2}\, f^{[abc]}\, \lambda^c \;+\; \tfrac{1}{2}\, d^{(abc)}\, \lambda^c \quad , \qquad (3.49)$$

which contains the u(n) Lie algebra with real structure constants f^{abc}. The color-enhanced spectrum and amplitudes of the open string are obviously invariant under global U(n) transformations acting as $\lambda^a \mapsto U\lambda^a U^\dagger$, under which the states transform in the adjoint representation. The U(1) phase factor acts trivially,[5] and the corresponding gauge boson decouples from all amplitudes, so that we end up with scattering for string states all living in the adjoint representation of SU(n). We shall see that this symmetry is actually a gauge symmetry in the spacetime dynamics.

Three-point. The full scattering amplitude for colored excitations carries an adjoint label for each external leg. Accordingly, complete the three-tachyon amplitude (3.30) gets modified to

$$\begin{aligned}
T^{\mathrm{tree}}_{TTT} &= \mathrm{tr}\,(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3})\, A^{\mathrm{tree}}_{TTT}(123) \;+\; \mathrm{tr}\,(\lambda^{a_2}\lambda^{a_1}\lambda^{a_3})\, A^{\mathrm{tree}}_{TTT}(213) \\
&= \mathrm{tr}\,(\{\lambda^{a_1},\lambda^{a_2}\}\,\lambda^{a_3})\ g \;=\; d^{a_1a_2a_3}\, g \quad .
\end{aligned} \qquad (3.50)$$

For three gluons, we get

$$T^{\mathrm{tree}}_{ggg} \;=\; -f^{a_1a_2a_3}\, g(2\alpha')^2\, (\varepsilon_1{\cdot}k_2\, \varepsilon_2{\cdot}\varepsilon_3 + \varepsilon_2{\cdot}k_3\, \varepsilon_3{\cdot}\varepsilon_1 + \varepsilon_3{\cdot}k_1\, \varepsilon_1{\cdot}\varepsilon_2 + 2\alpha'\, \varepsilon_1{\cdot}k_2\, \varepsilon_2{\cdot}k_3\, \varepsilon_3{\cdot}k_1) \qquad (3.51)$$

which is now totally symmetric. The leading order in α' has precisely the structure of the cubic Yang-Mills coupling for U(n) vector gauge bosons! Thus, g is really the gauge coupling.

[5] in pure Minkowski spacetime background

Four-point. The colored complete Veneziano amplitude takes the form

$$\begin{aligned} T^{\text{tree}}_{TTTT} &= g^2 \left[\text{tr}\left(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4} + \lambda^{a_1}\lambda^{a_4}\lambda^{a_3}\lambda^{a_2}\right) A^{\text{tree}}_{TTTT}(s,t) \right. \\ &+ \text{tr}\left(\lambda^{a_1}\lambda^{a_3}\lambda^{a_2}\lambda^{a_4} + \lambda^{a_1}\lambda^{a_4}\lambda^{a_2}\lambda^{a_3}\right) A^{\text{tree}}_{TTTT}(t,u) \\ &+ \left. \text{tr}\left(\lambda^{a_1}\lambda^{a_2}\lambda^{a_4}\lambda^{a_3} + \lambda^{a_1}\lambda^{a_3}\lambda^{a_4}\lambda^{a_2}\right) A^{\text{tree}}_{TTTT}(u,s) \right] \quad . \end{aligned} \tag{3.52}$$

Factorization has become nontrivial: The residue of the pole at $s = -\frac{1}{\alpha'}$ corresponding to an intermediate on-shell tachyon carries a color factor

$$\text{tr}\left(\{\lambda^{a_1},\lambda^{a_2}\}\{\lambda^{a_3},\lambda^{a_4}\}\right) = \sum_a \text{tr}\left(\{\lambda^{a_1},\lambda^{a_2}\}\lambda^a\right) \text{tr}\left(\{\lambda^{a_3},\lambda^{a_4}\}\lambda^a\right) \quad , \tag{3.53}$$

which factorizes properly due to the completeness of the λ^a.

Exercise 3.6 *Employ the technique of Exercise 3.4 to calculate the four-gluon tree amplitude to leading order in* α'. *Thereby note that* $\alpha' m_i^2 = 0$ *and hence* $2\alpha' k_3 \cdot k_4 = -\alpha(s) + 1$ *etc. for gluons. Identify the constant which remains after subtracting all poles in* s, t *and* u.

Effective action. Since a string in a physical stationary state propagates like the particle associated to the corresponding excitation, interactions of strings in such states can be described by an effective point-particle theory. Although the full effective theory describes an infinite tower of massive interacting particles, we may restrict ourselves to a finite subset, e.g. the tachyonic and massless excitations. An effective quantum field theory for these degrees of freedom must reproduce the string tree amplitudes we computed earlier. Hence, from the knowledge of the string amplitudes we can reconstruct the fundamental vertices of the effective point-particle field theory, i.e. its action.

If we ignore the tachyon (it will be absent in the superstring) and concentrate on the (hermitian u(n)-valued) gluon field A_μ only, the result (3.51) implies that

$$\begin{aligned} S_{\text{eff}}[A_\mu] = \int \mathrm{d}^D x \ \text{tr}\big[&\tfrac{1}{2} A_\mu (\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu) A_\nu + \mathrm{i}g\,[A_\mu, A_\nu]\,\partial^\mu A^\nu \\ &- \tfrac{2}{3}\mathrm{i}g\,\alpha' (\partial^\mu A_\nu - \partial_\nu A^\mu)(\partial^\nu A_\rho - \partial_\rho A^\nu)(\partial^\rho A_\mu - \partial_\mu A^\rho) + O(g^2) \big] \end{aligned} \tag{3.54}$$

where we have disregarded the overall normalization for the string amplitudes.[6] To order $(\alpha')^0$, the $O(g^2)$ term may be inferred from the result of Exercise 3.6 as $\frac{1}{4} g^2 \int \text{tr}\,[A_\mu, A_\nu][A^\mu, A^\nu]$ which precisely completes the first two terms under the trace in (3.54) to

$$-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad \text{with} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu, A_\nu] \quad . \tag{3.55}$$

It is nontrivial that the relative normalization of the three- and four-point string amplitudes reproduces the coupling constant universality, i.e. the proper ratio

[6] With unitarity considerations this can be fixed.

between the three- and four-gluon coupling in the Yang-Mills action. Summarizing, in the *zero-slope limit*, $\alpha' \to 0$, the massless open string dynamics reduces exactly to the nonabelian gauge dynamics of SU(n) gluons. Remarkably, the global Chan-Paton U(n) invariance of the worldsheet theory gave rise to a local gauge invariance in the spacetime dynamics!

Anticipating gauge invariance also for the higher orders in α', we are led to complete the $O(\alpha')$ term in (3.54). Pushing one order further one gets

$$\begin{aligned} S_{\text{eff}}[A_\mu] \;=\; \int\!\mathrm{d}^D x\;\; \mathrm{tr}\Big[& -\tfrac14 F_{\mu\nu}F^{\mu\nu} - \tfrac23 \mathrm{i} g\,\alpha'\,F^\mu{}_\nu\,F^\nu{}_\rho\,F^\rho{}_\mu - \tfrac{\pi^2}{6}g^2\alpha'^2\big(F_{\mu\nu}F^{\nu\rho}F_{\rho\lambda}F^{\lambda\mu} \\ & + 2F_{\mu\nu}F_{\rho\lambda}F^{\nu\rho}F^{\lambda\mu} - (\tfrac12{+}\tfrac{3}{\pi^2})F_{\mu\nu}F^{\mu\nu}F_{\rho\lambda}F^{\rho\lambda} - (\tfrac14{-}\tfrac{3}{\pi^2})F_{\mu\nu}F_{\rho\lambda}F^{\mu\nu}F^{\rho\lambda}\big) \\ & + O(\alpha'^3 g^3 F^5)\Big] \quad . \end{aligned} \tag{3.56}$$

Since the combination $g\alpha' F$ is dimensionless, the remaining part of the α' expansion starts as indicated. This is our first example of string corrections to gauge theories (and gravity). We close by noting that the gauge coupling has dimension $[g] = [L^{\frac{D}{2}-2}]$ and may be scaled out of S_{eff} as an overall factor of g^{-2}.

Unoriented strings

Worldsheet parity. So far our open and closed strings are oriented: the σ parametrization increases from the left to the right end of the open string and determines a direction along the closed string. In the transversal gauge fixing, we were left with constant σ shifts for the closed strings, but did not consider the worldsheet orientation reversal

$$\sigma \;\mapsto\; \beta\pi - \sigma \qquad \text{and} \qquad \tau \;\mapsto\; \tau \quad . \tag{3.57}$$

This transformation is generated by the worldsheet parity or twist operator Ω, which squares to unity and (ignoring the ghosts) acts as

$$\begin{aligned} \Omega\,\alpha_n^\mu\,\Omega \;&=\; (-1)^n\,\alpha_n^\mu \qquad && \text{in the open string} \quad , \\ \Omega\,\alpha_n^\mu\,\Omega \;&=\; \widetilde{\alpha}_n^\mu && \text{in the closed string} \quad . \end{aligned} \tag{3.58}$$

It can be employed to attach open-string vertex operators at the $\sigma{=}\pi$ end via

$$V(\sigma{=}\pi) \;=\; \Omega\,V(\sigma{=}0)\,\Omega \quad . \tag{3.59}$$

We assign the eigenvalue $\Omega = +1$ to the ground states $|k\rangle$ and note that Ω interchanges the left and right end Chan-Paton labels. For a given mass level $N = \widetilde{N} = 1{+}\frac{1}{\beta^2}\alpha' M^2$, this implies

$$\begin{aligned} \Omega\,|*,k;ij\rangle \;&=\; (-1)^N\,|*,k;ji\rangle \qquad && \text{in the open string} \quad , \\ \Omega\,|*,k\rangle \;&=\; |*,k\rangle|_{L\leftrightarrow R} && \text{in the closed string} \quad . \end{aligned} \tag{3.60}$$

In the closed string massless sector, the exchange of left and right movers induces a map $e_{ij} \mapsto e_{ji}$, thus from (2.103) we see that $\Omega = +1$ on the graviton and the scalar (the dilaton) while $\Omega = -1$ on the antisymmetric two-tensor. To construct Ω eigenstates for the open string, we take a basis for the λ^a matrices in which $\frac{1}{2}n(n-1)$ of them are antisymmetric (spanning an so(n) Lie subalgebra), $s^a = -1$, and the remaining $\frac{1}{2}n(n+1)$ ones are symmetric, $s^a = +1$. Then,

$$\Omega\,|*,k;a\rangle \;=\; (-1)^N\, s^a\,|*,k;a\rangle \quad , \tag{3.61}$$

so that the gluon states have eigenvalue $\Omega = -s^a$.

Other gauge groups. With the above ground state assignment, worldsheet parity is multiplicatively conserved in string interactions. Hence, it makes sense to project onto the $\Omega{=}+1$ part of the spectrum, i.e. removing all $\Omega{=}-1$ states and keeping only the $\Omega{=}+1$ excitations. The result is called an *unoriented* (open or closed) string. It still contains the tachyon as its lowest mode. On the massless level, the unoriented closed string lacks the antisymmetric two-tensor excitation, and the unoriented open string keeps the gluons for the SO(n) subgroup of U(n) only. The ficticious quark and antiquark at the two ends transform in the fundamental and antifundamental representation of SO(n), respectively, so that their antisymmetric tensor product yields the adjoint representation as required.

In fact, more orientation-reversing symmetries can be constructed by combining Ω with a U(n) rotation of the Chan-Paton labels. After some analysis, one concludes that for the gluons of an unoriented string the only other possibility besides SO(n) is the symplectic group Sp(n). The same list results from a unitarity consideration, which concludes that a subset of matrices $\{\lambda^a\}$ is consistent only if the combinations

$$\lambda^{a_1}\lambda^{a_2}\cdots\lambda^{a_p} \;-\; (-1)^p\lambda^{a_p}\cdots\lambda^{a_2}\lambda^{a_1} \qquad \forall p \tag{3.62}$$

is also in the subset. For $p = 2$ this condition requires the subset to form a Lie algebra, but for $p > 2$ it is more subtle. A theorem of Wedderburn yields the above list as the only solutions. We remark that unoriented strings do not couple to oriented ones.

At the one-loop level, however, all these gauge interactions become anomalous except for the group SO(32). In the closed strings with supersymmetry, we shall find other ways to implement Yang-Mills symmetry, including exceptional gauge groups. Yet, also in this case quantum anomalies rule out most of them, leaving only SO(32)/$\mathbb{Z}_2$ and $E_8{\times}E_8$.

Closed-String Tree-Level Correlations

Left-right factorization. In a heuristic sense, closed strings may be viewed as a marriage of two open strings, one left- and one right-moving. This fact shows in the construction of closed-string vertex operators,

$$V_{\rm cl}(k;z,\bar z) \;=\; V_R(\tfrac{k}{2};z)\;V_L(\tfrac{k}{2};\bar z) \quad , \tag{3.63}$$

or linear combinations thereof, where now

$$z \;=\; {\rm e}^{{\rm i}(\tau-\sigma)} \;=\; {\rm e}^{t-{\rm i}\sigma} \;=\; y\,{\rm e}^{-{\rm i}\sigma} \;\in\mathbb{C} \quad , \tag{3.64}$$

and V_R (V_L) is a function of X_R and c^- (X_L and c^+) only (see (2.37)). Ignoring the constraint $w^\mu{=}0$ for a moment, we may regard left- and right-movers including their zero modes as independent, provided we put

$$[q^\mu_L\,,\,p^\nu_L] \;=\; 2{\rm i}\,\eta^{\mu\nu} \;=\; [q^\mu_R\,,\,p^\nu_R] \qquad \text{and others vanish} \quad . \tag{3.65}$$

Then, closed-string correlators factorize into a left- and right-moving part, which depend on the location variables in a meromorphic and anti-meromorphic fashion, respectively. Comparing (2.37) with (2.44) we observe that $X^\mu_R(\tau{-}\sigma)$ has the same quantum properties as $X^\mu_{\rm open}(\tau,\sigma{=}0)|_{\tau\to\tau-\sigma}$, and likewise for $X^\mu_L(\tau{+}\sigma)$. Therefore, we may recycle the open-string correlators computed earlier and write

$$\begin{aligned} &\langle 0|\,V^1_{\rm cl}(k_1;z_1,\bar z_1)\cdots V^r_{\rm cl}(k_r;z_r,\bar z_r)\,|0\rangle \;= \\ &=\; \langle 0|\,V^1_{\rm op}(\tfrac{k_1}{2};z_1)\cdots V^r_{\rm op}(\tfrac{k_r}{2};z_r)\,|0\rangle\;\langle 0|\,\bar V^1_{\rm op}(\tfrac{k_1}{2};\bar z_1)\cdots \bar V^r_{\rm op}(\tfrac{k_r}{2};\bar z_r)\,|0\rangle \end{aligned} \tag{3.66}$$

with obvious notation and without position ordering. Both factors live on the full complex plane.

Beyond the convenient factorization (3.66) of the closed-string correlators it is possible to derive relations between open- and closed-string amplitudes, *after* integrating over $r{-}3$ vertex operator locations, by exploiting identities like $\Gamma(x)\Gamma(1{-}x)\sin(\pi x)=\pi$ [**?**].

Closed-string vertex operators. Recalling the form of closed-string states, the corresponding vertex operators at mass level $N=\widetilde N$ take the form

$$\begin{aligned} V_{\rm cl}(k;z,\bar z) \;&=\; z\bar z\;:c^-\,c^+\,\mathcal{P}_{N,\widetilde N}(\partial^*_z X_{\rm closed},\partial^*_{\bar z}X_{\rm closed})\,{\rm e}^{{\rm i}k\cdot X_{\rm closed}}:(z,\bar z) \\ &=\; \big(z:c^-\,\mathcal{P}_N(\tfrac12\partial^*_z X_R)\,{\rm e}^{\frac{\rm i}{2}k\cdot X_R}:(z)\big)\;\big(\bar z:c^+\,\mathcal{P}_{\widetilde N}(\tfrac12\partial^*_{\bar z} X_L)\,{\rm e}^{\frac{\rm i}{2}k\cdot X_L}:(\bar z)\big) \end{aligned} \tag{3.67}$$

or linear combinations thereof. The graviton, for instance, is generated by

$$V_G(e,k;z,\bar z) \;=\; z\bar z\;c^-\,c^+\;\tfrac14 e_{(\mu\nu)}\;:\partial_z X^\mu_R\;\partial_{\bar z}X^\nu_L\;{\rm e}^{\frac{\rm i}{2}k\cdot(X_R+X_L)}:(z,\bar z) \tag{3.68}$$

with a symmetric, traceless and transverse polarization tensor $(e_{(\mu\nu)})$. Ubiquitous is the r-tachyon correlator

$${}_X\langle 0|\prod_{i=1}^r :{\rm e}^{\frac{\rm i}{2}k_i\cdot(X_R+X_L)}:(z_i,\bar z_i)\,|0\rangle_X \;=\; \delta^{(26)}(k_1{+}\ldots{+}k_r)\;\prod_{i<j}|z_i{-}z_j|^{\alpha' k_i\cdot k_j} \tag{3.69}$$

as well as the ghost correlator

$$
{}_{\rm gh}\langle 0|\, c^-(z_1)c^+(\bar z_1)\, c^-(z_2)c^+(\bar z_2)\, c^-(z_r)c^+(\bar z_r)\, |0\rangle_{\rm gh} \;=\; \Bigl|\frac{(z_1{-}z_2)(z_1{-}z_r)(z_2{-}z_r)}{z_1\ z_2\ z_r}\Bigr|^2 \tag{3.70}
$$

resulting from putting the c-type vertex operators at the locations z_1, z_2 and z_r. Again, the prescription to keep three factors of c^-c^+ and to integrate

$$
2{\rm i} \textstyle\int {\rm d}t_\ell\, {\rm d}\sigma_\ell\ z_\ell \bar z_\ell \ldots \;=\; \textstyle\int {\rm d}z_\ell\, {\rm d}\bar z_\ell \ldots \qquad \text{for} \quad \ell = 3,\ldots,r{-}1 \tag{3.71}
$$

can be motivated from string field theory. In summary, the ghost-number count is twice that of the open string.

From the disk to the sphere. The closed-string Fock space admits an action of two copies of the sl(2,$\mathbb{R}$) subalgebra of the Virasoro algebra, generated by $\{L_{-1},L_0,L_{+1}\}$ and by $\{\widetilde L_{-1},\widetilde L_0,\widetilde L_{+1}\}$, which kill the ground state. The corresponding group SO(2,2) turns into SL(2,$\mathbb{C}$) after the Wick rotation (3.64) and acts via fractional linear transformations on the complex z plane including the point at infinity, i.e. the Riemann sphere. By a conformal rescaling, ${\rm d}z{\rm d}\bar z \mapsto (1{+}z\bar z)^{-2}{\rm d}z{\rm d}\bar z$, we obtain the standard round unit sphere. Indeed, SL(2,$\mathbb{C}$) is the conformal automorphism group of the sphere, and the integration over the vertex operator locations overcounts by its infinite volume, unless we fix three points on the sphere, typically

$$
z_1 \;=\; \infty \quad , \qquad z_2 \;=\; 1 \quad , \qquad z_r \;=\; 0 \quad . \tag{3.72}
$$

Again, the first location ($z_1{=}\infty$) comes with a vanishing power in the amplitude, due to the closed-string mass-shell condition $\alpha' k_1^2 = 4(1{-}N)$. In contrast to the open string, the vertex operator locations are not ordered, and so integrating over them already takes care of the permutation invariance for identical external legs. Finally, we denote the cubic closed-string coupling by κ, hence the r-string tree-level amplitude comes with a factor of κ^{r-2}.

Tachyons and Gravitons

Three-point. Three closed-string tachyons obviously couple with a strength $T^{\rm tree}_{TTT} = \kappa$. For three gravitons, we can essentially square the three-photon correlator (3.32) and (with $k_i{\cdot}e_i = 0 = e_i{\cdot}k_i$ and $\sum_i k_i = 0$) obtain

$$
\begin{aligned}
T^{\rm tree}_{GGG} \;=\; \kappa\, (\tfrac{\alpha'}{2})^4 \bigl[& e_1{:}e_2\ k_2{\cdot}e_3{\cdot}k_1 \;+\; k_3{\cdot}e_1{\cdot}e_2{\cdot}e_3{\cdot}k_1 \;+\; k_1{\cdot}e_3{\cdot}e_2{\cdot}e_1{\cdot}k_3 \\
& -\; 2\tfrac{\alpha'}{2}\, k_3{\cdot}e_1{\cdot}e_2{\cdot}k_3\ k_2{\cdot}e_3{\cdot}k_1 \;+\; \text{cyclic} \;+\; (\tfrac{\alpha'}{2})^2\, k_3{\cdot}e_1{\cdot}k_2\ k_1{\cdot}e_2{\cdot}k_3\ k_2{\cdot}e_3{\cdot}k_1 \bigr]
\end{aligned} \tag{3.73}
$$

where the dots signify Lorentz contractions so that $e_1{:}e_2 = e_{1(\mu\nu)} e_{2(\rho\lambda)} \eta^{\mu\rho}\eta^{\nu\lambda}$. It is totally symmetric in the three leg labels; no Chan-Paton factors are possible or needed. It can be shown that for a judicious gauge choice the three-graviton coupling of Einstein relativity is precisely proportional to the first line of (3.73). This is nontrivial evidence that the dynamics of the spin-two closed-string excitation is indeed the standard gravitational one! In other words, the closed-string effective action for the space-time metric field $G_{\mu\nu}$ should take the form

$$S_{\text{eff}}[G_{\mu\nu}] \;=\; \tfrac{1}{2\kappa^2}\int\!\mathrm{d}^D x\;\sqrt{-\det G_{..}}\;\big[R(G)\;+\;O(\alpha')\big] \quad , \tag{3.74}$$

where κ is the gravitational coupling. From (3.73) we infer that the part of the string correction to the Einstein-Hilbert action which is cubic in the graviton consists of terms only quadratic and cubic in the Riemann tensor.

Exercise 3.7 *Verify* (3.73) *starting with* (3.32).

Four-point: Virasoro-Shapiro amplitude. The closed-string analog of the Veneziano amplitude is a slightly more complicated integral. Recalling that $\alpha' m_i^2 = -4$ in (3.35) and $\beta = 2$ in (3.36), we obtain (writing $z_3 \equiv z$),

$$\begin{aligned} T^{\text{tree}}_{TTTT} \;&=\; \kappa^2\int_{\mathbb{C}}\mathrm{d}^2 z\;|1{-}z|^{\alpha' k_2\cdot k_3}\;|z|^{\alpha' k_3\cdot k_4} \\ &=\; \kappa^2\int_{\mathbb{C}}\mathrm{d}^2 z\;|z|^{-\alpha' s/2-4}\;|1{-}z|^{-\alpha' t/2-4} \\ &=\; \kappa^2\int_{\mathbb{C}}\mathrm{d}^2 z\;|z|^{-2\alpha(s)-2}\;|1{-}z|^{-2\alpha(t)-2} \\ &=\; \kappa^2\;\frac{2\pi\;\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))\,\Gamma(-\alpha(u))}{\Gamma(-\alpha(s)-\alpha(t))\,\Gamma(-\alpha(t)-\alpha(u))\,\Gamma(-\alpha(u)-\alpha(s))} \\ &=\; \kappa^2\;\frac{2\pi\;\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))\,\Gamma(-\alpha(u))}{\Gamma(1+\alpha(s))\,\Gamma(1+\alpha(t))\,\Gamma(1+\alpha(u))} \quad , \end{aligned} \tag{3.75}$$

with the kinematical relation

$$\alpha(s)+\alpha(t)+\alpha(u) \;=\; \tfrac{\alpha'}{4}(s+t+u)\;+\;3 \;=\; -1 \quad . \tag{3.76}$$

Clearly, this expression is *totally* symmetric in all four legs and hence manifestly crossing symmetric, so no external leg sum is required. It also features the correct closed-string poles at $\frac14\alpha' s = n{-}1$ and $\frac14\alpha' t = n{-}1$ and $\frac14\alpha' u = n{-}1$ for $n \in \mathbb{N}_0$. For completeness, we mention the Regge limit

$$T^{\text{tree}}_{TTTT}(s{\to}\infty, t) \;\sim\; \tfrac{\Gamma(-\alpha(t))}{\Gamma(1+\alpha(t))}\,s^{2\alpha(t)} \;\sim\; s^{\frac12\alpha' t+2} \tag{3.77}$$

as well as the hard-scattering limit

$$T^{\text{tree}}_{TTTT}(s{\to}\infty, \theta) \;\sim\; \mathrm{e}^{-\frac12\alpha'(s\ln s+t\ln t+u\ln u)} \;\sim\; |f(\theta)|^{-2\alpha(s)} \quad . \tag{3.78}$$

The computation of the four-graviton amplitude is a more tedious task. As we shall show later, the result is actually simpler in the superstring.

Target-Space Field Theory

Strings in curved spacetime. The scattering amplitudes we have derived are well defined only on-shell. The mass-shell condition was enforced by the Virasoro constraints and also arises naturally from considering asymptotic pointlike string sources, as we implemented using vertex operators and computing S-matrix elements. It is possible, however, to define off-shell amplitudes after fixing a gauge or directly in string field theory after picking a particular string-field interaction. Here, we shall remain on-shell but remember that the reconstruction of the target-space effective action from the S matrix is subject to the ambiguity of field redefinition.

The example of (3.68) and (3.74) illustrates that the excitation of a coherent superposition of plane-wave gravitons via inserting the exponential

$$\mathrm{e}^{\kappa \int \mathrm{d}^2 z\, V_G^X(e,k;z,\bar{z})} \;=\; \mathrm{e}^{-2\kappa \int \mathrm{d}^2\xi\, \eta^{\alpha\beta} :\partial_\alpha X^\mu\, \partial_\beta X^\nu\, \mathrm{e}^{\mathrm{i}k\cdot X}:\, e_{(\mu\nu)}} \tag{3.79}$$

into the correlator is equivalent to modifying in the corresponding path integral the gauge-fixed Polyakov action (2.17) according to

$$S_0^{\mathrm{gf}}[X;\eta_{\mu\nu} \to G_{\mu\nu}(X)] \qquad \text{with} \qquad G_{\mu\nu}(X) \;=\; \eta_{\mu\nu} + 8\pi\alpha'\kappa\, e_{(\mu\nu)}\, \mathrm{e}^{\mathrm{i}k\cdot X} \quad . \tag{3.80}$$

Performing the same shift in the Polyakov action $S_0[X,h;\eta_{\mu\nu}]$ defined in (2.11), we can even read off the graviton vertex operator on a curved worldsheet. Furthermore, the plane wave may be replaced by a more general graviton wave function with impunity. We learn that an appropriately excited string in flat spacetime behaves like a string in a curved target space! This suggests that spacetime itself is somehow built from strings. Similar to general relativity, not only is the string motion governed by the spacetime background, but the target space itself is also severely constrained by admitting the string, as we shall see.

Closed-string backgrounds. It is natural to generalize the relationship of the graviton vertex operator with the spacetime metric to the other massless closed-string excitations, i.e. the Kalb-Ramond state and the dilaton state.[7] The corresponding background degrees of freedom, $B_{\mu\nu}(X)$ and $\Phi(X)$, extend the target spacetime metric $G_{\mu\nu}(X)$ by additional data. In this way, one obtains the Polyakov action in a general (massless) background,

$$S[X,h] = -\tfrac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\xi\, \sqrt{-h}\Big[\big(h^{\alpha\beta}G_{\mu\nu}(X) + \epsilon^{\alpha\beta}B_{\mu\nu}(X)\big)\partial_\alpha X^\mu\, \partial_\beta X^\nu + \alpha' R\,\Phi(X)\Big] \,, \tag{3.81}$$

where $(\epsilon^{\alpha\beta})$ is the Levi-Civita tensor on the worldsheet. The dilaton Φ couples to the worldsheet curvature scalar $R{=}R(h_{\alpha\beta})$ because its (dimensionally regularized) vertex operator reads

$$V_\Phi^X(k) \;\sim\; \sqrt{-h}\,\big[h^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,\eta_{\mu\nu} \;+\; \tfrac{D}{4}\,\alpha' R(h)\big]\,\mathrm{e}^{\mathrm{i}k\cdot X} \quad . \tag{3.82}$$

[7]Again we ignore the tachyons. Open-string backgrounds will be discussed later.

Exercise 3.8 *Show that $S[X,h]$ is the most general action invariant under rigid Weyl transformations, This limits the possible background degrees of freedom to G, B and Φ.*

It is intriguing that there exists a one-to-one correspondence between background data and (massless) string excitations. Actions like (3.81) are known from nonlinear sigma models; therefore, string theory in such backgrounds is also named the "string sigma model". Its background functionals can be interpreted as an infinite collection of coupling constants. For anything but very special backgrounds, this is an interacting theory which does not lend itself to exact solutions.

Symmetries. By construction, the sigma-model action (3.81) is invariant under worldsheet reparametrizations, but also under global transformations which leave the background invariant, e.g. Poincaré transformations (2.31) in Minkowski space. Generic backgrounds, however, do not admit such isometries, except for

$$\delta B_{\mu\nu}(X) \;=\; \partial_\mu \zeta_\nu(X) - \partial_\nu \zeta_\mu(X) \quad , \tag{3.83}$$

which generalizes the usual (spacetime) electromagnetic gauge transformations. Note that global worldsheet symmetries are tied to local spacetime invariances. The gauge invariant object is the field strength

$$H_{\mu\nu\rho} \;=\; \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \quad . \tag{3.84}$$

Exercise 3.9 *Perform a general string coordinate redefinition $X^\mu \to X'^\mu(X)$ and verify that the action (3.81) is not invariant but retains its form, i.e.*

$$S[X,h;G,B,\Phi] \;=\; S[X',h;G',B',\Phi'] \quad , \tag{3.85}$$

where the background fields have been subject to the corresponding tensorial transformations induced by the coordinate change.

Sigma models related by coordinate redefinitions are different theories from the worldsheet point of view, but they describe string dynamics on the same target spacetime and may thus be identified.

Local Weyl invariance and background constraints. Less obvious is the fate of local Weyl invariance (2.14) in general backgrounds. We have seen that Weyl invariance is broken by a quantum anomaly already on the flat background, with a coefficient of $D-26$. In curved backgrounds this anomaly is more serious because background-field dependent. However, for the right choice of background it can be made to cancel with the explicit Weyl non-invariance of the dilaton term in (3.81),

$$R(\mathrm{e}^\Lambda h_{..}) \;=\; \mathrm{e}^{-\Lambda}\big[R(h_{..}) - h^{\alpha\beta}\nabla_\alpha\partial_\beta\Lambda\big] \quad . \tag{3.86}$$

The explicit α' factor of the Φ term in (3.81) is not a convention. Noting that α' takes the role of $\hbar$ in the worldsheet actions, a power series in α' is nothing but a semiclassical or loop expansion. In this worldsheet sense, the dilaton term is introduced in (3.81) and (3.82) as a one-loop counterterm, at $O(\alpha')$ just like the Weyl anomaly.

By the little argument in (2.24) the Weyl variation of the action is proportional to the (integrated) trace of the energy-momentum tensor. A careful quantum computation of the latter in a general massless closed-string background finally yields

$$\delta_\Lambda S_{\rm qu}[X,h] \;=\; \tfrac{1}{8\pi\alpha'}\int_\Sigma \mathrm{d}^2\xi\,\sqrt{-h}\Big[\big(h^{\alpha\beta}\beta^G_{\mu\nu}+\epsilon^{\alpha\beta}\beta^B_{\mu\nu}\big)\partial_\alpha X^\mu\,\partial_\beta X^\nu+\alpha'\beta^\Phi R(h)\Big]\Lambda \tag{3.87}$$

with the so-called beta functions

$$\begin{aligned}
\beta^G_{\mu\nu} &= \alpha' R_{\mu\nu}(G) \;+\; 2\alpha'\nabla_\mu\partial_\nu\Phi \;-\; \tfrac{\alpha'}{4}H_\mu{}^{\rho\lambda}H_{\nu\rho\lambda} \;+\; O(\alpha'^2) \quad , \\
\beta^B_{\mu\nu} &= -\tfrac{\alpha'}{2}\nabla^\rho H_{\rho\mu\nu} \;+\; \alpha'(\partial^\rho\Phi)H_{\rho\mu\nu} \;+\; O(\alpha'^2) \quad , \\
\beta^\Phi &= \tfrac{D-26}{6} \;-\; \tfrac{\alpha'}{2}\nabla^\mu\partial_\mu\Phi \;+\; \alpha'(\partial^\mu\Phi)(\partial_\mu\Phi) \;-\; \tfrac{\alpha'}{24}H^{\mu\nu\rho}H_{\mu\nu\rho} \;+\; O(\alpha'^2) \quad ,
\end{aligned} \tag{3.88}$$

where ∇_μ denotes the gravitationally covariant derivative in spacetime and $R_{\mu\nu}$ is the associated Ricci tensor. Note that the dilaton beta function is defined one order in α' higher than the other ones; its $O(\alpha')$ contributions above required a two-loop computation in the sigma model.

Local Weyl invariance is essential for the quantum consistency of the worldsheet theory. Therefore, we are forced to impose on the background fields the conditions

$$\beta^G_{\mu\nu} \;=\; \beta^B_{\mu\nu} \;=\; \beta^\Phi \;=\; 0 \quad . \tag{3.89}$$

These constraints are very reasonable to leading order in α'. The first one, for example, can be rearranged into

$$R_{\mu\nu}(G)-\tfrac{1}{2}G_{\mu\nu}R(G) \;=\; T_{\mu\nu}(B,\Phi) \quad , \tag{3.90}$$

which is nothing but Einstein's equation in the presence of B and Φ "matter" sources. Also the other two equations are precisely the equations of motion for $B_{\mu\nu}$ and for Φ with the standard couplings to the other fields.

When pushing the computation of the beta functions to higher order in α' we find stringy corrections to the familiar contraints (3.89) on the background. The prime example is

$$\beta^G_{\mu\nu} \;=\; \alpha' R_{\mu\nu} \;+\; \tfrac{1}{2}\alpha'^2 R_\mu{}^{\rho\lambda\sigma}R_{\nu\rho\lambda\sigma} \;+\; O(\alpha'^3) \;+\; O(\partial\Phi,H) \tag{3.91}$$

and involves a square of the Riemann tensor. On a target space with characteristic curvature radius R_c the effective dimensionless expansion parameter is α'/R_c^2.

Therefore, the α' expansion will be reasonable if the string scale $\sqrt{\alpha'}$ is much smaller than R_c. In this situation, the internal structure of the string is unimportant because the excitation of massive states is strongly suppressed, and it makes sense to restrict ourselves to massless backgrounds.

The string coupling. So far the dilaton enters only with its gradient, but the average $\langle\Phi\rangle$ plays an important role as well. Clearly, for the "empty" flat spacetime ($G{=}\eta$, $B{=}0$, $\Phi{=}\Phi_0{=}$const) one recovers the Polyakov action (2.11),

$$S[X,h] \quad \stackrel{\text{flat}}{\longrightarrow} \quad S_0[X,h] \; - \; \chi\,\Phi_0 \quad , \tag{3.92}$$

plus a topological term because

$$\tfrac{1}{4\pi}\int_\Sigma \mathrm{d}^2\xi\,\sqrt{-h}\,R(h) \;=\; \chi(\Sigma) \tag{3.93}$$

is the Euler number $\chi = 2-2g$ of a genus-g worldsheet Σ. Since the Euclidean action is minus the Minkowskian action, this implies that the weight of e^{+S} in the Euclidean path integrals which compute the closed-string scattering amplitudes produces a factor of $\mathrm{e}^{-\chi\Phi_0}$. This topological weight allows us to generate the closed-string coupling κ: An r-string g-loop amplitude needs $r{+}2g{-}2$ string interactions, hence carries a weight of κ^{r+2g-2}. By using rescaled vertex operators κV, we can produce a factor of κ^r. The remaining factor appears when we identify

$$(\tfrac{\kappa}{\kappa_0})^{2g-2} \;=\; \mathrm{e}^{(2g-2)\Phi_0} \qquad\longrightarrow\qquad \kappa \;=\; \mathrm{e}^{\Phi_0}\,\kappa_0 \quad , \tag{3.94}$$

where κ_0 takes care of the dimension $[\kappa^2]=[L^{D-2}]$ by setting $\kappa_0=\alpha'^{(D-2)/4}$.

In this way, we have absorbed the dimensionless coupling constant $\kappa/\alpha'^{(D-2)/4}$ into the average value of the dilaton field, e.g. its vacuum expectation value. This means that the strength of the string interaction is not a free parameter but determined by the dynamics, which for instance might provide an effective potential for the dilaton. Again we see that different backgrounds (distinguished by the value of $\langle\Phi\rangle$) describe the same theory, only at different coupling strength. As a result, there are no dimensionless parameters in string theory. The only free parameter is the Regge slope α' which sets the string scale and thus provides dimensionalities.

Spacetime actions. At first sight it is intriguing that the background conditions (3.89) can be obtained from a variational principle. They are the equations of motion for the "spacetime action"

$$\begin{aligned} S_{\text{string}} \;=\; \tfrac{1}{2\kappa_0^2}\int\mathrm{d}^D x\,\sqrt{-\det G_{..}}\;\mathrm{e}^{-2\Phi}\big[-\tfrac{D-26}{3\alpha'/2} \;+\; R(G) \;+\; 4(\partial^\mu\Phi)(\partial_\mu\Phi) \\ -\;\tfrac{1}{12}H^{\mu\nu\rho}H_{\mu\nu\rho} \;+\; O(\alpha')\big] \end{aligned} \tag{3.95}$$

for the massless closed-string background fields, where X has been degraded to a coordinate x.

Exercise 3.10 *Derive the background field equations* (3.89) *with* (3.88) *by extremizing the action* (3.95)*. Hint: It is convenient to consider the combination*

$$\beta^\Phi - \tfrac14 G^{\mu\nu}\beta^G_{\mu\nu} \;=\; \tfrac{D-26}{6} - \alpha'\nabla^\mu\partial_\mu\Phi + \alpha'(\partial^\mu\Phi)(\partial_\mu\Phi) + \tfrac{\alpha'}{48}H^{\mu\nu\rho}H_{\mu\nu\rho} - \tfrac{\alpha'}{4}R(G) + \ldots\;. \tag{3.96}$$

For $H{=}0$ and $\Phi{=}\Phi_0$ in $D{=}26$ this action coincides with the effective action $S_{\text{eff}}[G_{\mu\nu}]$ of (3.74). This nontrivial match between strings coupling to a background (or external potential) and strings interacting with one another is further evidence that the background itself is made of string!

For a non-constant dilaton, (3.95) differs from the standard Einstein-Hilbert form by a conformal field redefinition

$$\widetilde{G}_{\mu\nu}(x) \;=\; \mathrm{e}^{\Omega(x)}\, G_{\mu\nu}(x) \tag{3.97}$$

with a suitable function Ω. One refers to $G_{\mu\nu}$ as the string metric and to $\widetilde{G}_{\mu\nu}$ as the Einstein metric.

Exercise 3.11 *Employ the formula*

$$R(\widetilde{G}) \;=\; e^{-\Omega}\left[R(G) \;-\; (D{-}1)\nabla^\mu\partial_\mu\Omega \;-\; \tfrac14(D{-}1)(D{-}2)(\partial^\mu\Omega)(\partial_\mu\Omega)\right] \tag{3.98}$$

with the choice $\Omega = \frac{4}{D-2}(\Phi_0{-}\Phi) =: \frac{-4}{D-2}\widetilde{\Phi}$ *to compute* $S_{\text{EH}}[\widetilde{G}_{\mu\nu}] = S_{\text{string}}[G_{\mu\nu}(\widetilde{G})]$.

Using (3.94) the result of this spacetime Weyl transformation is

$$\begin{aligned} S_{\text{EH}} \;=\; \tfrac{1}{2\kappa^2}\!\int\!\mathrm{d}^D x\,\sqrt{-\det\widetilde{G}_{\cdot\cdot}}\;\Big[&-\tfrac{D-26}{3\alpha'/2}\,\mathrm{e}^{\frac{4}{D-2}\widetilde{\Phi}} \;+\; R(\widetilde{G}) \;+\; \tfrac{4}{D-2}(\widetilde{\partial}^\mu\widetilde{\Phi})(\partial_\mu\widetilde{\Phi}) \\ &-\; \tfrac{1}{12}\mathrm{e}^{-\frac{8}{D-2}\widetilde{\Phi}}\,\widetilde{H}^{\mu\nu\rho}H_{\mu\nu\rho} \;+\; O(\alpha')\Big] \quad , \end{aligned} \tag{3.99}$$

where the tildes serve to remind us that indices have been raised with $\widetilde{G}^{\mu\nu}$. It is noteworthy that the non-minimal coupling of the dilaton violates the equivalence principle. To remain phenomenologically viable, higher-order effects must (and do!) provide a dilaton mass, thus keeping this violation under control.

Finally, the average value of the dilaton appears in (3.95) just in the overall factor $\mathrm{e}^{-2\Phi}$, which is precisely the correct form required for a tree-level amplitudes ($g{=}0$ for the sphere topology). We therefore expect eventual contributions from other worldsheet topologies to the spacetime action to come with a factor of $\mathrm{e}^{-\chi\Phi}$. Since in (3.99) the role of $\hbar$ is played by κ^2, the closed-string loop expansion (in powers of κ^2) is indeed the topological one (in powers of $\mathrm{e}^{2\Phi_0}$).

Open-string backgrounds. The previous considerations can be extended to open strings, by allowing for a photon background degree of freedom $A_\mu(X)$, via adding to the Polyakov action the term

$$\int_{\partial\Sigma} \mathrm{d}\tau \; A_\mu(X)\, \partial_\tau X^\mu \quad . \tag{3.100}$$

As an open-string excitation, the gauge boson A_μ is attached to the worldsheet boundary.[8] Generalizing (3.92) and (3.93) to worldsheets with boundaries, the geodesic curvature of $\partial\Sigma$ also contributes to the dilaton coupling, which we subsume in the bulk term $-\frac{1}{4\pi}\int_\Sigma \sqrt{-h}R\,\Phi$. Evident is the spacetime gauge transformation and field strength, namely

$$\delta A_\mu(X) \;=\; \partial_\mu \lambda(X) \qquad \text{and} \qquad F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \tag{3.101}$$

Note that a full massless background (G, B, Φ, A) is attributed to a theory of both open and closed strings. The worldsheet equation of motion (2.18) and the Neumann boundary conditions (2.42) in such a case generalize to

$$\Big\{\big[G_{\mu\nu}\partial_\alpha + \Gamma_{\mu\nu\lambda}(\partial_\alpha X^\lambda)\big]\sqrt{-h}\,h^{\alpha\beta} \;+\; \tfrac{1}{2}H_{\mu\nu\lambda}(\partial_\alpha X^\lambda)\,\epsilon^{\alpha\beta}\Big\}\,\partial_\beta X^\nu \;=\; 0\;, \tag{3.102}$$

$$\big[\partial_n X^\mu \;+\; \mathcal{B}^\mu{}_\nu \partial_t X^\nu\big]\big|_{\sigma=0,\pi} \;=\; 0\;, \tag{3.103}$$

respectively, where $\Gamma^\mu{}_{\nu\lambda}$ are the Christoffel symbols for the metric G,

$$\partial_n \;=\; n_\alpha \sqrt{-h}\,h^{\alpha\beta}\,\partial_\beta \qquad \text{and} \qquad \partial_t \;=\; n_\alpha\,\epsilon^{\alpha\beta}\,\partial_\beta \tag{3.104}$$

denote the worldsheet derivatives normal and tangent to the boundary, and

$$\mathcal{B}_{\mu\nu} \;=\; B_{\mu\nu} \;+\; 2\pi\alpha'\,F_{\mu\nu} \tag{3.105}$$

is a combination invariant under

$$\delta B_{\mu\nu} \;=\; \partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu \qquad \text{and} \qquad \delta A_\mu \;=\; -\tfrac{1}{2\pi\alpha'}\,\zeta_\mu \quad , \tag{3.106}$$

which generalizes the gauge transformation (3.83) for worldsheets with boundaries. Similarly, in the conformal gauge, the Γ and H terms in (3.102) combine to the generalized connections $\Gamma \pm \frac{1}{2}H$. It is noteworthy that the boundary condition (3.103) interpolates between Neumann and Dirichlet.

Brane actions. Interestingly, when gradient-expanding the photon beta function $\beta^A_\mu(F)$, the leading term can be computed to all orders in α',

$$\beta^A_\mu \;=\; (\nabla^\nu \mathcal{B}_\mu{}^\lambda)\big[G - \mathcal{B}^2\big]^{-1}_{\lambda\nu} + \tfrac{1}{2}\mathcal{B}_{\mu\nu}H^{\nu\lambda\rho}\big[\tfrac{\mathcal{B}}{G-\mathcal{B}^2}\big]_{\lambda\rho} + \tfrac{1}{2}(\nabla^\nu\Phi)\mathcal{B}_{\mu\nu} + O(\nabla^2) \tag{3.107}$$

[8]For gluons we should instead consider the Wilson loop $P\mathrm{e}^{-S}$.

which involves B and A only in the combinations H and $\mathcal{B}$. Thus for slowly-varying fields, the vanishing of β^A_μ is the equation of motion for the spacetime action

$$\begin{aligned} S_{\rm DBI} &= -T_{D-1}\int\! {\rm d}^D x\ {\rm e}^{-\Phi}\,\sqrt{-\det(G_{..}+\mathcal{B}_{..})} \\ &= -T_{D-1}\tfrac{\kappa_0}{\kappa}\int\! {\rm d}^D x\ {\rm e}^{\frac{D+2}{D-2}\widetilde{\Phi}}\,\sqrt{-\det(\widetilde{G}_{..}+{\rm e}^{-\frac{4}{D-2}\widetilde{\Phi}}\mathcal{B}_{..})}\quad , \end{aligned} \tag{3.108}$$

known as the Dirac-Born-Infeld action. T_{D-1} is the tension of the space-filling $(D{-}1)$-brane. This action is also appropriate for branes of lower space dimension $p < D{-}1$. One simply substitutes for G and $\mathcal{B}$ the fields induced on the p-brane via the embedding of the brane worldvolume into the ambient spacetime, integrates over the worldvolume and multiplies with the brane tension T_p.

Expanding the square root for $\mathcal{B} \ll G$ and abbreviating $G^{\mu\rho}\mathcal{B}_{\rho\nu} =: M^\mu{}_\nu$,

$$S_{\rm DBI} \;=\; T_{D-1}\int\! {\rm d}^D x\ {\rm e}^{-\Phi}\sqrt{-\det G_{..}}\Big[-1+\tfrac14\,{\rm tr}\,M^2+\tfrac18\,{\rm tr}\,M^4-\tfrac{1}{32}({\rm tr}\,M^2)^2+\ldots\Big] \tag{3.109}$$

since odd powers of M trace to zero. For purely open-string (abelian) backgrounds $(G,\Phi,\mathcal{B})=(\eta,\Phi_0,2\pi\alpha' F)$, this spacetime action precisely matches with the effective action (3.56) in the abelian case ($F^3{=}0$), after a rescaling of the fields and identification of the couplings.

Exercise 3.12 *Vary the purely open-string spacetime action*

$$S^0_{\rm DBI} \;=\; -T_{D-1}\tfrac{\kappa_0}{\kappa}\int\! d^D x\ \sqrt{-\det(1+2\pi\alpha' F_{..})} \tag{3.110}$$

with respect to the gauge potential. You should arrive at

$$\frac{\delta S^0_{\rm DBI}}{\delta A^\mu} \;\sim\; \big[1-(2\pi\alpha' F)^2\big]^{-1}_{\mu\nu}\ \big(\partial^\rho F_\mu{}^\lambda\big)\ \big[1-(2\pi\alpha' F)^2\big]^{-1}_{\lambda\rho}\quad . \tag{3.111}$$

For a common theory of closed and open strings, one expects the total spacetime action to be the sum of $S_{\rm string}$ and $S_{\rm DBI}$. The Dirac-Born-Infeld term then modifies the closed-string spacetime equations of motion, because the gauge field A_μ acts as a source for the closed-string fields. However, such a modification of the closed-string beta functions cannot arise from a tree-level worldsheet computation, because these beta functions do not get altered by worldsheet boundary effects. Therefore, the influence of the open string on the closed-string beta functions must be a string loop effect, i.e. due to divergences of the string sigma model on higher topologies in the string loop expansion. Indeed, the prefactors

$${\rm e}^{-2\Phi}\ \sim\ \kappa^{-2}\qquad\text{and}\qquad {\rm e}^{-\Phi}\ \sim\ \kappa^{-1} \tag{3.112}$$

of $S_{\rm string}$ and $S_{\rm DBI}$, respectively, already reflect the different topologies of the relevant leading worldsheets, namely the sphere (χ=2) versus the disk (χ=1).

By investigating the divergences from a small handle and from the sphere with a shrinking hole, one may in fact reproduce the expected modifications: The metric beta function, for example, picks up a cosmological constant and a photon energy-momentum tensor, to leading order in α'. The Weyl anomaly created by such a background shift just cancels the one stemming from the worldsheet topological fluctuations. Thus, a consistent string sigma model requires the topological expansion.

Tachyon potential. So far we have ignored the presence of the open- and closed-string tachyons among the background fields. Since the tachyons couple to the worldsheet volume and boundary perimeter via

$$\int_\Sigma \mathrm{d}^2\xi\, \sqrt{-h}\; T_{\rm cl}(X) \qquad \text{and} \qquad \int_{\partial\Sigma} \mathrm{d}\tau\; T_{\rm op}(X) \quad , \tag{3.113}$$

respectively, they violate the conformal invariance of the string sigma model but represent a relevant perturbation of it. String-field computations yield a potential for the open-string tachyon which is consistent with the decay of the ground state (the space-filling brane) into a new vacuum devoid of open-string excitations!

Open-Closed String Interactions

Higher topologies. The possible worldsheet topologies are realized by Riemann surfaces, oriented or not, with or without boundaries, and with marked points in the interior and/or on the boundaries. The oriented surfaces without boundary are fully classified by the Euler number $\chi = 2-2g$ where the genus g counts the number of handles. The simplest ones are the sphere S_2 (g=0) and the torus T_2 (g=1). Cutting a hole (window) into such a surface produces a boundary and lowers the Euler number by one unit. To generate an unoriented surface, one replaces at least one boundary component with a cross-cap, meaning that one identifies antipodal points along the boundary component, which eliminates it. The Euler number of a surface with g handles, b boundary components and c cross caps becomes

$$\chi \;=\; 2 \;-\; 2g \;-\; b \;-\; c \quad . \tag{3.114}$$

Combining the attributes closed/open and oriented/unoriented we have four types of string theories, with different topological expansions, because open-string legs require $b \neq 0$ and oriented strings need $c = 0$. For Euler numbers 2, 1 and 0, the following table emerges:

string type χ:	2	1	0
closed oriented	S_2	D_2	T_2, C_2
open oriented		D_2	C_2
closed unoriented	S_2	D_2, P_2	T_2, C_2, M_2, K_2
open unoriented		D_2	C_2, M_2

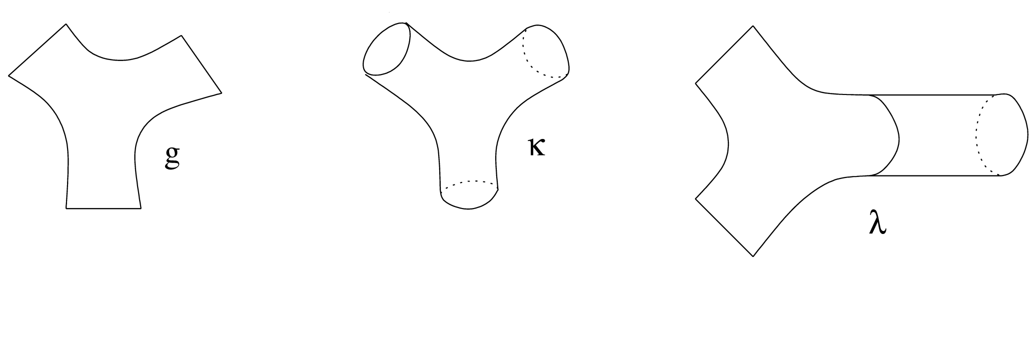
with the surfaces

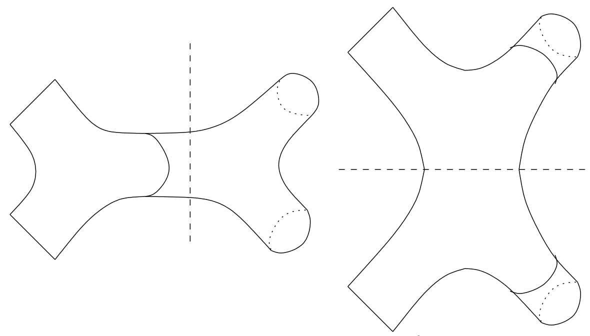
name	symbol	g	b	c
sphere	S_2	0	0	0
disk	D_2	0	1	0
projective plane	P_2	0	0	1
torus	T_2	1	0	0
cylinder=annulus	C_2	0	2	0
Möbius strip	M_2	0	1	1
Klein bottle	K_2	0	0	2

Note that by admitting holes in closed-string topologies we allow closed strings to form open-string intermediate states. Excluding this possibility eliminates D_2, C_2 and M_2 from the closed-string topologies in the table above. In contrast, open strings always give rise to intermediate closed strings at the loop level by a suitable choice of the temporal slices of the worldsheets.

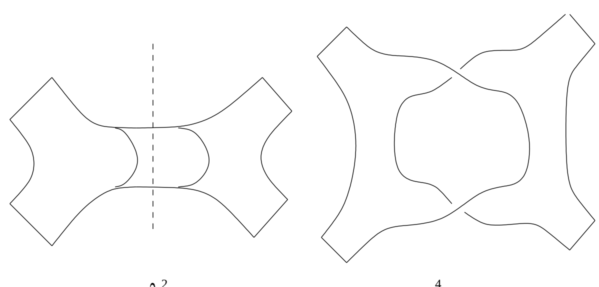
Exercise 3.13 *Write down simplicial decompositions of all above surfaces with the appropriate edge identifications and convince yourself that the list is complete for $\chi \geq 0$.*

Open-closed string coupling. In addition to the fundamental closed-string and open-string cubic couplings κ and g (not to be confused with the genus), mixed scattering amplitudes introduce a third basic coupling λ of one closed with two open strings, stemming from the disk with one bulk and two boundary vertex operators inserted. The breaking up of a closed string into an open one is therefore weighted with λ/g. The three string vertices are displayed below. Unitarity considerations relate those three coupling. First, look at the four-point amplitude with two open and two closed strings in leading topology, i.e. on the disk. Depending on the channel, the intermediate state is an open or a closed





string. The respective decompositions into two cubic interactions connected by the intermediate string are depicted above; equating the corresponding weights yields $\lambda\kappa \sim \lambda^2$. Second, study the four-point open-string amplitude in subleading topology, i.e. on the cylinder, with two open-string legs attached to either boundary component. As shown below, the string diagram can be viewed in two ways, either as two open strings joining to form a closed string and breaking again into two open strings (a tree-level configuration), or else as two open strings scattering via exchanging two open strings (a one-loop configuration!) with a couple of twists put in (a nonplanar diagram). This diagram is the prototype for demonstrating that closed-string channels necessarily appear on the one-loop level in open-string theories. Unitarity here dictates $\lambda^2 \sim g^4$. Hence together, putting in



the dimensions for space-filling branes we have (up to numerical coefficients)

$$\lambda \;=\; \kappa \qquad \text{and} \qquad \kappa \;=\; \alpha'^{\frac{6-D}{4}}\, g^2 \quad . \tag{3.115}$$

The first relation underscores the universality of gravity (everything couples to closed strings with the same strength), while the second relation connects the gauge to the gravitational coupling, another surprise of string theory!

3.3 Loop Amplitudes

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4 The Superstring Theories [incomplete]

5 Compactification and duality [incomplete]

A Solutions to the Exercises [incomplete]