BRST Symmetry and Cohomology

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We present the mathematical considerations which determine all gauge invariant actions and anomaly candidates in gauge theories of standard type such as ordinary or gravitational Yang Mills theories. Starting from elementary concepts of field theory the discussion tries to be explicit and complete, only the cohomology of simple Lie algebras it quoted from the literature.

After a short introduction to jet spaces chapter 1 deals with the “raison d'être” of gauge symmetries: the problem to define the subspace of physical states in a Lorentz invariant theory with higher spin. The operator Q_s which characterizes the physical states was found by Carlo Becchi, Alain Rouet and Raymond Stora as a symmetry generator of a fermionic symmetry, the BRST symmetry, in gauge theories with covariant gauge fixing [1]. Independently Igor Tjutin described the symmetry in a Lebedev Institute report which however remained unpublished for political reasons. For a derivation of the BRST symmetry from the gauge fixing in path integrals the reader may consult the literature [2, 4, 21]. Chapter 1 is supplemented by a discussion of free vectorfields for gauge parameter $\lambda \neq 1$. This is not a completely trivial exercise [3] and rarely discussed in detail [4].

Chapter 2 deals with the requirement that the physical subspace remains physical if interactions are switched on. This restricts the action to be BRST invariant. Consequently the Lagrange density has to satisfy a cohomological equation similar to the physical states. Quantum corrections may violate the requirement of BRST symmetry because the naive evaluation of Feynman diagrams leads to divergent loop integrals which have to be regularized. This regularization can lead to an anomalous symmetry breaking. It has to satisfy a cohomological equation, the Wess Zumino consistency condition [5].

In chapter 3 we study some elementary cohomological problems of a nilpotent fermionic derivative d,

$$\mathrm{d}^2 = 0 \quad , \quad \mathrm{d}\,\omega = 0 \quad , \quad \omega \bmod \mathrm{d}\eta \ .$$

We derive the Poincaré lemma as the basic lemma of all the investigations to come. In particular one has to consider Lagrange densities as jet functions, i.e. functions of the fields and their derivatives and not only of the coordinates. We investigate differential forms depending on these jet variables and derive the algebraic Poincaré lemma which is where Lagrangians of local actions enter the stage. The relative cohomology, which characterizes Lagrange densities and candidate anomalies, is shown to lead to the descent equations which can again be written compactly as a cohomological problem. The chapter concludes with Künneth's formula which allows to tackle cohomological problems in smaller bits if the complete problem factorizes.

Chapter 4 presents a formulation [6] of the gravitational BRST transformations in which the cohomology factorizes. Consequently one has to deal only with the subalgebra of tensors and undifferentiated ghosts. It is shown that the ghosts which correspond to translations can be removed from anomalies (if the space-time dimension exceeds two)[1], i.e. coordinate transformations are not anomalous.

In chapter 5 we solve the cohomology of the BRST transformations acting on ghosts and tensors. The tensors have to couple together with the translation ghosts to invariants and also the ghosts for spin and isospin transformations have to couple to invariants. The invariant ghost polynomials generate the Lie algebra cohomology which we quote from the mathematical literature [8]. Moreover the tensors are restricted by the covariant Poincaré lemma [10], for which we give a simplified proof. This lemma introduces the Chern forms. They are the integrands of all local actions which do not change under a smooth change of the fields and therefore give topological informations about classes of fields which are related by smooth deformations.

In chapter 6 we exhibit the Chern forms as the BRST transformation of the Chern Simons forms. Chern Simons forms can contribute to local gauge invariant actions though they are not gauge invariant. They are independent of the metric and do not contribute to the energy momentum tensor but nevertheless influence the field equations. We conclude by giving examples of Lagrange densities and anomaly candidates.

In chapter 7 we sketch how the cohomological analysis presented in chapters 3 to 6 can be extended to include antifields and how the cohomology is affected by the inclusion of antifields.

[1] The two-dimensional case can be special [7].

Contents

1 The Space of Physical States

1.1 Indefinite Fock Space

BRST symmetry is indispensable in Lorentz covariant theories with fields with higher spin because it allows to construct an acceptable space of physical states out of the Fock space which contains states with negative norm.

Before we demonstrate the problem, we recollect some elementary definitions and concepts. A (bosonic) field ϕ is a map of a base space, which locally is some domain of $\mathbb{R}^D$ with points $x = (x^0, x^1, x^2, \dots x^{D-1})$ to a target space $\mathbb{R}^d$,

$$\phi : \begin{cases} \mathbb{R}^D & \to \quad \mathbb{R}^d \\ x & \mapsto \quad \phi(x) = (\phi^1(x), \phi^2(x) \dots \phi^d(x)) \ . \end{cases} \tag{1.1}$$

By assumption we consider fields, which are sufficiently differentiable. Each field defines a field $\hat{\phi}$, the prolongation of ϕ to the jet space $\mathcal{J}_1$,

$$\hat{\phi} : \begin{cases} \mathbb{R}^D & \to \quad \mathbb{R}^{D+d+D\,d} \\ x & \mapsto \quad (x, \phi(x), \partial_0\phi(x), \partial_1\phi(x), \partial_2\phi(x) \dots \partial_{D-1}\phi(x)) \end{cases} \ . \tag{1.2}$$

Locally the jet space $\mathcal{J}_1$ is the cartesian product of some domain of the base space, the target space and the tangent space of a point.

Analogously, the prolongation of ϕ to the jet space $\mathcal{J}_k$ maps x to x, the field $\phi(x)$ and its partial derivatives $\partial \dots \partial\phi(x)$ up to k^{th} order. The prolongation $\hat{\phi}$ of an infinitely differentiable field maps the base space to $\mathcal{J} = \mathcal{J}_\infty$ and each point x to x, $\phi(x)$ and all its derivatives at x .

Jet functions $\mathcal{L}$ are maps from some $\mathcal{J}_k$, where k is *finite*, to $\mathbb{R}$. By composition with the projection

$$\pi_k : \begin{cases} \mathcal{J} & \to \quad \mathcal{J}_k \\ (x, \phi, \partial\phi, \dots, \partial^k\phi, \dots) & \mapsto \quad (x, \phi, \partial\phi, \dots, \partial^k\phi) \end{cases} \tag{1.3}$$

each jet function can be constantly continued to the function $\mathcal{L} \circ \pi_k$ of $\mathcal{J}$.

In notation we do not distinguish between $\mathcal{L}$ and its constant continuation but consider jet functions as functions of some $\mathcal{J}_k$ or of $\mathcal{J}$ as needed.

The action W is a local functional of fields ϕ, which is to say it maps fields to the integral over a jet function, the Lagrange density $\mathcal{L}$, evaluated on the prologation of the fields,

$$W : \phi \mapsto W[\phi] = \int d^D x \, (\mathcal{L} \circ \hat{\phi})(x) \ . \tag{1.4}$$

The equations of motion are derived from the variational principle that for physical fields the action W be stationary up to boundary terms under all variations of the fields. This holds if and only if the Euler derivative of the Lagrangian [1]

$$\frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\phi^i} = \frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_n \frac{\partial\mathcal{L}}{\partial(\partial_n\phi^i)} + \dots \ , \tag{1.5}$$

vanishes on the prolongation of the physical field,

$$\frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\phi} \circ \hat{\phi}_{\text{physical}} = 0 \ . \tag{1.6}$$

In case of the massless vectorfield A, $D = d = 4$ and the Lagrangian is

$$\mathcal{L} : \begin{cases} \mathcal{J}_1 & \to \quad \mathbb{R} \\ (x, A, \partial A) & \mapsto \quad = -\frac{1}{4e^2}(\partial_m A_n - \partial_n A_m)(\partial^m A^n - \partial^n A^m) - \frac{\lambda}{2e^2}(\partial_m A^m)^2 \ . \end{cases} \tag{1.7}$$

Here we use the shorthand $A^m = \eta^{mk}A_k$ and $\partial^n = \eta^{nl}\partial_l$ where η is the diagonal matrix $\eta = \text{diag}(1, -1, -1, -1)$. To avoid technical complications at this stage we consider the case $\lambda = 1$, $\lambda \neq 1$ is discussed at the end of this section. We choose to introduce the gauge coupling e as normalization of the kinetic energies to avoid its appearance in Lie algebras, which we have to consider later.

The physical vectorfield has to satisfy the wave equation,

$$\frac{1}{e^2}\Box A_n(x) = 0 \quad , \quad \Box = \eta^{mn}\partial_m\partial_n = {\partial_0}^2 - {\partial_1}^2 - {\partial_2}^2 - {\partial_3}^2 \ , \tag{1.8}$$

with the solution

$$A_n(x) = e \int \tilde{d}k \, (e^{ikx} a_n^\dagger(\vec{k}) + e^{-ikx} a_n(\vec{k}))\Big|_{k^0 = \sqrt{\vec{k}^2}} \ . \tag{1.9}$$

Here we use the notation

$$\tilde{d}k = \frac{d^3k}{(2\pi)^3 2|\vec{k}|} \ , \ kx = k^0x^0 - k^1x^1 - k^2x^2 - k^3x^3 = k^m x^n \eta_{mn} \ . \tag{1.10}$$

The vectorfield is quantized by the requirement that the propagator, the vacuum expectation value of the time ordered product of two fields,

$$\langle \Omega | T A^m(x) A_n(0) \, \Omega \rangle \ , \tag{1.11}$$

be a Green function corresponding to the Euler derivative,

$$\frac{1}{e^2}\Box \langle \Omega | T A^m(x) A_n(0) \, \Omega \rangle = i \, \delta^4(x) \, \delta^m{}_n \ . \tag{1.12}$$

[1] The dots denote terms which occur if $\mathcal{L}$ depends on second or higher derivatives of ϕ .

The creation and annihilation operators $a^\dagger(\vec{k})$ and $a(\vec{k})$ are identified by their commutation relations with the momentum operators P^m ,

$$\left[P_m, a_n^\dagger(\vec{k})\right] = k_m a_n^\dagger(\vec{k}) \quad , \quad \left[P_m, a_n(\vec{k})\right] = -k_m a_n(\vec{k}) \ , \tag{1.13}$$

which follow because by definition the momentum operators P_m generate translations,

$$[iP_m, A_n(x)] = \partial_m A_n(x) \ . \tag{1.14}$$

$a_n^\dagger(\vec{k})$ adds and $a_n(\vec{k})$ subtracts energy $k_0 = \sqrt{\vec{k}^2} \geq 0$. Consequently the annihilation operators annihilate the lowest energy state, the vacuum $|\Omega\rangle$, and justify their denomination,

$$P_m|\Omega\rangle = 0 \ , \ a(\vec{k})|\Omega\rangle = 0 \ . \tag{1.15}$$

For $x^0 > 0$ the propagator (1.11) contains only positive frequencies from $e^{-ikx}a_m(\vec{k})$, for $x^0 < 0$ only negative frequencies from $e^{ikx}a_m^\dagger(\vec{k})$. These boundary conditions fix the solution to (1.12) to be

$$\langle \Omega | T A_m(x) A_n(0) \, \Omega \rangle = -i \, e^2 \, \eta_{mn} \lim_{\varepsilon \to 0+} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ipx}}{p^2 + i\varepsilon} \tag{1.16}$$

with $\eta = \text{diag}(1, -1, -1, -1)$. Evaluating the p^0 integral for positive and for negative x^0 and comparing with the explicit expression for the propagator (1.11) which results if one inputs the free fields (1.9) one can read off $\langle \Omega | a_m(\vec{k}) a_n^\dagger(\vec{k}') \, \Omega \rangle$ and the value of the commutator

$$\left[a_m(\vec{k}), a_n^\dagger(\vec{k}')\right] = - \, \eta_{mn}(2\pi)^3 2k^0 \delta^3(\vec{k} - \vec{k}') \ . \tag{1.17}$$

It is inevitable that the Lorentz metric η appears in such commutation relations in Lorentz covariant theories with fields with higher spin. The Fock space which results from such commutation relations necessarily contains negative norm states because the Lorentz metric is indefinite and contains both signs. In particular the state

$$|f_0\rangle = \int \tilde{d}k \, f(\vec{k}) \, a_0^\dagger(\vec{k}) \, |\Omega\rangle \tag{1.18}$$

has negative norm

$$\langle f_0 | f_0 \rangle = -\eta_{00} \int \tilde{d}k \, |f(\vec{k})|^2 < 0 \ . \tag{1.19}$$

1.2 Definiteness of the Scalar Product of Physical States

Such a space with an indefinite scalar product cannot be the space of physical states because in quantum mechanics

$$w(i, A, \Psi) = |\langle \Lambda_i | \Psi \rangle|^2 \tag{1.20}$$

is the probability for the measurement to yield the result number i (which for simplicity we take to be nondegenerate and discrete), if the state Ψ is measured with the apparatus A. Here the states Λ_j are the eigenstates of A, which yield the corresponding result number j with certainty

$$|\langle\Lambda_i|\Lambda_j\rangle|^2 = \delta^i{}_j \ . \tag{1.21}$$

Therefore different Λ_i are orthogonal to each other (and therefore linearly independent)

$$\langle\Lambda_i|\Lambda_j\rangle = 0 \ , \quad \text{if } i \neq j \ . \tag{1.22}$$

For $i = j$ the scalar product of the eigenstates is real, $\langle\Phi|\Psi\rangle^* = \langle\Psi|\Phi\rangle$, and has modulus 1,

$$\langle\Lambda_i|\Lambda_j\rangle = \eta_{ij} \ , \quad \eta = \text{diag}(1,1,\dots,-1,-1,\dots) \ . \tag{1.23}$$

In the space, which is spanned by the eigenstates, the scalar product therefore is of the form

$$\langle\Lambda|\Psi\rangle = (\Lambda|\eta\Psi) \ , \tag{1.24}$$

where the scalar product $(\Lambda|\Psi)$ is positive definite and η is the linear map which maps Λ_i to $\sum_j \Lambda_j \eta_{ji}$. In particular the eigenstates Λ_i of the measuring apparatus A and each other apparatus are eigenvectors of η.

But a superposition $\Gamma = a\Lambda_1 + b\Lambda_2$ with $a\,b \neq 0$ is an eigenvector of η only if the eigenvalues η_{11} and η_{22} coincide. Therefore, in the space of physical states, which contains the eigenvectors of all measuring devices and their superpositions, the scalar product has to be definite.

1.3 Classical Electrodynamics

In classical electrodynamics (in the vacuum) one does not have the troublesome amplitude $a_0^\dagger(\vec{k})$. There the wave equation $\Box A_n = 0$ results from Maxwell's equation $\partial_m(\partial^m A^n - \partial^n A^m) = 0$ and the Lorenz condition $\partial_m A^m = 0$. This gauge condition fixes the vectorfield up to the gauge transformation $A_m \mapsto A'_m = A_m + \partial_m C$ where C satisfies the wave equation $\Box C = 0$. In terms of the free fields A and C

$$C(x) = e\int \tilde{d}k \left(e^{ikx}c^\dagger(\vec{k}) + e^{-ikx}c(\vec{k})\right)_{|_{k^0=\sqrt{\vec{k}^2}}} \tag{1.25}$$

the Lorenz condition concerns the linear combination $k^m a_m$ of the amplitudes

$$\partial_m A^m = i\,e\int \tilde{d}k \left(e^{ikx}k^m a_m^\dagger(\vec{k}) - e^{-ikx}k^m a_m(\vec{k})\right)_{|_{k^0=\sqrt{\vec{k}^2}}} \tag{1.26}$$

and the gauge transformation changes the amplitudes by a contribution in direction k

$$A'_m - A_m = \partial_m C = i\,e\int \tilde{d}k \left(e^{ikx}k_m c^\dagger(\vec{k}) - e^{-ikx}k_m c(\vec{k})\right)_{|_{k^0=\sqrt{\vec{k}^2}}} \ . \tag{1.27}$$

To make this even more explicit, we decompose the creation operator $a_m^\dagger(\vec{k})$ into parts in the direction of the lightlike momentum k, in the direction $\bar{k}$ (which is k with reflected 3-momentum)

$$(\bar{k}^0,\bar{k}^1,\bar{k}^2,\bar{k}^3) = (k^0,-k^1,-k^2,-k^3) \tag{1.28}$$

and in two directions n^1 and n^2 which are orthogonal to k and $\bar{k}$ [2]

$$a_m^\dagger(\vec{k}) = \sum_{\tau=k,\bar{k},1,2} \varepsilon^*_m{}^\tau a_\tau^\dagger(\vec{k}) \ . \tag{1.29}$$

These polarization vectors $\varepsilon^\tau(\vec{k})$ are functions of the lightcone $\mathbb{R}^3 - \{0\}$

$$\varepsilon^*_m{}^\tau(\vec{k}) = \Big(\frac{1}{\sqrt{2}}\frac{k_m}{|\vec{k}|}, \frac{1}{\sqrt{2}}\frac{\bar{k}_m}{|\vec{k}|}, n^1_m, n^2_m\Big), \ \tau = k,\bar{k},1,2 \tag{1.30}$$

and have the scalar products

$$\varepsilon^{*\,\tau}\cdot\varepsilon^{\tau'} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \ . \tag{1.31}$$

The field $\partial_m A^m$ contains the amplitudes $a_{\bar{k}}^\dagger$, $a_{\bar{k}}$. The Lorenz gauge condition $\partial_m A^m = 0$ eliminates these amplitudes in classical electrodynamics.

The fields A'_m and A_m differ in the amplitudes $a_k^\dagger$, a_k in the direction of the momentum k. An appropriate choice of the remaining gauge transformation (1.27) cancels these amplitudes.

So in classical electrodynamics $a_m^\dagger$ can be restricted to 2 degrees of freedom, the transverse oscillations

$$a_m^\dagger(\vec{k}) = \sum_{\tau=1,2} \varepsilon^{*\tau}_m a_\tau^\dagger(\vec{k}) \ . \tag{1.32}$$

The corresponding quantized modes generate a positive definite Fock space.

We cannot, however, just require $a_k^\dagger = 0$ and $a_{\bar{k}}^\dagger = 0$ in the quantized theory, this would contradict the commutation relation

$$\left[a_k(\vec{k}), a_{\bar{k}}^\dagger(\vec{k}')\right] = -\,(2\pi)^3 2k^0\delta^3(\vec{k}-\vec{k}') \neq 0 \ . \tag{1.33}$$

To get rid of the troublesome modes we require, rather, that physical states do not contain $a_k^\dagger$ and $a_{\bar{k}}^\dagger$ modes. This requires the interactions to leave the subspace of physical states invariant, a requirement, which is not at all obviously satisfied, because the unphysical modes contribute to the propagator. As we shall see, both the selection rule of physical states and the restrictions on the interactions to respect the selection rule emerge from the BRST symmetry.

[2] For example $\vec{n}^1(\vec{k}) \propto \vec{w}\times\vec{k}$, $\vec{n}^2(\vec{k}) \propto \vec{n}^{1*}\times\vec{k}$, where $\vec{w}$ is a constant complex vector with linearly independent real and imaginary part.

1.4 The Physical States

To single out a physical subspace of the Fock space $\mathcal{F}$ we require that there exists a hermitean operator, the BRST operator,

$$Q_s = Q_s^\dagger \ , \tag{1.34}$$

which defines a subspace $\mathcal{N} \subset \mathcal{F}$, the gauge invariant states, by

$$\mathcal{N} = \{|\Psi\rangle : |Q_s\Psi\rangle = 0\} \ . \tag{1.35}$$

This requirement is no restriction at all, each subspace can be characterized as kernel of some hermitean operator.

Inspired by gauge transformations (1.27) we take the operator Q_s to act on one particle states according to

$$Q_s\, a^\dagger_m(\vec{k})|\Omega\rangle = k_m c^\dagger(\vec{k})|\Omega\rangle \ . \tag{1.36}$$

As a consequence the one particle states generated by $a^\dagger_\tau(\vec{k})$, $\tau = \bar{k}, 1, 2$, belong to $\mathcal{N}$,

$$Q_s\, a^\dagger_\tau(\vec{k})|\Omega\rangle = 0 \ , \ \tau = \bar{k}, 1, 2 \ . \tag{1.37}$$

The states created by the creation operator $a^\dagger_k$ in the direction of the momentum k are not invariant

$$Q_s\, a^\dagger_k(\vec{k})|\Omega\rangle = \sqrt{2}|\vec{k}|c^\dagger(\vec{k})|\Omega\rangle \ \neq 0 \tag{1.38}$$

and do not belong to $\mathcal{N}$.

The space $\mathcal{N}$ is not yet acceptable because it contains nonvanishing zero-norm states

$$|f\rangle = \int \tilde{d}k\, f(\vec{k})\, a^\dagger_{\bar{k}}(\vec{k})|\Omega\rangle \ , \ \langle f|f\rangle = 0 \ , \ \text{because} \quad \left[a_{\bar{k}}(\vec{k}), a^\dagger_{\bar{k}}(\vec{k}')\right] = 0 \ . \tag{1.39}$$

To get rid of these states the following observation is crucial:

Theorem 1.1:
Scalar products of gauge invariant states $|\psi\rangle \in \mathcal{N}$ *and* $|\chi\rangle \in \mathcal{N}$ *remain unchanged if the state* $|\psi\rangle$ *is replaced by* $|\psi + Q_s\Lambda\rangle$.

Proof:

$$\langle\chi|\psi + Q_s\Lambda\rangle = \langle\chi|\psi\rangle + \langle\chi|Q_s\Lambda\rangle = \langle\chi|\psi\rangle \tag{1.40}$$

The term $\langle\chi|Q_s\Lambda\rangle$ vanishes, because Q_s is hermitean and $Q_s\chi = 0$.

We obtain the BRST algebra from the seemingly innocent requirement that $|\psi + Q_s\Lambda\rangle$ belongs to $\mathcal{N}$ whenever $|\psi\rangle$ does. The requirement seems natural because $|\psi + Q_s\Lambda\rangle$ and $|\psi\rangle$ have the same scalar products with gauge invariant states and therefore cannot be distinguished experimentally. It is, nevertheless, a very restrictive condition, because it requires Q_s^2 to vanish on each state $|\Lambda\rangle$, i.e. Q_s is required to be nilpotent,

$$Q_s^2 = 0 \ . \tag{1.41}$$

Then the space $\mathcal{N}$ of gauge invariant states decomposes into equivalence classes

$$|\psi\rangle \sim |\psi + Q_s\Lambda\rangle \ . \tag{1.42}$$

These equivalence classes are the physical states,

$$\mathcal{H}_{\text{phys}} = \frac{\mathcal{N}}{Q_s\mathcal{F}} = \{|\psi\rangle : |Q_s\psi\rangle = 0 \ , |\psi\rangle \bmod |Q_s\Lambda\rangle\} \ . \tag{1.43}$$

$\mathcal{H}_{\text{phys}}$ inherits a scalar product from $\mathcal{F}$ because by theorem 1.1 the scalar product in $\mathcal{N}$ does not depend on the representative of the equivalence class.

The construction of $\mathcal{H}_{\text{phys}}$ by itself does not guarantee that $\mathcal{H}_{\text{phys}}$ has a positive definite scalar product. This will hold only if Q_s acts on the space $\mathcal{F}$ in a suitable manner. One has to check this positive definiteness in each class of models.

In the case at hand, the zero-norm states $|f\rangle$ (1.39) are equivalent to 0 in $\mathcal{H}_{\text{phys}}$ if there exists a massless, real field $\bar{C}(x)$

$$\bar{C}(x) = e\int \tilde{d}k \left(e^{ikx}\bar{c}^\dagger(\vec{k}) + e^{-ikx}\bar{c}(\vec{k})\right)_{|_{k^0=\sqrt{\vec{k}^2}}} \tag{1.44}$$

and if Q_s transforms the one-particle states according to

$$Q_s\, \bar{c}^\dagger(\vec{k})\, |\Omega\rangle = \sqrt{2}\, i\, |\vec{k}|\, a^\dagger_{\bar{k}}(\vec{k})\, |\Omega\rangle \ . \tag{1.45}$$

For the six one-particle states we conclude that $\bar{c}^\dagger(\vec{k})|\Omega\rangle$ and $a^\dagger_k(\vec{k})|\Omega\rangle$ are not invariant (not in $\mathcal{N}$), $a^\dagger_{\bar{k}}(\vec{k})|\Omega\rangle$ and $c^\dagger(\vec{k})|\Omega\rangle$ are of the form $Q_s|\Lambda\rangle$ and equivalent to 0, the remaining two transverse creation operators generate the physical one particle space with positive norm.

Notice the following pattern: states from the Fock space $\mathcal{F}$ are excluded in pairs from the physical Hilbert space $\mathcal{H}_{\text{phys}}$, one state, $|n\rangle$, is not invariant

$$Q_s|n\rangle = |t\rangle \neq 0 \tag{1.46}$$

and therefore not contained in $\mathcal{N}$, the other state, $|t\rangle$, is trivial and equivalent to 0 in $\mathcal{H}_{\text{phys}}$ because it is the BRST transformation of $|n\rangle$.

The algebra $Q_s^2 = 0$ enforces

$$Q_s|t\rangle = 0 \ . \tag{1.47}$$

If one uses $|t\rangle$ and $|n\rangle$ as basis then Q_s is represented by the matrix

$$Q_s = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ . \tag{1.48}$$

This is one of the two possible Jordan block matrices which can represent a nilpotent operator $Q_s^2 = 0$. The only eigenvalue is 0, so a Jordan block consists of a matrix with zeros and with 1 only in the upper diagonal

$$Q_{s\,ij} = \delta_{i+1,j} \ . \tag{1.49}$$

Because of $Q_s^2 = 0$ the blocks can only have the size 1×1 or 2×2. In the first case the corresponding vector on which Q_s acts is invariant and not trivial and contributes to $\mathcal{H}_{\text{phys}}$. The second case is given by (1.48), the corresponding vectors are not physical.

It is instructive to consider the scalar product of the states on which Q_s acts. If it is positive definite then Q_s has to vanish because Q_s is hermitean and can be diagonalized in a space with positive definite scalar product. Thereby the nondiagonalizable 2×2 block (1.48) would be excluded. It is, however, in Fock spaces with indefinite scalar product that we need the BRST operator and there it can act nontrivially. In the physical Hilbert space, which has a positive definite scalar product, Q_s vanishes. Nevertheless the existence of the BRST operator Q_s in Fock space severely restricts the possible actions of the models we are going to consider.

Reconsider the doublet (1.46, 1.47): if the scalar product is nondegenerate then by a suitable choice of $|n\rangle$ and $|t\rangle$ it can be brought to the standard form

$$\langle n|n\rangle = 0 = \langle t|t\rangle \quad \langle t|n\rangle = \langle n|t\rangle = 1 \ . \tag{1.50}$$

This is an indefinite scalar product of Lorentzian type

$$|e_\pm\rangle = \frac{1}{\sqrt{2}}(|n\rangle \pm |t\rangle) \quad \langle e_+|e_-\rangle = 0 \quad \langle e_+|e_+\rangle = -\langle e_-|e_-\rangle = 1 \ . \tag{1.51}$$

By the definition (1.43) pairs of states with wrong sign norm and with acceptable norm are excluded from the space $\mathcal{H}_{\text{phys}}$ of physical states.

1.5 Gauge Parameter $\lambda \neq 1$

If the gauge parameter λ is different from 1, then the vectorfield has to satisfy the coupled equations of motion

$$\frac{1}{e^2}(\Box A_n + (\lambda - 1)\partial_n\partial_m A^m) = 0 \ , \tag{1.52}$$

which imply

$$\Box\Box A_m = 0 \tag{1.53}$$

and its Fourier transformed version $(p^2)^2\tilde{A}_m = 0$. Consequently the Fourier tansformed field $\tilde{A}$ vanishes outside the light cone and the general solution $\tilde{A}$ contains a δ-function and its derivative.

$$\tilde{A}_m = a_m(p)\delta(p^2) + b_m(p)\delta'(p^2) \tag{1.54}$$

However, the derivative of the δ function is ill defined because spherical coordinates $p^2, \nu, \vartheta, \varphi$ are discontinuous at $p = 0$.

To solve $\Box\Box\phi = 0$ one can restrict $\phi(t, \vec{x})$ to $\phi(t)e^{i\vec{k}\vec{x}}$, the general solution can then be obtained as a wavepacket which is superposed out of solutions of this form. $\phi(t)$ has to satisfy the ordinary differential equation

$$\left(\frac{d^2}{dt^2} + k^2\right)^2\phi = 0 \tag{1.55}$$

which has the general solution

$$\phi(t) = (a + b\,t)e^{ikt} + (c + d\,t)e^{-ikt} \ . \tag{1.56}$$

Therefore the equations (1.53) are solved by

$$A_n(x) = e\int \tilde{d}k\, e^{ikx}\Big(a_n^\dagger(\vec{k}) + x^0 b_n^\dagger(\vec{k})\Big) + e^{-ikx}\Big(a_n(\vec{k}) + x^0 b_n(\vec{k})\Big)_{|_{k^0=\sqrt{\vec{k}^2}}} \ . \tag{1.57}$$

This equation makes the vague notion $\delta'(p^2)$ explicit. The amplitudes b_n, $b_n^\dagger$ are determined from the coupled equations (1.52),

$$b_n^\dagger(\vec{k}) = -i\frac{\lambda - 1}{\lambda + 1}\frac{k_n k^m}{k_0}a_m^\dagger(\vec{k}) \ , \ b_n(\vec{k}) = i\frac{\lambda - 1}{\lambda + 1}\frac{k_n k^m}{k_0}a_m(\vec{k}) \ . \tag{1.58}$$

From (1.14) one can deduce that the commutation relations

$$[P^i, a_m^\dagger(\vec{k})] = k^i a_m^\dagger(\vec{k}) \ , \ [P^i, a_m(\vec{k})] = -k^i a_m(\vec{k}) \ , \ i = 1,2,3 \ , \tag{1.59}$$

$$\text{and} \qquad [P_0, a_m^\dagger(\vec{k})] = k_0\, a_m^\dagger(\vec{k}) - \frac{(\lambda - 1)}{(\lambda + 1)}\frac{k_m k^n}{k_0}a_n^\dagger(\vec{k}) \tag{1.60}$$

have to hold. If we decompose $a_m^\dagger(\vec{k})$ according to (1.29) then we obtain

$$[P_0, a_t^\dagger(\vec{k})] = k_0\, a_t^\dagger(\vec{k}) \ , \ t = 1,2 \ , \tag{1.61}$$

for the transverse creation operators and also

$$[P_0, a_{\bar{k}}^\dagger(\vec{k})] = k_0\, a_{\bar{k}}^\dagger(\vec{k}) \tag{1.62}$$

for the creation operator in direction of $\bar{k}$. For the creation operator in the direction of the four momentum k one gets

$$[P_0, a_k^\dagger(\vec{k})] = k_0\, a_k^\dagger(\vec{k}) - 2\,k_0\frac{\lambda - 1}{\lambda + 1}a_{\bar{k}}^\dagger(\vec{k}) \ . \tag{1.63}$$

In particular, for $\lambda \neq 1$, $a_k^\dagger(\vec{k})$ does not generate energy eigenstates and the hermitean operator P_0 cannot be diagonalized in Fock space because the commutation relations are

$$[P_0, a^\dagger] = M\,a^\dagger \tag{1.64}$$

with a matrix M which contains a nondiagonalizable Jordan block

$$M \sim k_0\begin{pmatrix} 1 & -2\frac{\lambda-1}{\lambda+1} \\ 0 & 1 \end{pmatrix} \ . \tag{1.65}$$

That hermitean operators are not guaranteed to be diagonalizable is of course related to the indefinite norm in Fock space. For operators O_{phys} which correspond to measuring devices it is sufficient that they can be diagonalized in the physical Hilbert space. This is guaranteed if $\mathcal{H}_{\text{phys}}$ has positive norm. In Fock space it is sufficient that operators O_{phys} commute with the BRST operator Q_s and that they satisfy generalized eigenvector equations

$$O_{\text{phys}}|\psi_{\text{phys}}\rangle = c|\psi_{\text{phys}}\rangle + Q_s|\chi\rangle \ , \ c \in \mathbb{R} \ , \tag{1.66}$$

from which the spectrum can be read off.

The Hamilton operator $H = P_0$ which results from the Lagrange density,

$$\mathcal{L} = -\frac{1}{4e^2}F_{mn}F^{mn} - \frac{\lambda}{2e^2}(\partial_m A^m)^2 \ , \tag{1.67}$$

$$H = \frac{1}{2e^2}\int d^3x \ : \Big((\partial_0 A_i)^2 - (\partial_i A_0)^2 + \frac{1}{2}(\partial_j A_i - \partial_i A_j)(\partial_j A_i - \partial_i A_j) - \\ - \lambda(\partial_0 A_0)^2 + \lambda(\partial_i A_i)^2\Big) : \ , \ i,j \in \{1,2,3\} \ , \tag{1.68}$$

can be expressed in terms of the creation and annihilation operators,

$$H = \int \tilde{d}k \, k_0 \Big(\sum_{t=1}^{2} a_t^\dagger a_t - \frac{2\lambda}{\lambda+1}\big(a_k^\dagger a_{\bar{k}} + a_{\bar{k}}^\dagger a_k - 2\frac{\lambda-1}{\lambda+1} a_{\bar{k}}^\dagger a_{\bar{k}}\big)\Big) \ . \tag{1.69}$$

H generates time translations (1.60) because the creation and annihilation operators fulfil the commutation relations

$$[a_m(\vec{k}), a_n^\dagger(\vec{k}')] = 2k^0(2\pi)^3\delta^3(\vec{k}-\vec{k}')\Big(-\eta_{mn} + \frac{\lambda-1}{2\lambda k^0}(\eta_{m0}k_n + \eta_{n0}k_m - \frac{k_m k_n}{k^0})\Big) \tag{1.70}$$

which follow from the requirement that the propagator

$$\langle\Omega|TA^m(x)A_n(0)\Omega\rangle = -i\,e^2 \lim_{\epsilon\to 0+}\int \frac{d^4p}{(2\pi)^4}\frac{e^{ipx}}{(p^2+i\epsilon)^2}\Big(p^2\delta^m{}_n - \frac{\lambda-1}{\lambda}p^m p_n\Big) \tag{1.71}$$

is the Green function corresponding to the equation of motion (1.52), which for positive (negative) times contains positive (negative) frequencies only. If one decomposes the creation and annihilation operators according to (1.29) then the transverse operators satisfy

$$[a_i(\vec{k}), a_j^\dagger(\vec{k}')] = 2k^0(2\pi)^3\delta^3(\vec{k}-\vec{k}')\,\delta_{ij} \ , \ i,j \in \{1,2\} \ . \tag{1.72}$$

They commute with the other creation annihilation operators which have the following off diagonal commutation relations

$$[a_{\bar{k}}(\vec{k}), a_k^\dagger(\vec{k}')] = [a_k(\vec{k}), a_{\bar{k}}^\dagger(\vec{k}')] = -\frac{\lambda+1}{2\lambda}2k^0(2\pi)^3\delta^3(\vec{k}-\vec{k}') \ . \tag{1.73}$$

The other commutators vanish.

Just as for $\lambda = 1$ the analysis of the BRST transformations leads again to the result that physical states are generated only by the transverse creation operators.

2 BRST Symmetry

2.1 Graded Commutative Algebra

To choose the physical states one could have proceeded like Cinderella and could pick acceptable states by hand or have them picked by doves. Prescribing the action of Q_s on one particle states (1.36, 1.45) is not really different from such an arbitrary approach. From (1.36, 1.45) we know nothing about physical multiparticle states. Moreover we would like to know whether one can switch on interactions which respect our definition of physical states. Interactions should give transition amplitudes which are independent of the choice (1.42) of the representative of physical states. The time evolution should leave physical states physical.

All these requirements can be satisfied if the BRST operator Q_s belongs to a symmetry. We interpret the equation $Q_s^2 = 0$ as a graded commutator, an anticommutator, of a fermionic generator of a Lie algebra

$$\{Q_s, Q_s\} = 0 \ . \tag{2.1}$$

To require that Q_s be fermionic means that the BRST operator transforms fermionic variables into bosonic variables and vice versa. In particular we take the vectorfield A to be a bosonic field. Then the fields C and $\bar{C}$ have to be fermionic though they are real scalar fields and carry no spin. They violate the spin statistics relation which requires physical fields with half-integer spin to be fermionic and fields with integer spin to be bosonic. However, the corresponding particles do not occur in physical states, they are ghosts. We call C the ghost field and $\bar{C}$ the antighost field. Because the ghost fields C and $\bar{C}$ anticommute they contribute, after introduction of interactions, to each loop with the opposite sign as compared to bosonic contributions. The ghosts compensate in loops for the unphysical bosonic degrees of freedom contained in the vectorfield A.

We want to realize the algebra (2.1) as local transformations on fields and to determine actions which are invariant under these transformations. From this invariant action one can construct the BRST operator as Noether charge corresponding to the symmetry of the action.

The transformations act on polynomials in bosonic and fermionic variables ϕ^i. This means: on the vector space of linear combinations of these variables, there acts a linear map, the Grassmann reflection Π, $\Pi^2 = 1$. Each linear combination ϕ can be uniquely decomposed into its bosonic part, $(\phi + \Pi\phi)/2$, which by definition is even (invariant) under Grassmann reflection, and into its fermionic part $(\phi - \Pi\phi)/2$, which by definition changes sign under Grassmann reflection. For simplicity, we assume the variables ϕ^i chosen such, that they are either bosonic or fermionic and introduce the grading $|\phi^i|$

modulo 2, such that $\Pi\phi^i = (-1)^{|\phi^i|}\phi^i$,

$$|\phi^i| = \begin{cases} 0 \text{ if } \phi^i \text{ is bosonic} \\ 1 \text{ if } \phi^i \text{ is fermionic .} \end{cases} \tag{2.2}$$

By assumption the bosonic and fermionic variables have an associative product and are graded commutative,

$$\phi^i\phi^j = (-1)^{|\phi^i|\cdot|\phi^j|}\phi^j\phi^i =: (-)^{ij}\phi^j\phi^i \ , \tag{2.3}$$

i.e. bosons commute with bosons and fermions, fermions commute with bosons and anticommute with fermions.

For readability we often use the shorthand notation

$$(-)^{ij} := (-1)^{|\phi^i|\cdot|\phi^j|} \ . \tag{2.4}$$

By linearity and the product rule $\Pi(AB) = \Pi(A)\Pi(B)$ the Grassmann reflection extends to polynomials. The grading of products is the sum of the gradings,

$$|\phi^i\phi^j| = |\phi^i| + |\phi^j| \bmod 2 \ . \tag{2.5}$$

Like the elementary variables, also each polynomial can be decomposed into its bosonic and its fermionic parts. These parts have a definite grading and are graded commutative

$$AB = (-1)^{|A|\cdot|B|}BA \ . \tag{2.6}$$

Transformations and symmetries are operations O acting linearly, i.e. term by term, on polynomials, [1]

$$O(\lambda_1 A + \lambda_2 B) = \lambda_1 O(A) + \lambda_2 O(B) \ . \tag{2.7}$$

They are uniquely specified by their action on bosonic and on fermionic polynomials and can be decomposed into bosonic operations, which map bosons to bosons and fermions to fermions,

$$|O_{\text{bosonic}}(A)| = |A| \ , \tag{2.8}$$

and fermionic operations, which maps bosons to fermions and fermions to bosons,

$$|O_{\text{fermionic}}(A)| = |A| + 1 \bmod 2 \ . \tag{2.9}$$

We consider only bosonic or fermionic operations. They have a natural grading,

$$|O| = |O(A)| - |A| \bmod 2 \ . \tag{2.10}$$

The grading of composite operations is the sum of the gradings

$$|O_1\, O_2| = |O_1| + |O_2| \bmod 2 \ . \tag{2.11}$$

[1] We deal with the graded commutative algebra of fields and their derivatives and distinguish operations, acting on the algebra, from operators, acting in some Fock space.

First order derivatives ν are linear operations with a graded Leibniz rule[2]

$$\nu(AB) = (\nu A)B + (-)^{|\nu|\cdot|A|}A(\nu B) \ . \tag{2.12}$$

They are completely determined by their action on elementary variables, ϕ^i: $\nu(\phi^i) = \nu^i$, i.e. $\nu = \nu^i\partial_i$. The partial derivatives ∂_i are naturally defined by

$$\partial_i\phi^j = \delta_i{}^j. \tag{2.13}$$

They have the same grading as their corresponding variables,

$$|\partial_i| = |\phi^i| \ , \ \partial_i\partial_j = (-)^{ij}\partial_j\partial_i \ . \tag{2.14}$$

The grading of the components ν^i results naturally $|\nu^i| = |\nu| + |\phi^i| \bmod 2$.

An example of a fermionic derivative is given by the exterior derivative

$$d = dx^m\partial_m \ , \ |d| = 1 \ . \tag{2.15}$$

It transforms coordinates x^m into differentials dx^m which have opposite statistics

$$|dx^m| = |x^m| + 1 \bmod 2 \tag{2.16}$$

and which, considered as multiplicative operations, commute with ∂_n

$$[\partial_n, dx^m] = 0 \ . \tag{2.17}$$

Therefore and because of (2.14) the exterior derivative is nilpotent

$$d^2 = 0 \ . \tag{2.18}$$

The graded commutator of operations O and P

$$[O, P] = O\,P - (-1)^{|O||P|}P\,O \tag{2.19}$$

(i.e. the anticommutator, if both O and P are fermionic, or the commutator, if O is bosonic or P is bosonic,) is linear in both arguments, graded antisymmetric

$$[O, P] = -(-1)^{|O||P|}[P, O] \ , \tag{2.20}$$

and satisfies the product rule

$$[O, P\,Q] = [O, P]\,Q + (-1)^{|O||P|}P\,[O, Q] \ . \tag{2.21}$$

The graded commutator of first order derivatives is a first order derivative, i.e. satisfies the Leibniz rule (2.12).

[2] This Leibniz rule defines left derivatives: the left factor A is differentiated without a graded sign.

2.2 Conjugation

Lagrange densities have to be real polynomials to make the corresponding S-matrix unitary. This is why we have to discuss complex conjugation. We define conjugation such that hermitean conjugation of a time ordered operator corresponding to some polynomial gives the anti time ordered operator corresponding to the conjugate polynomial. We therefore require for all variables ϕ^i and complex numbers λ_i

$$(\phi^{i\,*})^* = \phi^i \ , \tag{2.22}$$
$$(\lambda_i \phi^i)^* = \lambda_i^* \phi^{i\,*} \ , \tag{2.23}$$
$$(\phi^i \phi^j)^* = \phi^{j\,*}\phi^{i\,*} = (-)^{ij}\phi^{i\,*}\phi^{j\,*} \ . \tag{2.24}$$

As a consequence, conjugation preserves the grading,

$$|\phi^{i\,*}| = |\phi^i| \ , \tag{2.25}$$

and by additivity is defined on polynomials.

The conjugation of operations O is defined by

$$O^*(A) = (-)^{|O||A|}(O(A^*))^* \ . \tag{2.26}$$

This definition ensures that O^* is linear and satisfies the Leibniz rule if O is a first order derivative. Both requirements have to hold in order to allow first order derivatives and their Lie algebra to be real i.e. self conjugate.

The exterior derivative d is real, $d = d^*$, if the conjugate differentials are related to the differentials of the conjugate variables by

$$(dx^m)^* = (-)^{|x^m|} d((x^m)^*) \ . \tag{2.27}$$

The partial derivative with respect to a real fermionic variable is purely imaginary. Also the operator δ is purely imaginary,

$$\delta = x^m \frac{\partial}{\partial(dx^m)} \ , \ \delta^* = -\delta \ . \tag{2.28}$$

The anticommutator of d and δ can be evaluated with the product rule of the graded commutator (2.21) and with the elementary graded commutator

$$[\frac{\partial}{\partial\phi^i}, \phi^j] = \delta_i{}^j \ , \tag{2.29}$$

of the partial derivative and the operation, which multiplies with the variable ϕ^j,

$$\Delta = \{d, \delta\} = x^m \frac{\partial}{\partial x^m} + dx^m \frac{\partial}{\partial(dx^m)} = N_x + N_{dx} \ . \tag{2.30}$$

The anticommutator counts the variables x and dx and is real as one can check with

$$(O_1 O_2)^* = (-)^{|O_1||O_2|} O_1^* O_2^* \tag{2.31}$$

which follows from (2.26).

Conjugation does *not* reverse the order of two operations O_1 and O_2.

We can now specify the main properties of the BRST transformation s: It is a real, fermionic, nilpotent first order derivative,

$$s = s^* \ , \ |s| = 1 \ , \ s^2 = 0 \ , \ s(A\,B) = (s\,A)\,B + (-1)^{|A|} A\,s\,B \ . \tag{2.32}$$

It acts on Lagrange densities and functionals of fields. Space-time derivatives ∂_m of fields are limits of differences of fields taken at neighbouring arguments. It follows from the linearity of s that it has to commute with space-time derivatives

$$[s, \partial_m] = 0 \ . \tag{2.33}$$

Linearity implies moreover that the BRST transformation of integrals is given by the integral of the transformed integrand. Therefore the differentials dx^m are BRST invariant,[3]

$$s(dx^m) = 0 = \{s, dx^m\} \ , \ ([s, dx^m] = 0 \text{ for fermionic } x^m) \ . \tag{2.34}$$

Taken together the last two equations imply that s and d (2.15) anticommute

$$\{s, d\} = 0 \ . \tag{2.35}$$

2.3 Independence of the Gauge Fixing

In the simplest multiplet s transforms a real anticommuting field $\bar{C} = \bar{C}^*$, the antighost field, into $\sqrt{-1}$ times a real bosonic field $B = B^*$, the auxiliary field. These denominations anticipate the roles which the fields will play in Lagrange densities,

$$s\,\bar{C}(x) = iB(x) \ , \ s\,B(x) = 0 \ . \tag{2.36}$$

The BRST transformation which corresponds to an abelian gauge transformation acts on a real bosonic vectorfield A and a real, fermionic ghost field C by

$$s\,A_m(x) = \partial_m C(x) \ , \ s\,C(x) = 0 \ . \tag{2.37}$$

We can attribute to the fields

$$\phi = (\bar{C}, B, A, C) \tag{2.38}$$

and to s and $\partial = (\partial_0, \partial_1, \ldots, \partial_{D-1})$ a ghostnumber, which adds on multiplication

$$gh(\bar{C}) = -1, \ gh(B) = 0, \ gh(A) = 0, \ gh(C) = 1, \ gh(s) = 1, \ gh(\partial) = 0, \tag{2.39}$$
$$gh(M\,N) = gh(M) + gh(N) \ . \tag{2.40}$$

[3]The first equation applies to the element dx^m of the graded commutative algebra, the second to the operation, which multiplies elements of the algebra with dx^m.

Our analysis of the algebra (2.36, 2.37) in $D = 4$ dimensions[4] will show:
All Lagrangians of BRST invariant local actions

$$W[\phi] = \int d^4x\, (\mathcal{L} \circ \hat{\phi})(x) \tag{2.41}$$

with ghostnumber 0 have the form

$$\mathcal{L} = \mathcal{L}_{\text{inv}}(F, \partial F, \dots) - i\, s\, X(\phi, \partial\phi, \dots) \ . \tag{2.42}$$

The part $\mathcal{L}_{\text{inv}}$ is real and depends only on the field strength

$$F_{mn} = -F_{nm} = \partial_m A_n - \partial_n A_m \tag{2.43}$$

and its partial derivatives. Therefore it is invariant under classical gauge transformations. Typically it is given by (1.7)

$$\mathcal{L}_{\text{inv}}(A, \partial A) = -\frac{1}{4e^2} F_{mn} F^{mn} \ . \tag{2.44}$$

$X(\phi, \partial\phi, \dots)$ is a real, fermionic polynomial with ghostnumber -1. Therefore, it has to contain a factor $\bar{C}$. In the simplest case it is

$$X = \frac{\lambda}{e^2}\, \bar{C}\, (\frac{1}{2} B - \partial_m A^m) \ . \tag{2.45}$$

λ is the gauge fixing parameter. The piece $i\, s\, X$ contributes the gaugefixing for the vectorfield and contains the action of the ghostfields C and $\bar{C}$,

$$-i\, s\, X = \frac{\lambda}{2e^2}(B - \partial_m A^m)^2 - \frac{\lambda}{2e^2}(\partial_m A^m)^2 - i\frac{\lambda}{e^2}\bar{C}\partial_m \partial^m C \ . \tag{2.46}$$

This Langrange density makes B an auxiliary field, its equation of motion fixes it algebraically, $B = \partial_m A^m$. C and $\bar{C}$ are free fields (1.25, 1.44).
The Lagrangian is invariant under scale transformations T_a, $a \in \mathbb{R}$,

$$T_a \bar{C} = e^{-a} \bar{C} \ , \ T_a C = e^a C \ , T_a A_m = A_m \ , T_a B = B \ . \tag{2.47}$$

The corresponding Noether charge is the ghostnumber.
To justify the name gauge fixing for the gauge breaking part $-\frac{\lambda}{2e^2}(\partial_m A^m)^2$ of the Lagrange density we show that a change of the fermionic function X cannot be measured in amplitudes of physical states as long as such a change leads only to a differentiable perturbation of amplitudes. This means that gauge fixing and ghostparts of the Lagrange density are unobservable. Only the parameters in the gauge invariant part $\mathcal{L}_{\text{inv}}$ are measurable.

Theorem 2.1:
Transition amplitudes of physical states are independent of the gauge fixing within perturbatively connected gauge sectors.

[4] In odd dimensions also Chern Simons forms can occur.

Proof: If one changes X by δX then the Lagrange density and the action change by

$$\delta\mathcal{L} = i\, s\, \delta X \ , \ \ \delta W = i \int d^4x \ s\, \delta X \ . \tag{2.48}$$

S-matrix elements of physical states $|\chi\rangle$ and $|\psi\rangle$ change to first order by

$$\delta\langle\chi_{\text{in}}|\psi_{\text{out}}\rangle = \langle\chi_{\text{in}}|i \cdot i \int d^4x \ s\, \delta X|\psi_{\text{out}}\rangle \tag{2.49}$$

where $s\, \delta X$ is an operator in Fock space. The transformation $s\, \delta X$ of the operator δX is generated by i times the anticommutator of the fermionic operator δX with the fermionic BRST operator Q_s

$$\langle\chi_{\text{in}}|\, s \int d^4x\, \delta X|\psi_{\text{out}}\rangle = \langle\chi_{\text{in}}|\{i\, Q_s, \int d^4x\, \delta X\}|\psi_{\text{out}}\rangle \ . \tag{2.50}$$

This matrix element vanishes because $|\chi\rangle$ and $|\psi\rangle$ are physical (1.43) and Q_s is hermitean.
The proof does not exclude the possible existence of different sectors of gauge fixing which cannot be joined smoothly by changing the parameters.

2.4 Invariance and Anomalies

Using this theorem we can concisely express the restriction which the Lagrange density of a local, BRST invariant action in D dimensions has to satisfy.
It is advantageous to combine $\mathcal{L}$ with the differential $d^D x$ and consider the Lagrange density as a D-form [5] $\omega^0_D = \mathcal{L}\, d^D x$ with ghostnumber 0. The BRST transformation of the Lagrange density ω^0_D has to give a (possibly vanishing) total derivative $d\, \omega^1_{D-1}$.
With this notation the condition for an invariant local action is

$$s\, \omega^0_D + d\, \omega^1_{D-1} = 0 \ . \tag{2.51}$$

It is sufficient to determine this Lagrange density ω^0_D up to a piece of the form $s\eta^{-1}_D$, where η^{-1}_D carries ghostnumber -1. Such a piece contributes only to gaugefixing and to the ghostsector and cannot be observed. It is trivially BRST invariant because s is nilpotent. A total derivative part $d\eta^0_{D-1}$ of the Lagrange density contributes only boundary terms to the action and is also neglected. This means that we look for the solutions of the equation

$$s\, \omega^0_D + d\, \omega^1_{D-1} = 0 \ , \ \omega^0_D \bmod \ (s\eta^{-1}_D + d\eta^0_{D-1}) \ . \tag{2.52}$$

This is a cohomological equation, similar to (1.43) which determines the physical states. The equivalence classes of solutions ω^0_D of this equation span a linear space: the relative cohomology of s modulo d at ghostnumber 0 and form degree D.

[5] We indicate the ghostnumber by the superscript and denote the form degree by the subscript.

If we use a Lagrange density which solves this equation, then the action is invariant under the continuous symmetry

$$\phi \to \phi + \alpha\, s\, \phi \tag{2.53}$$

with an arbitrary fermionic parameter α. In classical field theory Noether's theorem guarantees that there exists a current j^m which is conserved as a consequence of the equations of motion,

$$\partial_m j^m = 0 \ . \tag{2.54}$$

The Noether charge (from which we strip the parameter α)

$$Q_s = \int d^3x\, j^0(t,x) \tag{2.55}$$

is independent of the time t and generates the nilpotent BRST transformations of functionals $A[\phi,\pi]$ of the phase space variables $\phi^i(x)$ and $\pi_i(x) = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi^i}(x)$ by the graded Poisson bracket

$$\{A,B\}_P = \int d^3x \left((-1)^{|i|(|i|+|A|)} \frac{\delta A}{\delta \phi^i(x)} \frac{\delta B}{\delta \pi_i(x)} - (-1)^{|i||A|} \frac{\delta A}{\delta \pi_i(x)} \frac{\delta B}{\delta \phi^i(x)} \right)$$
$$s\,A = \{Q_s, A\}_P \ . \tag{2.56}$$

If one investigates the quantized theory then in the simplest of all conceivable worlds the classical Poisson brackets would be replaced by (anti-) commutators of quantized operators. In particular the BRST operator Q_s would commute with the scattering matrix S,

$$S = \text{“}\, T\, e^{i\int d^4x\, \mathcal{L}_{\text{int}}}\,\text{”} \quad , \quad [Q_s, S] = 0 \ , \tag{2.57}$$

and scattering processes would map physical states unitarily to physical states

$$S\mathcal{H}_{\text{phys}} = \mathcal{H}_{\text{phys}} \ . \tag{2.58}$$

Classically an invariant action is sufficient to ensure this property. The perturbative evaluation of scattering amplitudes, however, suffers from the problem, that the S-matrix (2.57) has ill defined contributions from products of $\mathcal{L}_{\text{int}}(x_1)\dots\mathcal{L}_{\text{int}}(x_n)$ if arguments x_i and x_j coincide. Though upon integration $\int d^4x_1 \dots d^4x_n$ this is a set of measure zero these products of fields at coinciding space time arguments are the reason for all ultraviolet divergencies which emerge upon the naive application of the Feynman rules. More precisely the S-matrix is a time ordered series in $i\int d^4x\, \mathcal{L}_{\text{int}}$ and a set of prescriptions, indicated by the quotes in (2.57), to define in each order the products of $\mathcal{L}_{\text{int}}(x)$ at coinciding space-time points. To analyze these divergencies it is sufficient to consider only connected diagrams. In momentum space they decompose into products of one particle irreducible n-point functions $\tilde{G}_{1PI}(p_1,\dots,p_n)$ which define the effective action.

$$\begin{aligned}\Gamma[\phi] &= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n\, \phi(x_1)\dots\phi(x_n)\, G_{1PI}(x_1,\dots,x_n) \\ &= \int d^4x\, \mathcal{L}_0(\phi, \partial\phi, \dots) + \sum_{n\geq 1} \hbar^n \Gamma_n[\phi]\end{aligned} \tag{2.59}$$

To lowest order in $\hbar$ the effective action Γ is the classical action $\Gamma_0[\phi] = \int d^4x\, (\mathcal{L}_0 \circ \hat{\phi})(x)$. This is a local functional, in particular $\mathcal{L}_0$ is a series in the fields and a polynomial in the partial derivatives of the fields. The Feynman diagrams fix the expansion of the nonlocal effective action $\Gamma = \sum \hbar^n \Gamma_n$ up to local functionals which can be chosen in each loop order, i.e. the Lagrange density can be chosen as a series in $\hbar$.

$$\mathcal{L} = \mathcal{L}_0 + \sum_{n\geq 1} \hbar^n \mathcal{L}_n \tag{2.60}$$

The condition that the effective action be BRST invariant

$$s\,\Gamma[\phi] = 0 \tag{2.61}$$

has to be satisfied in each loop order. To lowest order it requires the Lagrange density $\mathcal{L}_0$ to be a solution of (2.52).

Assume the invariance condition to be satisfied up to n-loop order. The naive calculation of $n+1$-loop diagrams contains divergencies which make it necessary to introduce a regularization, e.g. the Pauli-Villars regularization, and counterterms (or use a prescription such as dimensional regularization or the BPHZL prescription which is a shortcut for regularization and counterterms). No regularization respects locality, unitarity and symmetries simultaneously, otherwise it would not be a regularization but an acceptable theory. The Pauli-Villars regularization is local. It violates unitarity for energies above the regulator masses and also because it violates BRST invariance. If one cancels the divergencies of diagrams with counterterms and considers the limit of infinite regulator masses then unitarity is obtained if the BRST symmetry guarantees the decoupling of the unphysical gauge modes. Locality was preserved for all values of the regulator masses. What about BRST symmetry?

One cannot argue that one has switched off the regularization and that therefore the symmetry should be restored. There is the phenomenon of hysteresis. A spherically symmetric iron ball exposed to a symmetry breaking magnetic field will usually not become spherically symmetric again if the magnetic field is switched off. Analogously in the calculation of Γ_{n+1} we have to be prepared that the regularization and the cancellation of divergencies by counterterms does not lead to an invariant effective action but rather to

$$s\,\Gamma = \hbar^{n+1} a + \sum_{k\geq n+2} \hbar^k a_k \ . \tag{2.62}$$

If the functional a cannot be made to vanish by an appropriate choice of $\mathcal{L}_{n+1}$ then the BRST symmetry is broken by the anomaly a.

Because s is nilpotent the anomaly a has to satisfy

$$s\,a = 0 \ . \tag{2.63}$$

This is the celebrated consistency condition of Wess and Zumino [5]. The consistency condition has acquired an outstanding importance because it allows to calculate all possible anomalies a as the general solution to $s\,a = 0$ and to check in each given model whether the anomaly actually occurs. At first sight one would not expect that the consistency equation has comparatively few solutions. The BRST transformation $a = s\,\Gamma$ of arbitrary functionals Γ satisfies $s\,a = 0$. The anomaly a, however, arises from the divergencies of Feynman diagrams where all subdiagrams are finite and compatible with BRST invariance. These divergencies can be isolated in parts of the n-point functions which depend polynomially on the external momenta, i.e. in local functionals. Therefore it turns out that the anomaly is a local functional.

$$a = \int d^4x\, \mathcal{A}^1(x, \phi(x), \partial\phi(x), \dots) \tag{2.64}$$

The anomaly density $\mathcal{A}^1$ is a jet function, i.e. a series in the fields ϕ and a polynomial in the partial derivatives of the fields comparable to a Lagrange density but with ghostnumber $+1$. The integrand $\mathcal{A}^1$ represents an equivalence class. It is determined only up to terms of the form $s\,\mathcal{L}$ because we are free to choose contributions to the Lagrange density at each loop order, in particular we try to choose $\mathcal{L}_{n+1}$ such that $s\,\mathcal{L}_{n+1}$ cancels $\mathcal{A}^1$ in order to make Γ_{n+1} BRST invariant. Moreover $d^4x\,\mathcal{A}^1$ is determined only up to derivative terms of the form $d\eta^1$.

$\mathcal{A}^1$ transforms into a derivative because the anomaly a satisfies the consistency condition. We combine the anomaly density $\mathcal{A}^1$ with d^Dx to a volume form ω^1_D and denote the ghostnumbers as superscripts and the form degree as subscript. Then the consistency condition and the description of the equivalence class read

$$s\,\omega^1_D + d\,\omega^2_{D-1} = 0\ ,\ \ \omega^1_D \bmod \left(s\eta^0_D + d\eta^1_{D-1}\right)\ . \tag{2.65}$$

This equation determines all possible anomalies. Its solutions depend only on the field content and the BRST transformations s and not on other particular properties of the model under consideration.

The determination of all possible anomalies is again a cohomological problem just as the determination of all BRST invariant local actions (2.52) but now with ghostnumbers shifted by $+1$. We will deal with both equations and consider the equation

$$s\,\omega^g_D + d\,\omega^{g+1}_{D-1} = 0\ ,\ \ \omega^g_D \bmod \left(s\eta^{g-1}_D + d\eta^g_{D-1}\right)\ , \tag{2.66}$$

for arbitrary ghostnumber g.

3 Cohomological Problems

3.1 Basic Lemma

In the preceding sections we have encountered repeatedly the cohomological problem to solve the linear equation

$$s\,\omega = 0\ ,\ \ \omega \bmod s\,\eta\ , \tag{3.1}$$

where s is a nilpotent operator $s^2 = 0$, acting on the elements of an algebra A. The equivalence classes of solutions ω form a linear space, the cohomology $H(A, s)$ of s. The equivalence classes of solutions ω^g_p of the problem

$$s\,\omega^g_p + d\,\omega^{g+1}_{p-1} = 0\ ,\ \ \omega^g_p \bmod s\,\eta^{g-1}_p + d\eta^g_{p-1}\ , \tag{3.2}$$

where $s^2 = 0 = d^2 = \{s, d\}$ form the relative cohomology $H^g_p(A, s|\,d)$ of s modulo d of ghostnumber g and form degree p.

Let us start to solve such equations and consider the problem to determine the physical multiparticle states. Multiparticle states can be written as a polynomial P of the creation operators acting on the vacuum

$$P(a^\dagger, c^\dagger, \bar{c}^\dagger)|\Omega\rangle\ , \tag{3.3}$$

if one neglects the notational complication that all these creation operators depend on momenta $\vec{k}$ and have to be smeared with normalizable functions. The BRST operator Q_s acts on these states in the same way as the fermionic derivative

$$s = \sqrt{2}|\vec{k}|\Big(i a^\dagger_k \frac{\partial}{\partial \bar{c}^\dagger} + c^\dagger \frac{\partial}{\partial a^\dagger_k}\Big) \tag{3.4}$$

acts on polynomials in commuting and anticommuting variables. For one particle states, i.e. linear homogeneous polynomials P, we had concluded that the physical states, the cohomology of Q_s with particle number 1, are generated by the transverse creation operators $a^\dagger_i$, i.e. by variables which are neither generated by s such as $a^\dagger_k$ or $c^\dagger$ nor transformed as $\bar{c}^\dagger$ and $a^\dagger_k$.

To investigate the action of s on polynomials, we simplify our notation and denote the variables with respect to which s differentiates collectively by x and their transformation by dx. Then the derivative s becomes the exterior derivative d (2.15). It maps the variables x to dx with opposite statistics (grading),

$$d = dx^m \frac{\partial}{\partial x^m}\ ,\ |dx^m| = |x^m| + 1 \mod 2\ . \tag{3.5}$$

The cohomology of the exterior derivative d acting on polynomials in x and dx is described by the basic lemma,

Theorem 3.1: *Basic Lemma*

$$\mathrm{d}\, f(x, \mathrm{d}x) = 0 \Leftrightarrow f(x, \mathrm{d}x) = f_0 + \mathrm{d}\, g(x, \mathrm{d}x) \ . \tag{3.6}$$

f_0 denotes the polynomial which is homogeneous of degree 0 in x and $\mathrm{d}x$ and is therefore independent of these variables.

Applied to the Fock space the basic lemma implies that physical n-particle states are generated by polynomials f_0 of creation operators which contain no operators $a^{\dagger}_{\bar{k}}$, $a^{\dagger}_{k}$, $c^{\dagger}$, $\bar{c}^{\dagger}$. Physical multiparticle states are generated by physical (transverse) creation operators $a^{\dagger}_{i}$, $i = 1, 2$.

This result seems to be trivial, but it is strikingly different from the consequences of a bosonic symmetry, e.g. a rotation of a vector with components (x, y) leaves the polynomial $x^2 + y^2$ invariant though neither x nor y are invariant.

The basic lemma determines all functions ω of the vector potential A, the ghost C and their derivatives, which are invariant under the abelian gauge transformation (2.37)

$$s A_m = \partial_m C \ , \ s\, C = 0 \ , \ s^2 = 0 \ , \ [s, \partial_m] = 0 \ . \tag{3.7}$$

The symmetrized partial derivatives of the vectorfield (symmetrization is indicated by the braces) are not invariant and transform into the derivatives of C,

$$s\, \partial_{(m_1} \dots \partial_{m_{k-1}} A_{m_k)} = \partial_{m_1} \dots \partial_{m_k} C \ . \tag{3.8}$$

By the basic lemma only trivial solutions of $s\,\omega = 0$ can depend on these variables. The nontrivial solutions depend on the remaining jet variables, the partial derivatives of antisymmetrized derivatives $F_{mn} = \partial_m A_n - \partial_n A_m$ and the undifferentiated ghost C. These variables are annihilated by s just as constants.

$$s\,\omega = 0 \Leftrightarrow \omega = f(C, F, \partial F, \partial \dots \partial F) + s\,\eta \ . \tag{3.9}$$

The sum is direct, because f and $s\,\eta$ depend on different variables,

$$f(C, F, \partial F, \partial \dots \partial F) + s\,\eta = 0 \Leftrightarrow f(C, F, \partial F, \partial \dots \partial F) = 0 \ \wedge \ s\,\eta = 0 \ . \tag{3.10}$$

To prove the basic lemma (3.6) we introduce the operation

$$\delta = x^m \frac{\partial}{\partial(\mathrm{d}x^m)} \ . \tag{3.11}$$

The anticommutator Δ of d and δ counts the variables x^m and $\mathrm{d}x^m$ (2.30),

$$\{\mathrm{d}, \delta\} = \Delta = x^m \frac{\partial}{\partial x^m} + \mathrm{d}x^m \frac{\partial}{\partial(\mathrm{d}x^m)} = N_x + N_{\mathrm{d}x} \ . \tag{3.12}$$

Because d is nilpotent it commutes with $\{\mathrm{d}, \delta\}$, no matter what δ is,

$$\mathrm{d}^2 = 0 \Rightarrow [\mathrm{d}, \{\mathrm{d}, \delta\}] = 0 \ . \tag{3.13}$$

Of course we can easily check explicitly that d does not change the overall number of variables x and $\mathrm{d}x$ in a polynomial. We can decompose each polynomial f into pieces f_n of definite homogeneity n in the variables x and $\mathrm{d}x$, i.e. $(N_x + N_{\mathrm{d}x}) f_n = n f_n$. Using (3.12) we can write f in the following form,

$$\begin{aligned} f &= f_0 + \sum_{n \geq 1} f_n = f_0 + \sum_{n \geq 1} (N_x + N_{\mathrm{d}x}) \frac{1}{n} f_n \\ &= f_0 + \mathrm{d}\Big(\delta \sum_{n \geq 1} \frac{1}{n} f_n\Big) + \delta\Big(\mathrm{d} \sum_{n \geq 1} \frac{1}{n} f_n\Big) \\ f &= f_0 + \mathrm{d}\eta + \delta\,\chi \ . \end{aligned} \tag{3.14}$$

This is the Hodge decomposition of an arbitrary polynomial in x and $\mathrm{d}x$ into a zero mode f_0, a d-exact[1] part $\mathrm{d}\eta$ and a δ-exact part $\delta\,\chi$. If f is d-closed, i.e. if it solves $\mathrm{d}\, f = 0$, then the equations $\mathrm{d}\, f_n = 0$ have to hold for each piece $\mathrm{d}\, f_n$ separately because the pieces are eigenpolynomials of $N_x + N_{\mathrm{d}x}$ with different eigenvalues and therefore linearly independent. But $\mathrm{d}\, f_n = 0$ implies that the last term in the Hodge decomposition, the δ-exact term, vanishes. This proves the lemma. If one writes $\frac{1}{n}$ as $\int_0^1 \mathrm{d}t\, t^{n-1}$ one obtains Poincaré 's lemma for forms in a star shaped domain

Theorem 3.2: *Poincaré 's lemma*

$$\mathrm{d}\, f(x, \mathrm{d}x) = 0 \Leftrightarrow f(x, \mathrm{d}x) = f(0, 0) + \mathrm{d}\,\delta \int_0^1 \frac{\mathrm{d}t}{t} (f(tx, t\mathrm{d}x) - f(0, 0)) \tag{3.15}$$

In this form the lemma is not restricted to polynomials but applies to all differentiable differential forms f which are defined along all rays tx for $0 \leq t \leq 1$ and all x, i.e. in a star shaped domain. Note that the integral is not singular at $t = 0$.

We chose to present the Poincaré lemma in the algebraic form – though it applies only to polynomials and to analytic functions – because we will follow a related strategy to solve the cohomological problems to come: given a nilpotent operation d we inspect operations δ and the anticommutators Δ. Only the zero modes of Δ can contribute to the cohomology of d.

3.2 Algebraic Poincaré Lemma

The basic lemma for forms with component functions which are functions of the base manifold does not apply to jet forms, i.e. differential forms ω with component functions which are functions of some jet space $\mathcal{J}_k$, $k < \infty$. The jet forms, which we consider, are series in fields ϕ, polynomials in derivatives of fields $\partial\phi, \partial\partial\phi, \dots, \partial \dots \partial\phi$, polynomials in $\mathrm{d}x$ and series in the coordinates x,

$$\omega : (x, \mathrm{d}x, \phi, \partial\phi, \partial\partial\phi, \dots) \to \omega(x, \mathrm{d}x, \phi, \partial\phi, \partial\partial\phi, \dots) \ . \tag{3.16}$$

[1] A polynomial g is called d-exact (or, shorter, exact, if the nilpotent operator d is evident) if it is of the form $g = \mathrm{d}\eta$ for some polynomial η.

Jet forms occur as integrands of local functionals. Because they depend polynomially on derivatives of fields they contain only terms with a finite number of derivatives, though there is no bound on the number of derivatives which is common to all forms ω.

We use curly brackets around a field to denote it and its derivatives

$$\{\phi\} = (\phi, \partial\phi, \partial\partial\phi, \dots) \ . \tag{3.17}$$

For Lagrange densities $\omega = \mathcal{L}(x, \{\phi\})\, d^D x$ the basic lemma cannot hold: they satisfy $d\,\omega = 0$ because they are volume forms, but they cannot be total derivatives, $\omega \neq d\eta$, if their Euler derivative does not vanish.

Let us show that constants and Lagrange densities with nonvanishing Euler derivative constitute the cohomology of the exterior derivative d in the space of jet forms.

The exterior derivative $d = dx^m \partial_m$ of jet forms differentiates the coordinates x. Acting on derivatives of a field, the partial derivatives $\partial_1, \partial_2 \dots$ map them to the next higher derivative with an additional label, just as a creation operator acts on a Fock state,

$$\partial_m x^n = \delta_m{}^n \ , \ \partial_m dx^m = 0 \ , \ \partial_k(\partial_l \dots \partial_m \phi) = \partial_k \partial_l \dots \partial_m \phi \ . \tag{3.18}$$

The jet variables satisfy no differential equation, i.e. $\partial_k \partial_l \dots \partial_m \phi$ are independent variables up to the fact that partial derivatives commute

$$\partial_k \dots \partial_m \phi = \partial_m \dots \partial_k \phi \ . \tag{3.19}$$

On these jet variables we define the operations t^n which annihilate a derivative and act like a derivative with respect to ∂_n, i.e. $t^n = \frac{\partial}{\partial(\partial_n)}$,

$$\begin{aligned} &t^n(x^m) = 0 \ , && t^n(dx^m) = 0 \ , \\ &t^n(\phi) = 0 \ , && t^n(\partial_{m_1} \dots \partial_{m_l} \phi) = \sum_{i=1}^{l} \partial_{m_1} \dots \partial_{m_{i-1}} \delta^n_{m_i} \partial_{m_{i+1}} \dots \partial_{m_l} \phi \ . \end{aligned} \tag{3.20}$$

The action of t^n on polynomials in the jet variables is defined by linearity and the Leibniz rule. Then t^n are vector fields on the jet space $\mathcal{J}$ which act on jet functions.

Obviously the operations t^n commute, $[t^m, t^n] = 0$. Less trivial is

$$[t^n, \partial_m] = \delta^n_m N_{\{\phi\}} \ . \tag{3.21}$$

$N_{\{\phi\}}$ counts the (differentiated) fields. The equation holds for linear polynomials, i.e. for the jet variables and coordinates and differentials, and extends to arbitrary polynomials because both sides of this equation satisfy the Leibniz rule.

To determine the cohomology of $d = dx^m \partial_m$ we consider separately forms ω with a fixed form degree p,

$$N_{dx} = dx^m \frac{\partial}{\partial(dx^m)} \ , \ N_{dx}\, \omega = p\, \omega \ , \tag{3.22}$$

which are homogeneous of degree N in $\{\phi\}$. We assume $N > 0$, the case $N = 0$ is covered by Poincaré 's lemma (theorem 3.2).

Consider the operation

$$b = t^m \frac{\partial}{\partial(dx^m)} \tag{3.23}$$

and calculate its anticommutator with the exterior derivative d as an exercise in graded commutators (2.19):

$$\begin{aligned} \{b, d\} &= \{t^m \frac{\partial}{\partial(dx^m)}, dx^n\} \partial_n - dx^n [t^m \frac{\partial}{\partial(dx^m)}, \partial_n] \\ &= t^m \delta_m{}^n \partial_n - dx^n \delta_n{}^m N \frac{\partial}{\partial dx^m} \\ &= \partial_n t^n + \delta_n{}^n N - N\, N_{dx} \ . \end{aligned} \tag{3.24}$$

So we get

$$\{d, b\} = N(D - N_{dx}) + P_1 \ . \tag{3.25}$$

$D = \delta^n_n$ is the dimension of the base manifold, the operator P_1 is

$$P_1 = \partial_k t^k \ . \tag{3.26}$$

Consider more generally the operations P_n

$$P_n = \partial_{k_1} \dots \partial_{k_n} t^{k_1} \dots t^{k_n} \tag{3.27}$$

which take away n derivatives and redistribute them afterwards. For each polynomial ω in the jet variables there exists a $\bar{n}(\omega)$ such that

$$\forall n > \bar{n}(\omega) : \ P_n \omega = 0 \ , \tag{3.28}$$

because each monomial of ω has a bounded number of derivatives.

Using the commutation relation (3.21) one proves the recursion relation

$$P_1 P_k = P_{k+1} + k N P_k \tag{3.29}$$

which can be used iteratively to express P_k in terms of P_1 and N

$$P_k = \prod_{l=0}^{k-1} (P_1 - l N) \ . \tag{3.30}$$

Using the argument (3.13) that a nilpotent operation commutes with all its anticommutators we conclude from (3.25)

$$[d, N(D - N_{dx}) + P_1] = 0 \ . \tag{3.31}$$

Therefore $d\,\omega = 0$ implies $d(P_1 \omega) = 0$ and from (3.30) we conclude $d(P_k \omega) = 0$. We use the relation (3.25) to express these closed forms $P_k \omega$ as exact forms up to terms $P_{k+1}\omega$.

$$\begin{aligned} d(b\,\omega) &= P_1 \omega + N(D - p)\, \omega \\ d(b\, P_k \omega) &= P_1 P_k \omega + N(D - p) P_k \omega \\ &= P_{k+1} \omega + k N P_k \omega + N(D - p) P_k \omega \\ d(b\, P_k \omega) &= P_{k+1} \omega + N(D - p + k) P_k \omega \ , \ k = 0, 1, \dots \end{aligned} \tag{3.32}$$

If $p < D$ then we can solve for ω in terms of exact forms $d(b\,\omega)$ and $P_1\omega$ which can be expressed as exact form and a term $P_2\omega$ and so on. This recursion terminates because $P_n\omega = 0\ \forall n \geq \bar{n}(\omega)$ (3.28). Explicitly we have for $p < D$ and $N > 0$:

$$d\,\omega = 0 \Rightarrow\ ,\ \omega = d\Big(b\sum_{k=0}^{\bar{n}(\omega)}\frac{(-)^k}{N^{k+1}}\frac{(D-p-1)!}{(D-p+k)!}P_k\omega\Big) = d\eta\ . \tag{3.33}$$

To complete the investigation of the cohomology of d we have to consider volume forms $\omega = \mathcal{L}\,d^Dx$. We treat separately pieces $\mathcal{L}_N$ which are homogeneous of degree $N > 0$ in the jet variables $\{\phi\}$. These pieces can be written as

$$\begin{aligned} N\mathcal{L}_N &= \phi^i\frac{\partial\mathcal{L}_N}{\partial\phi^i} + \partial_m\phi^i\frac{\partial\mathcal{L}_N}{\partial(\partial_m\phi^i)} + \dots \\ &= \phi^i\frac{\hat{\partial}\mathcal{L}_N}{\hat{\partial}\phi^i} + \partial_m X^m_N\ ,\ X^m_N = \phi^i\frac{\partial\mathcal{L}_N}{\partial(\partial_m\phi^i)} + \dots\ . \end{aligned} \tag{3.34}$$

Here we use the notation (1.5)

$$\frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\phi^i} = \frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_m\frac{\partial\mathcal{L}}{\partial(\partial_m\phi^i)} + \dots \tag{3.35}$$

for the Euler derivative of the Lagrange density with respect to ϕ^i. The dots denote terms which come from higher derivatives. The derivation of (3.34) is analogous to the derivation of the Euler Lagrange equations from the action principle. Eq.(3.34) implies that the volume form $\omega_N = \mathcal{L}_N\,d^Dx$ is an exact term and a piece proportional to the Euler derivative

$$\mathcal{L}_N\,d^Dx = \frac{1}{N}\phi^i\frac{\hat{\partial}\mathcal{L}_N}{\hat{\partial}\phi^i}d^Dx + d\Big(\frac{1}{N}X^m_N\frac{\partial}{\partial(dx^m)}d^Dx\Big)\ . \tag{3.36}$$

If we combine this equation with Poincaré's lemma (theorem 3.2) and with (3.33), combine terms with different degrees of homogeneity N and different form degree p we obtain the algebraic Poincaré lemma for forms of the coordinates, differentials and jet variables

Theorem 3.3: *Algebraic Poincaré Lemma*

$$\begin{aligned} &d\,\omega(x, dx, \{\phi\}) = 0 \quad \Leftrightarrow \\ &\omega(x, dx, \{\phi\}) = \text{const} + d\eta(x, dx, \{\phi\}) + \mathcal{L}(x, \{\phi\})\,d^Dx\ . \end{aligned} \tag{3.37}$$

The Lagrange form $\mathcal{L}(x, \{\phi\})\,d^Dx$ is trivial, i.e. of the form $d\eta$, if and only if its Euler derivative vanishes identically in the fields.

The algebraic Poincaré lemma does not hold if the base manifold is not starshaped or if the fields ϕ take values in a topologically nontrivial target space. In these cases the operations $\delta = x\frac{\partial}{\partial(dx)}$ and $b = t^n\frac{\partial}{\partial(dx^n)}$ cannot be defined because a relation like $x \cong x + 2\pi$, which holds for the coordinates on a circle, would lead to the contradiction $0 \cong 2\pi\frac{\partial}{\partial(dx)}$. Here we restrict our investigations to topologically trivial base manifolds and topologically trivial target spaces. It is the topology of the invariance groups and the Lagrangian solutions in the algebraic Poincaré lemma which give rise to a nontrivial cohomology of the exterior derivative d and the BRST transformation s.

The operations t^m and b used in the above proof of the algebraic Poincaré lemma (3.37) do not affect the dependence on the variables x^m. Therefore the algebraic Poincaré lemma holds for $N > 0$ also for the part of d which differentiates the fields only but not the x^m. Hence, using the decomposition

$$d = d_x + d_\phi\ ,\quad d_x = dx^m\frac{\partial}{\partial x^m}\ , \tag{3.38}$$

where d_ϕ denotes the part of d which differentiates only the fields, we obtain:

Theorem 3.4: *Algebraic Poincaré Lemma for* d_ϕ

$$\begin{aligned} &d_\phi\,\omega(x, dx, \{\phi\}) = 0 \quad \Leftrightarrow \\ &\omega(x, dx, \{\phi\}) = \chi(x, dx) + d_\phi\,\eta(x, dx, \{\phi\}) + \mathcal{L}(x, \{\phi\})\,d^Dx. \end{aligned} \tag{3.39}$$

The algebraic Poincaré lemma is modified if the jet space contains in addition variables which are space time constants. This occurs for example if one treats rigid transformations as BRST transformations with constant ghosts C, i.e. $\partial_m C = 0$. If these ghosts occur as variables in forms ω then they are not counted by the number operators N which have been used in the proof of the algebraic Poincaré lemma and can appear as variables in η, in $\mathcal{L}$ and in the constant solution of $d\,\omega = 0$.

3.3 Descent Equation

We are now prepared to investigate the relative cohomology and derive the so called descent equations. We recall that we deal with two nilpotent derivatives, the exterior derivative d and the BRST transformation s, which anticommute with each other

$$d^2 = 0\ ,\ s^2 = 0\ ,\ \{s, d\} = 0\ . \tag{3.40}$$

s leaves the form degree N_{dx} invariant, d raises it by 1

$$[N_{dx}, s] = 0\ ,\ [N_{dx}, d] = d\ . \tag{3.41}$$

To derive necessary conditions on the solution of (2.51)

$$s\,\omega_D + d\,\omega_{D-1} = 0\ ,\ \omega_D \mod (s\eta_D + d\eta_{D-1})\ , \tag{3.42}$$

where the subscript denotes the form degree, we apply s and use (3.40)

$$0 = s(s\,\omega_D + d\,\omega_{D-1}) = s\,d\,\omega_{D-1} = -\,d(s\,\omega_{D-1})\ . \tag{3.43}$$

By (3.37) $s\,\omega_{D-1}$ is of the form const$+d\eta(\{\phi\})+\mathcal{L}(\{\phi\})\,d^Dx$. The piece $\mathcal{L}(\{\phi\})\,d^Dx$ has to vanish because ω_{D-1} has form degree $D-1$ and if $D>1$ then also the constant piece vanishes because ω_{D-1} contains $D-1>0$ differentials and is not constant. Therefore we conclude

$$s\,\omega_{D-1}+d\,\omega_{D-2}=0\ ,\ \omega_{D-1}\ \mathrm{mod}\ (s\eta_{D-1}+d\eta_{D-2}) \tag{3.44}$$

where we denoted η by ω_{D-2} to indicate its form degree. Adding to ω_{D-1} a piece of the form $s\eta_{D-1}+d\eta_{D-2}$ changes ω_D only within its class of equivalent representatives. Therefore ω_{D-1} is naturally a representative of an equivalence class. From (3.42) we have derived (3.44) which is nothing but (3.42) with form degree lowered by 1. Iterating the arguments we lower the form degree step by step and obtain the descent equations

$$s\,\omega_i+d\,\omega_{i-1}=0\ ,\ i=D,D-1,\ldots,1\ ,\ \omega_i\ \mathrm{mod}\,(s\eta_i+d\eta_{i-1}) \tag{3.45}$$

until the form degree drops to zero. It cannot become negative. For $i=0$ one has

$$s\,\omega_0=\ \text{const}\ ,\ \omega_0\ \mathrm{mod}\ s\eta_0 \tag{3.46}$$

because this is the solution to $d\,s\,\omega_0=0$ for 0-forms.

If, however, the BRST transformation is not spontaneously broken i.e. if s does not transform fields into numbers, $s\,\phi_{|(\phi=0)}=0$, then $s\,\omega_0$ has to vanish. This follows most easily if one evaluates both sides of $s\,\omega_0=$ const for vanishing fields. We assume for the following that the BRST transformations are not spontaneously broken,

$$s\,\omega_0=0\ ,\ \omega_0\ \mathrm{mod}\ s\eta_0\ . \tag{3.47}$$

We will exclude from our considerations also spontaneously broken rigid symmetries. There we cannot apply these arguments because $s\,\phi_{|(\phi=0)}=C$ gives ghosts which are space time constant and $s\,\omega_0=f(C)\neq 0$ can occur.

The descent equations (3.45, 3.47) are just another cohomological equation for a nilpotent operator $\tilde{s}$ and a form $\tilde{\omega}$

$$\tilde{s}=s+d\ ,\ \tilde{s}^2=0\ , \tag{3.48}$$

$$\tilde{\omega}=\sum_{i=0}^{D}\omega_i\ ,\ \tilde{\eta}=\sum_{i=0}^{D}\eta_i\ , \tag{3.49}$$

$$\tilde{s}\ \tilde{\omega}=0\ ,\ \tilde{\omega}\,\mathrm{mod}\ \tilde{s}\tilde{\eta}\ . \tag{3.50}$$

That $\tilde{s}$ is nilpotent follows from (3.40). The descent equations (3.45, 3.47) imply $\tilde{s}\,\tilde{\omega}=0$ and $\tilde{\omega}$ is equivalent to $\tilde{\omega}+\tilde{s}\,\eta$. So (3.50) is a consequence of the descent equations. On the other hand if (3.48) holds then the equation (3.50) implies the descent equations. This follows if one splits $\tilde{s}$, $\tilde{\omega}$ and $\tilde{\eta}$ with respect to the form degree (3.41).

Theorem 3.5:
Let $\tilde{s}=s+d$ be a sum of two nilpotent, anticommuting fermionic derivatives where s preserves the form degree and d raises it by one, then each solution $(\omega_0,\ldots,\omega_D)$ of the descent equations

$$s\,\omega_i+d\,\omega_{i-1}=0\ ,\ i=0,1,\ldots D\ ,\ \omega_i\ \mathrm{mod}\,(s\eta_i+d\eta_{i-1})\ , \tag{3.51}$$

corresponds one to one to an element $\tilde{\omega}=\sum\omega_i$ of the cohomology

$$H(\tilde{s})=\{\tilde{\omega}:\ \tilde{s}\ \tilde{\omega}=0\ ,\ \tilde{\omega}\,\mathrm{mod}\ \tilde{s}\tilde{\eta}\}\ . \tag{3.52}$$

The forms ω_i are the parts of $\tilde{\omega}$ with form degree i.

The formulation of the descent equations as a cohomological problem of the operator $\tilde{s}$ has several virtues. The solutions to $\tilde{s}\ \tilde{\omega}=0$ can obviously be multiplied to obtain further solutions. They form an algebra, not just a vector space. Moreover, for the BRST operator in gravitational Yang Mills theories we will find that the equation $\tilde{s}\ \tilde{\omega}=0$ can be cast into the form $s\,\omega=0$ by a change of variables, where s is the original BRST operator. This equation has to be solved anyhow as part of the descent equations. Once one has solved it one can recover the complete solution of the descent equations, in particular one can read off ω_D as the D form part of $\tilde{\omega}$. These virtues justify to consider with $\tilde{\omega}$ a sum of forms of different form degrees which in traditional eyes would be considered to add apples and oranges.

3.4 Künneth's Theorem

If the nilpotent derivative d acts on a tensor product

$$A=A_1\otimes A_2 \tag{3.53}$$

of vectorspaces which are separately invariant under d

$$d\,A_1\subset A_1\ ,\ d\,A_2\subset A_2\ , \tag{3.54}$$

then Künneth's theorem states that the cohomology $H(A,d)$ of d acting on A is given by the product of the cohomology $H(A_1,d)$ of d acting on A_1 and $H(A_2,d)$ of d acting on A_2.

Theorem 3.6: *Künneth-formula*
Let $d=d_1+d_2$ be a sum of nilpotent differential operators which leave their vectorspaces A_1 and A_2 invariant

$$d_1\,A_1\subset A_1\ ,\ d_2\,A_2\subset A_2 \tag{3.55}$$

and which are defined on the tensor product $A=A_1\otimes A_2$ by the Leibniz rule

$$d_1(kl)=(d_1\,k)l\ ,\ d_2(kl)=(-)^{|k|}k(d_2\,l)\ ,\ \forall k\in A_1,\ l\in A_2\ . \tag{3.56}$$

Then the cohomology $H(A,d)$ of d acting on A is the tensor product of the cohomologies of d_1 acting on A_1 and d_2 acting on A_2

$$H(A_1\otimes A_2,d_1+d_2)=H(A_1,d_1)\otimes H(A_2,d_2)\ . \tag{3.57}$$

The formula justifies to count numbers as nontrivial solution of $\mathrm{d}\,\omega = 0$ rather than to exclude them for simplicity from the definition of $H(A, \mathrm{d})$.

To prove the theorem we consider an element $f \in H(\mathrm{d})$

$$f = \sum_i k_i l_i \tag{3.58}$$

given as a sum of products of elements $k_i \in A_1$ and $l_i \in A_2$. Without loss of generality we assume that the elements k_i are taken from a basis of A_1 and the elements l_i are taken from a basis of A_2.

$$\sum c_i k_i = 0 \Leftrightarrow c_i = 0 \;\forall\; i \tag{3.59}$$

$$\sum c_i l_i = 0 \Leftrightarrow c_i = 0 \;\forall\; i \tag{3.60}$$

Otherwise one has a relation like $l_1 = \sum_i' \alpha_i l_i$ or $k_1 = \sum_i' \beta_i k_i$, where $\sum'$ does not contain $i = 1$, and can rewrite f with fewer terms $f = \sum_i'(k_i + \alpha_i k_1) \cdot l_i$ or $f = \sum_i' k_i \cdot (l_i + \beta_i l_1)$. We can even choose $f \in H(\mathrm{d})$ in such a manner that the elements k_i are taken from a basis of a complement to the space $\mathrm{d}_1 A_1$. In other words we can choose f such that no linear combination of the elements k_i combines to a d_1-exact form.

$$\sum_i c_i k_i = \mathrm{d}_1 g \Leftrightarrow \mathrm{d}_1 g = 0 = c_i \;\forall\; i \tag{3.61}$$

Otherwise we have a relation like $k_1 = -\mathrm{d}_1 \kappa + \sum_i' \beta_i k_i$, where $\sum'$ does not contain $i = 1$, and we can rewrite $f \in H(\mathrm{d})$ up to an irrelevant piece as sum of products with elements $k_i' = \kappa, k_2, \ldots$, where κ is not in $\mathrm{d}_1 A_1$,

$$f = (-)^{|\kappa|} \kappa \,\mathrm{d}_2\, l_1 + \sum_{i>1} k_i \cdot (l_i + \beta_i l_1) - \mathrm{d}(\kappa\; l_1) \; . \tag{3.62}$$

We can iterate this argument until no linear combination of the elements k_i' combines to a d_1-exact form.

By assumption f solves $\mathrm{d}\, f = 0$ which implies

$$\sum_i \Big((\mathrm{d}_1\, k_i) l_i + (-)^{k_i} k_i (\mathrm{d}_2\, l_i)\Big) = 0 \; . \tag{3.63}$$

In this sum $\sum_i (\mathrm{d}_1\, k_i) l_i$ and $\sum_i (-)^{k_i} k_i (\mathrm{d}_2\, l_i)$ have to vanish separately because the elements k_i are linearly independent from the elements $\mathrm{d}_1\, k_i \in \mathrm{d}_1 A_1$. $\sum_i (\mathrm{d}_1\, k_i) l_i = 0$, however, implies

$$\mathrm{d}_1\, k_i = 0 \tag{3.64}$$

because the elements l_i are linearly independent and $\sum_i (-)^{k_i} k_i (\mathrm{d}_2\, l_i) = 0$ leads to

$$\mathrm{d}_2\, l_i = 0 \tag{3.65}$$

analogously. So we have shown

$$\mathrm{d}\, f = 0 \Rightarrow f = \sum_i k_i l_i + \mathrm{d}\,\chi \text{ where } \mathrm{d}_1\, k_i = 0 = \mathrm{d}_2\, l_i \;\forall i \; . \tag{3.66}$$

Changing k_i and l_i within their equivalence class $k_i \bmod \mathrm{d}_1\, \kappa_i$ and $l_i \bmod \mathrm{d}_2\, \lambda_i$ does not change the equivalence class $f \bmod \mathrm{d}\,\chi$:

$$\sum_i (k_i + \mathrm{d}_1\, \kappa_i)(l_i + \mathrm{d}_2\, \lambda_i) = \sum_i k_i l_i + \mathrm{d} \sum_i \Big(\kappa_i (l_i + \mathrm{d}_2\, \lambda_i) + (-)^{k_i} k_i \lambda_i\Big) \tag{3.67}$$

Therefore $H(A, \mathrm{d})$ is contained in $H_1(A_1, \mathrm{d}_1) \otimes H_2(A_2, \mathrm{d}_2)$. On the other hand, the inclusion $H_1(A_1, \mathrm{d}_1) \otimes H_2(A_2, \mathrm{d}_2) \subset H(A, \mathrm{d})$ is trivial. This concludes the proof of Künneth's theorem.

4 BRST Algebra of Gravitational Yang Mills Theories

4.1 Covariant Operations

Gauge theories such as gravitational Yang Mills theories rely on tensor analysis. The set of tensor components is a subalgebra of the polynomials in the graded commutative jet variables.

$$\Big(\text{Tensors}\Big) \subset \Big(\text{Polynomials}\,(\phi, \partial\phi, \partial\partial\phi, \dots)\Big) \tag{4.1}$$

The covariant operations Δ_M which occur in tensor analysis

$$\Delta_M : \Big(\text{Tensors}\Big) \to \Big(\text{Tensors}\Big) \tag{4.2}$$

map tensors to tensors and satisfy the graded Leibniz rule (2.12). These covariant operations have a basis consisting of the real bosonic covariant space time derivatives $D_a,\ a = 0, \dots, D-1$, the complex covariant, fermionic spinor derivatives $D_\alpha,\ D_{\dot\alpha} = (D_\alpha)^*$, in supergravitational theories and real bosonic spin and isospin transformations δ_i, which correspond to a basis of the Lie algebra of the gauge group and of the Lorentz group, possibly including dilatations and so-called R-transformations

$$(\Delta_M) = (D_a,\ D_\alpha,\ D^{\dot\alpha},\ \delta_i)\ . \tag{4.3}$$

The grading of the covariant operations can be red off from the index, $|\Delta_M| = |M|$, only spinor derivatives are fermionic.

By assumption, the space of covariant operations is closed with respect to graded commutation: the graded commutator of covariant operations is a covariant operation which can be linearly combined from the basic covariant operations,

$$[\Delta_M, \Delta_N] := \Delta_M\Delta_N - (-)^{MN}\Delta_N\Delta_M = \mathcal{F}_{MN}{}^K\Delta_K\ . \tag{4.4}$$

with structure functions

$$\mathcal{F}_{MN}{}^K = -(-)^{|M||N|}\mathcal{F}_{NM}{}^K \tag{4.5}$$

which are components of graded antisymmetric tensorfields and which are graded according to their index picture, $|\mathcal{F}_{MN}{}^K| = |M| + |N| + |K|$.

We raise and lower spinor indices with $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} = \epsilon_{\dot\alpha\dot\beta}$, $\epsilon_{12} = 1$, $Y_\alpha = \epsilon_{\alpha\beta}Y^\beta$, $Y_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}Y^{\dot\beta}$ and use the summation convention

$$X^M Y_M := X^a Y_a + X^\alpha Y_\alpha + X_{\dot\alpha} Y^{\dot\alpha} + X^i Y_i\ , \tag{4.6}$$

that in a spinor sum the first *un*dotted index is *up* and the first *do*tted index is *do*wn. If then the components are graded according to their index picture

$$|X^M| = |X| + |M| \ , \ |Y_M| = |Y| + |M| \tag{4.7}$$

and define components of the conjugate quantities $\bar{X}$ and $\bar{Y}$ by

$$\bar{X}^{\bar{M}} = (-)^{|X|+|M|}(X^M)^* \ , \ \bar{Y}_{\bar{M}} = (-)^{|Y|+|M|}(Y_M)^* \tag{4.8}$$

then because of $(-)^{|M|}\bar{X}^{\bar{M}}\bar{Y}_{\bar{M}} = X^M Y_M$ the conjugate of the sum turns out to be the sum over the conjugate products,

$$(X^M Y_M)^* = (-)^{|X||Y|}\bar{X}^M \bar{Y}_M \ . \tag{4.9}$$

The structure functions turn out to be graded real,

$$(\mathcal{F}_{MN}{}^K)^* = (-)^{|K|(|M|+|N|)+|M||N|}\mathcal{F}_{\bar{M}\bar{N}}{}^{\bar{K}} \ , \tag{4.10}$$

i.e. conjugation maps the graded commutator algebra to itself, it is real.

Some of the structure functions have purely numerical values as for example the structure constants in the commutators of infinitesimal Lorentz or isospin transformations

$$[\delta_i, \delta_j] = f_{ij}{}^k \delta_k \ . \tag{4.11}$$

Other constant structure functions are the elements of matrices G_i, which represent isospin or Lorentztransformations on the covariant space time derivatives

$$[\delta_i, D_a] = -G_{i\,a}{}^b D_b \ . \tag{4.12}$$

Other components of the tensors $\mathcal{F}_{MN}{}^K$ are given by the Riemann curvature, the Yang Mills field strength and in supergravity the Rarita Schwinger field strength and auxiliary fields of the supergravitational multiplet. We use the word field strength also to denote the Riemann curvature and the Yang Mills field strength collectively.

The commutator algebra (4.4) implies the Jacobi identity. If we denote the graded sum over the cyclic permutations of an expression X_{MNP} by

$$\sum\!\!\!\!\!\!\bigcirc_{MNP} X_{MNP} := X_{MNP} + (-)^{|M|(|N|+|P|)}X_{NPM} + (-)^{|P|(|M|+|N|)}X_{PMN} \ , \tag{4.13}$$

then the Jacobi identity can be written as

$$\sum\!\!\!\!\!\!\bigcirc_{MNP} [\Delta_M, [\Delta_N, \Delta_P]] = 0 \ . \tag{4.14}$$

Inserting (4.4) one obtains the first Bianchi identity for the structure functions

$$\sum\!\!\!\!\!\!\bigcirc_{MNP} (\Delta_M \mathcal{F}_{NP}{}^K - \mathcal{F}_{MN}{}^L \mathcal{F}_{LP}{}^K) = 0 \ . \tag{4.15}$$

The covariant operations are not defined on arbitrary polynomials of the jet variables. In particular one cannot realize the commutator algebra (4.4) on connections, on ghosts or on auxiliary fields.

To keep the discussion simple we will not consider fermionic covariant derivatives in the following. Then the commutator algebra (4.4) has more specifically the structure given by (4.11) and (4.12) and

$$[D_a, D_b] = -T_{ab}{}^c D_c + F_{ab}{}^i \delta_i \ . \tag{4.16}$$

We will simplify this algebra even more and choose the spin connection by the requirement that the torsion $T_{ab}{}^c$ vanishes.

4.2 Transformation and Exterior Derivative

The fields ϕ in gravitational Yang Mills theories are the ghosts C^N, antighosts $\bar{C}^N$, auxiliary fields B^N, gauge fields $A_m{}^N$, $m = 0, \ldots, D-1$, also called connections, and elementary tensor fields T. The gauge potentials, ghosts and auxiliary fields are real and correspond to a basis of the covariant operations Δ_M, i.e. there are connections, ghosts and auxiliary fields for translations, Lorentz transformations and isospin transformations. Matter fields are tensors and denoted by T.

$$\phi = (C^N, \bar{C}^N, B^N, A_m{}^N, T) \tag{4.17}$$

We define the BRST transformation of the antighosts and the auxiliary fields by

$$s\,\bar{C}^N = iB^N \ , \ s\,B^N = 0 \ . \tag{4.18}$$

The BRST transformation of tensors is a sum of covariant operations with ghosts as coefficients [6]

$$s\,T = -C^N \Delta_N T \ . \tag{4.19}$$

We require that partial derivatives ∂_m of tensors can be expressed as a linear combination of covariant operations with coefficients which by their definition are the connections or gauge fields. For the exterior derivative this means

$$\begin{aligned} d\,T &= dx^m \partial_m T = -dx^m A_m{}^N \Delta_N T = -A^N \Delta_N T \ , \\ A^N &= dx^m A_m{}^N \ . \end{aligned} \tag{4.20}$$

s acts on tensors strikingly similar to d: $s\,T$ contains ghosts C^N where $d\,T$ contains composite connection one forms A^N.

Let us check that (4.20) is nothing but the usual definition of covariant derivatives. We spell out the sum over covariant operations and denote the connection $-A_m{}^a$ which correspond to covariant space-time derivatives by $e_m{}^a$, the vielbein,

$$e_m{}^a = -A_m{}^a \ . \tag{4.21}$$

The index i enumerates a basis of spin and isospin transformations,

$$\partial_m = -A_m{}^N \Delta_M = e_m{}^a D_a - A_m{}^i \delta_i \ . \tag{4.22}$$

If the vielbein has an inverse $E_a{}^m$, which we take for granted,

$$e_m{}^a E_a{}^n = \delta_m{}^n \ , \tag{4.23}$$

then we can solve for the covariant space time derivative and obtain as usual

$$D_a = E_a{}^m(\partial_m + A_m{}^i \delta_i) \ . \tag{4.24}$$

We require that s and d anticommute and be nilpotent (3.40). This fixes the BRST transformation of the ghosts and the connection and identifies the curvature and field strength. In particular $s^2 = 0$ implies

$$0 = s^2\, T = s\,(-C^N \Delta_N T) = -(s\, C^N)\, \Delta_N T + C^N s(\Delta_N T) \ . \tag{4.25}$$

$\Delta_N T$ is a tensor so

$$C^N s\,(\Delta_N T) = -C^N C^M \Delta_M \Delta_N T = -\frac{1}{2} C^N C^M [\Delta_M, \Delta_N] T \ . \tag{4.26}$$

The commutator is given by the algebra (4.4) and we conclude

$$0 = (s\, C^N + \frac{1}{2} C^K C^L \mathcal{F}_{LK}{}^N)\Delta_N T \ , \ \forall T \ . \tag{4.27}$$

This means that the operation $(s\, C^N + \frac{1}{2} C^K C^L \mathcal{F}_{LK}{}^N)\Delta_N$ vanishes. The covariant operations Δ_N are understood to be linearly independent. Therefore $s\, C^N$ is determined

$$s\, C^N = -\frac{1}{2} C^K C^L \mathcal{F}_{LK}{}^N \ . \tag{4.28}$$

The BRST transformation of the ghosts is given by a polynomial which is quadratic in the ghosts with expansion coefficients given by the structure functions $\mathcal{F}_{LK}{}^N$. s transforms the algebra of polynomials generated by ghosts (not derivatives of ghosts) and tensors into itself (4.19, 4.28).

The requirement that s and d anticommute fixes the transformation of the connection,

$$\begin{aligned} 0 &= \{s, d\} T = s(-A^N \Delta_N T) + d(-C^N \Delta_N T) \\ &= -(s\, A^N)\Delta_N T - A^N C^M \Delta_M \Delta_N T - (d\, C^N)\Delta_N T - C^N A^M \Delta_M \Delta_N T \\ &= -(s\, A^N + d\, C^N + A^K C^L \mathcal{F}_{LK}{}^N)\Delta_N T \ , \ \forall T \ . \end{aligned} \tag{4.29}$$

So we conclude

$$s\, A^N = -\, d\, C^N - A^K C^L \mathcal{F}_{LK}{}^N \tag{4.30}$$

for the connection one form A^N. For the gauge field $A_m{}^N$ we obtain [1]

$$s\, A_m{}^N = \partial_m C^N + A_m{}^K C^L \mathcal{F}_{LK}{}^N \ . \tag{4.31}$$

[1] Anticommuting dx^m through s changes the signs.

The BRST transformation of the connection contains the characteristic inhomogeneous piece $\partial_m C^N$.

$d^2 = 0$ identifies the field strength as curl of the connection,

$$\begin{aligned} 0 = d^2\, T &= dx^m dx^n \partial_m \partial_n T = -dx^m dx^n \partial_m (A_n{}^N \Delta_N T) \\ &= -dx^m dx^n \Big((\partial_m A_n{}^N)\Delta_N T + A_n{}^N \partial_m(\Delta_N T) \Big) \\ &= -dx^m dx^n \Big((\partial_m A_n{}^N)\Delta_N T - A_n{}^N A_m{}^M \Delta_M \Delta_N T \Big) \ . \end{aligned} \tag{4.32}$$

Because the differentials anticommute, the antisymmetric part of the bracket vanishes,

$$0 = \partial_m A_n{}^K - \partial_n A_m{}^K - A_m{}^M A_n{}^N \mathcal{F}_{MN}{}^K \ . \tag{4.33}$$

We split the summation over $M\,N$, employ the definition of the vielbein, denote by i and j collectively spin and isospin values

$$\begin{aligned} 0 = \partial_m A_n{}^K - \partial_n A_m{}^K - e_m{}^a e_n{}^b \mathcal{F}_{ab}{}^K + e_m{}^a A_n{}^i \mathcal{F}_{ai}{}^K \\ + A_m{}^i e_n{}^a \mathcal{F}_{ia}{}^K - A_m{}^i A_n{}^j \mathcal{F}_{ij}{}^K \end{aligned} \tag{4.34}$$

and solve for the structure functions $\mathcal{F}_{ab}{}^K$ with two space time indices. Up to nonlinear terms, they are the antisymmetrized derivatives of the gauge fields,

$$\mathcal{F}_{ab}{}^K = E_a{}^m E_b{}^n \Big(2\partial_{[m} A_{n]}{}^K + 2 e_{[m}{}^c A_{n]}{}^i \mathcal{F}_{ci}{}^K - A_m{}^i A_n{}^j \mathcal{F}_{ij}{}^K \Big) \ . \tag{4.35}$$

They are the torsion, $\mathcal{F}_{ab}{}^c = -T_{ab}{}^c$, if $K = c$ corresponds to space-time translations,

$$T_{ab}{}^c = E_a{}^m E_b{}^n \big(\partial_m e_n{}^c - \partial_n e_m{}^c + \omega_{m\,d}{}^c e_n{}^d - \omega_{n\,d}{}^c e_m{}^d \big) \ , \tag{4.36}$$

the Riemann curvature $R_{ab}{}^{cd}$, if $K = cd = -dc$ corresponds to Lorentz transformations,

$$R_{abc}{}^d = E_a{}^m E_b{}^n \big(\partial_m \omega_{n\,c}{}^d - \partial_n \omega_{m\,c}{}^d - \omega_{m\,c}{}^e \omega_{n\,e}{}^d + \omega_{n\,c}{}^e \omega_{m\,e}{}^d \big) \ , \tag{4.37}$$

and the Yang Mills field strength $F_{ab}{}^i$, if $K = i$ ranges over isospin indices,

$$F_{ab}{}^i = E_a{}^m E_b{}^n \big(\partial_m A_n{}^i - \partial_n A_m{}^i - A_m{}^j A_n{}^k f_{jk}{}^i \big) \ . \tag{4.38}$$

The formula applies, however, also to supergravity, which has a more complicated algebra (4.4). It allows in a surprisingly simple way to identify the Rarita Schwinger field strength $\Psi_{ab}{}^\alpha$ when $K = \alpha$ corresponds to supersymmetry transformations.

We choose the spin connection $\omega_{a\,bc} = E_a{}^k \omega_{k\,bc}$ such that the torsion vanishes,

$$\omega_{a\,bc} = \frac{1}{2} \big(\eta_{ad} E_b{}^m E_c{}^n + \eta_{bd} E_a{}^m E_c{}^n - \eta_{cd} E_a{}^m E_b{}^n \big) \big(\partial_m e_n{}^d - \partial_n e_m{}^d \big) \ . \tag{4.39}$$

This choice simplifies the algebra. It does not restrict the generality of our considerations, because a different spin connection differs by a tensor only and leaves the algebra of all tensors unchanged.

We have used that s and d are nilpontent and anticommute if applied to tensors. This has fixed the transformations of the ghosts and connections and identified the structure functions $\mathcal{F}_{ab}{}^{N}$. That s and d are nilpontent and anticommute also if applied to connections and ghosts follows from the Bianchi identity (4.15).

The formulas

$$s\,T = -C^N \Delta_N T\ ,\ d\,T = -A^N \Delta_N T \tag{4.40}$$

for the nilpotent, anticommuting operations s and d not only encrypt the basic geometric structures. They allow also to prove easily that the cohomologies of s and s + d acting on tensors and ghosts, (*not* on connections, derivatives of ghosts, auxiliary fields and antighosts) differ only by a change of variables. Inspection of (s + d), acting on tensors T, shows (note (4.21))

$$\begin{aligned} &\tilde{s}\,T = (s+d)\,T = -(C^N + A^N)\Delta_N T = -\tilde{C}^N \Delta_N T\ , \\ \text{where}\quad &\tilde{C}^i = C^i + A^i = C^i + dx^m A_m{}^i\ , \tilde{C}^a = C^a + A^a = C^a - dx^m e_m{}^a\ , \end{aligned} \tag{4.41}$$

that the $\tilde{s}$-transformation of tensors is the s-transformation with the ghosts C replaced by $\tilde{C}$.

The $\tilde{s}$-transformation of $\tilde{C}$ follows from $\tilde{s}^2 = 0$ and the transformation of tensors (4.41) by the same arguments which determined s C from $s^2 = 0$ and from (4.19). So we obtain

$$\tilde{s}\,\tilde{C}^N = -\frac{1}{2}\tilde{C}^K \tilde{C}^L \mathcal{F}_{LK}{}^N\ . \tag{4.42}$$

This is just the tilded version of (4.28).

Define the map $\rho : C \mapsto \tilde{C} = C + A$ to translate the ghosts C by the connection 1-forms A and to leave A and tensors T invariant. Jet functions P, which depend on ghosts and tensors, and are constant as functions of A are transformed by the corresponding pullback ρ^* to

$$\rho^*(P) = P \circ \rho\ . \tag{4.43}$$

Then (4.41, 4.42) and (4.19, 4.28) state that the BRST cohomologies of s and $\tilde{s}$ acting on functions of ghosts and tensors are invertibly related

$$\tilde{s} \circ \rho^* = \rho^* \circ s\ . \tag{4.44}$$

Theorem 4.1:
A form $\omega(C,T)$ *solves* $s\,\omega(C,T) = 0$ *if and only if* $\omega(\tilde{C},T)$ *solves* $\tilde{s}\ \omega(\tilde{C},T) = 0$.

If we combine this result with theorem 3.5 then the solutions to the descent equations can be found from the cohomology of s if we can restrict the jet functions, which contribute to the cohomology of $\tilde{s}$, to functions of the ghosts and tensors.

4.3 Factorization of the Algebra

If the base manifold and the target space of the fields have trivial topology, then we can restrict the jet functions, which contribute to the cohomology of $\tilde{s}$, to functions of the

ghosts and tensors, because the algebra of jet variables is a product of algebras on which $\tilde{s}$ acts separately and trivially on all factors, apart of the algebra of ghosts and tensors. Using Künneth's formula (theorem 3.6) we can determine nontrivial Lagrange densities and anomaly candidates from solutions of $\tilde{s}\,\omega(\tilde{C},T) = 0$. To establish this result we prove the following theorem:

Theorem 4.2:
The algebra $\mathcal{A}$ *of series in* x^m *and the fields* ϕ (4.17) *and of polynomials in* dx^m *and the partial derivatives of the fields is a product algebra*

$$\mathcal{A} = \mathcal{A}_{\tilde{C},T} \otimes \prod_l \mathcal{A}_{u_l, \tilde{s} u_l} \tag{4.45}$$

where the variables u_l *are enlisted by* $(k = 1, 2, \dots)$

$$\left(x^m, e_m{}^a, \omega_m{}^{ab}, A_m{}^i, \partial_{(m_k} \dots \partial_{m_1} A_{m_0)}{}^N, \bar{C}^N, \partial_{m_k} \dots \partial_{m_1} \bar{C}^N\right) \tag{4.46}$$

$\tilde{s}$ *acts on each factor* $\mathcal{A}_{u_l, \tilde{s} u_l}$ *separately,* $\tilde{s}\,\mathcal{A}_{u_l, \tilde{s} u_l} \subset \mathcal{A}_{u_l, \tilde{s} u_l}$.

The braces around indices, $\partial_{(m_k} \dots \partial_{m_1} A_{m_0)}{}^N$, denote symmetrization. The subscript of the algebras denote the generating elements, e.g. $\mathcal{A}_{e_m{}^a, \tilde{s} e_m{}^a}$ is the algebra of series in the vielbein $e_m{}^a$ and in $\tilde{s}\,e_m{}^a$. $\tilde{s}$ leaves $\mathcal{A}_{u_l, \tilde{s} u_l}$ invariant by construction because of $\tilde{s}^2 = 0$.

To prove the theorem we inspect the variables u_l and $\tilde{s}\,u_l$ to lowest order[2] in the differentials and fields. To this order the variables $\tilde{s}\,u_l$ are

$$\left(dx^m, \partial_m C^a, \partial_m C^{ab}, \partial_m C^i, \partial_{m_k} \dots \partial_{m_0} C^N, iB^N, i\partial_{m_k} \dots \partial_{m_1} B^N\right) \tag{4.47}$$

Also to lowest order the covariant derivatives of the field strengths are

$$(T) \approx \left(E_{a_k}{}^{m_k} \dots E_{a_0}{}^{m_0} \partial_{m_k} \dots \partial_{[m_1} A_{m_0]}{}^N,\ k = 1, 2, \dots\right)\ . \tag{4.48}$$

The brackets denote antisymmetrization of the enclosed indices. In linearized order we find all jet variables as linear combinations of the variables $\tilde{C}, T, u_l$ and $\tilde{s}\,u_l$: the symmetrized derivatives of the connections belong to (u_l), the antisymmetrized derivatives of the connections belong to the field strengths listed as T. The derivatives of the vielbein are slightly tricky. The symmetrized derivatives are the variables $-\partial_{(m_k} \dots \partial_{m_1} A_{m_0)}{}^N$ for $N = a$. The antisymmetrized derivatives of the vielbein are in one to one correspondence to the spin connection because we have chosen it such that the torsion vanishes.

So the transformation of the jet variables $\psi = \{\phi\} = \phi, \partial\phi, \dots$ to the variables $\psi' = \left(\tilde{C}, T, u_l, \tilde{s}\,u_l\right)$ has the structure

$$\psi'^i = M^i{}_j \psi^j + O^i(\psi^2)\ , \tag{4.49}$$

where M is invertible,

$$M^i{}_j = \frac{\partial \psi'^i}{\partial \psi^j}_{|\psi=0}\ . \tag{4.50}$$

[2]We do not count powers of the vielbein $e_m{}^a$ or its inverse. They are not affected by the change of variables, which we investigate. Derivatives of the vielbein, however, are counted.

Therefore, the map $\psi \to \psi'$ is invertible in a neigbourhood of $\psi = 0$, analytic functions $f(\psi)$ are analytic functions $F(\psi') = f(\psi(\psi'))$ of ψ' and the algebra, generated by x, dx and the jet variables $\{\phi\}$, coincides with the algebra, generated by ψ',

$$A_{x,dx,\{\phi\}} = A_{\tilde{C},T} \otimes \prod_l A_{u_l,\tilde{s}\,u_l}. \tag{4.51}$$

Because $\tilde{s}$ leaves each factor of the product algebra invariant, Künneth's theorem (theorem 3.6) applies and the cohomology of $\tilde{s}$ acting on the algebra $A_{x,dx,\{\phi\}}$ of the jet variables is given by the product of the cohomologies of $\tilde{s}$ acting on the ghost tensor algebra $A_{\tilde{C},T}$ and on the algebras $A_{u_l,\tilde{s}\,u_l}$

$$H(A,\tilde{s}) = H(A_{\tilde{C},T}\,,\tilde{s}) \otimes \prod_l H(A_{u_l,\tilde{s}\,u_l},\tilde{s})\ . \tag{4.52}$$

By the basic lemma (theorem 3.6) the cohomology of d acting on an algebra $A_{x,dx}$ of differential forms $f(x, dx)$ which depend on generating and independent variables x and dx is given by numbers f_0. The algebra $A_{u_l,\tilde{s}\,u_l}$ and the action of $\tilde{s}$ on this algebra differ only in the denomination. Therefore the cohomology $H(A_{u_l,\tilde{s}\,u_l},\tilde{s})$ is given by numbers, at least as long as the variables u_l and $\tilde{s}\,u_l$ are independent and not subject to constraints.

Whether the variables $u_l, \tilde{s}\,u_l$ are subject to constraints is a matter of choice of the theory which one considers. This choice influences the cohomology. For example, one could require that two coordinates x^1 and x^2 satisfy $(x^1)^2 + (x^2)^2 = 1$ because one wants to consider a theory on a circle. Then the differential $d(\arctan\frac{y}{x}) = d\varphi$ is closed $(d\,d\varphi = 0)$ but not exact, because the angle φ is not a function on the circle, $d\varphi$ is just a misleading notation for a one form which is not d of a function φ. In this example the periodic boundary condition $\varphi \sim \varphi + 2\pi$ gives rise to a nontrivial cohomology of d acting on φ and $d\varphi$. Nontrivial cohomologies also arise if the fields take values in nontrivial spaces. For example if in nonlinear sigma models one requires scalar fields ϕ^i to take values on a sphere $\sum_{i=1}^{n+1} {\phi^i}^2 = 1$ then the volume form $d^n\phi$ is nontrivial. More complicated is the case where scalar fields are restricted to take values in a general coset G/H of a group G with a subgroup H. Also the relation

$$\det e_{\cdot}^{\ \cdot} \neq 0 \tag{4.53}$$

(where $e_{\cdot}^{\ \cdot}$ is the matrix with elements ${e_m}^a$ in the $m^{\rm th}$ row and $a^{\rm th}$ column) restricts the vielbeine to take values in the group $GL(D)$ of invertible real $D \times D$ matrices. This group has the nontrivial cohomology of $O(D)$.

In our investigation we neglect the cohomologies coming from a nontrivial topology of the base manifold with coordinates x^m or the target space with coordinates ϕ or ${e_m}^a$. We have to determine the cohomology of $\tilde{s}$ on the ghost tensor variables anyhow and start with this problem. To obtain the complete answer we can determine the cohomology of the base space and the target space in a second step which we postpone. So we choose to investigate topologically trivial base manifolds and target spaces. We combine eq. (4.52) with theorem 3.5 and theorem 4.1 and conclude

Theorem 4.3:
If the target space and the base manifold have trivial topology then the nontrivial solutions of the descent equations in gravitational theories are in one to one correspondence to the nontrivial solutions $\omega(C, T)$ *of the equation* $s\,\omega = 0$*. Up to trivial terms the solution* ω_D *of the descent equation* (3.42) *is given by the* D*-form part of the form* $\omega(C + A, T)$*.*

ω depends only on the ghosts, not on their derivatives. Therefore the ghostnumber of ω is bounded by the number of ghosts for translations and spin and isospin transformations $D + \frac{D(D-1)}{2} + \dim(G)$. If we take the D-form part of $\omega(C + A, T)$ then D differentials dx^m rather than ghosts have to be picked. Therefore the ghostnumber of nontrivial solutions of the relative cohomology is bounded by $\frac{D(D-1)}{2} + \dim(G)$.

From the theorem one concludes that anomaly candidates which one expresses in terms of the ghost variables $\hat{C}$, (the index i enumerates spin and isospin values collectively, translation ghosts are denoted by c, to distinguish them from spin and isospin ghosts C)

$$\hat{C}^i = C^i + c^a {E_a}^m {A_m}^i\ ,\ \hat{c}^m = c^a {E_a}^m\ , \tag{4.54}$$

can be chosen such that they contain no ghosts $\hat{c}^m$ of coordinate transformations or, in other words, that coordinate transformations are not anomalous.

This holds, because the variables $\hat{C}^i$ are invariant under the shift $\rho : C \mapsto C + A$ (4.43), only the translation ghosts are shifted,

$$\rho(\hat{C}^i) = \hat{C}^i\ ,\ \rho(\hat{c}^m) = \hat{c}^m - dx^m\ . \tag{4.55}$$

Therefore, if one expresses a form $\omega(C + A, T)$ by ghost variables $\hat{c}, \hat{C}$ then ω depends on dx^m only via the combination $\hat{c}^m - dx^m$ (4.41). The D form part ω_D originates from a coefficient function multiplying

$$(\hat{c}^1 - dx^1)(\hat{c}^2 - dx^2)\ldots(\hat{c}^D - dx^D) = (-)^D\,(dx^1dx^2\ldots dx^D + \ldots)\ . \tag{4.56}$$

This coefficient function of d^Dx cannot contain a translation ghost $\hat{c}^m$ because $\hat{c}^m$ enters only in the combination $\hat{c}^m - dx^m$ and $D + 1$ factors of $\hat{c}^m - dx^m$ vanish.

In our formulation s maps the subalgebra of ghosts and tensors to itself,

$$-\,s\,T = C^N\Delta_N T = c^a {E_a}^m(\partial_m + {A_m}^i\delta_i)\,T + C^i\,\delta_i T\ . \tag{4.57}$$

In terms of the ghosts $\hat{C}$ this is a shift term $\hat{c}^m\partial_m T$ and the BRST transformation of a Yang Mills theory

$$-\,s\,T = \hat{c}^m\partial_m T + \hat{C}^i\delta_i T\ . \tag{4.58}$$

This formulation arises naturally if one enlarges the BRST transformation of Yang Mills theories to allow also general coordinate transformations. However, $\partial_m T$ is not a tensor and it is not manifest, that s leaves a subalgebra invariant.

5 BRST Cohomology on Ghosts and Tensors

5.1 Invariance under Adjoint Transformations

In the preceding section the problem to determine Lagrange densities and anomaly candidates has been reduced to the calculation of the cohomology of s acting on tensors and ghosts,

$$s\,\omega(C, c, T) = 0\ ,\ \omega \ \text{mod}\ s\,\eta(C, c, T)\ . \tag{5.1}$$

Let us recall the transformation s explicitly [1]

$$s\,T = -(c^a D_a + C^i \delta_i) T\ , \tag{5.2}$$

$$s\,c^a = -C^i c^b G_{ib}{}^a\ , \tag{5.3}$$

$$s\,C^i = \frac{1}{2} C^k C^l f_{kl}{}^i + \frac{1}{2} c^a c^b F_{ab}{}^i\ . \tag{5.4}$$

To determine the cohomology of s, we proceed as in the derivation of the basic lemma and investigate the anticommutator of s with other fermionic operations. Here we consider the partial derivatives with respect to the spin and isospin ghosts C^i. These anticommutators are the generators δ_i of spin and isospin transformations

$$\delta_i = -\{s, \frac{\partial}{\partial C^i}\} \tag{5.5}$$

which on the ghosts c are represented by G_i and on the ghosts C by the adjoint representation

$$\delta_i c^a = G_{ib}{}^a c^b\ ,\ \delta_i C^j = f_{ki}{}^j C^k\ . \tag{5.6}$$

Eq. (5.5) is easily verified on the elementary variables c, C and T. It extends to arbitrary polynomials because both sides of the equation are linear operations with the same product rule.

Arbitrary linear combinations $\delta = \alpha^i \delta_i$ of the spin and isospin transformations commute with s because each anticommutator $\{s, r\}$ of a nilpotent s commutes with s no matter what operation r is (3.13),

$$[\delta, s] = 0\ . \tag{5.7}$$

[1] By choice of the spin connection $T_{ab}{}^c$ vanishes. Translation ghosts are denoted by c to distinguish them from spin and isospin ghosts C, which are enumerated by i.

The representation of the isospin transformations on the algebra of ghosts and tensors is completely reducible because the isospin transformations belong to a semisimple group or to abelian transformations which decompose the algebra into polynomials of definite charge and definite dimension. Therefore the following theorem applies.

Theorem 5.1:
If the representation of δ_i on the ghost and tensor algebra is completely reducible then each solution of $s\,\omega = 0$ is invariant under all $\delta = a^i\delta_i$ up to an irrelevant piece,

$$s\,\omega = 0 \Rightarrow \omega = \omega_{\text{inv}} + s\,\eta\ ,\ \delta\,\omega_{\text{inv}} = 0\ . \tag{5.8}$$

The theorem is proven by the following argument. The null space of s,

$$Z = \{\omega :\ s\,\omega = 0\}\ , \tag{5.9}$$

is mapped by spin and isospin transformations to itself,

$$s\,\delta_i\omega = \delta_i\, s\,\omega = 0\ ,\ \delta_i Z \subset Z\ . \tag{5.10}$$

Z contains the subspace Z_δ of elements which can be written as sum of isospin transformations applied to some other elements $\kappa^i \in Z$,

$$Z_\delta = \{\omega \in Z :\ \omega = \delta_i(\kappa^i)\ ,\ s\,\kappa^i = 0\}\ . \tag{5.11}$$

Z_δ is mapped by isospin transformations to itself. A second invariant subspace is given by Z_{inv}, the subspace of δ invariant elements,

$$Z_{\text{inv}} = \{\omega \in Z :\ a^i\delta_i\omega = 0\ \}\ . \tag{5.12}$$

If the representation of δ_i is completely reducible then the space Z decomposes as a sum

$$Z = Z_{\text{inv}} \oplus Z_\delta \oplus Z_{\text{comp}} \tag{5.13}$$

with a complement Z_{comp} which is also mapped to itself. This complement, however, contains only $\omega = 0$ because if there were a nonvanishing element $\omega \in Z_{\text{comp}}$ it would not be invariant because it is not from Z_{inv}. ω would be mapped to $\delta\omega \in Z_\delta$ and Z_{comp} would not be an invariant subspace,

$$Z = Z_{\text{inv}} \oplus Z_\delta\ . \tag{5.14}$$

Each ω which solves $s\,\omega = 0$ can therefore be decomposed as

$$\omega = \omega_{\text{inv}} + \delta_i\kappa^i\ ,\ s\,\kappa^i = 0\ ,\ \delta_i\omega_{\text{inv}} = 0\ . \tag{5.15}$$

We replace δ_i by $-\{s, \frac{\partial}{\partial C^i}\}$ (5.5), use $s\,\kappa^i = 0$ and verify the theorem,

$$\omega = \omega_{\text{inv}} + s\,\eta\ ,\ \eta = -\frac{\partial}{\partial C^i}\kappa^i\ . \tag{5.16}$$

The theorem restricts nontrivial solutions to $s\,\omega = 0$ to δ-invariant ω. If we decompose it as a sum of products of polynomials Θ_l of the spin and isospin ghosts C and of forms f^l of the translations ghosts c, which depend on tensors T, it has the form

$$\omega(C, c, T) = \sum_l \Theta_l(C) f^l(c, T)\ , \tag{5.17}$$

where $\Theta_1, \Theta_2 \dots$ as well as $f^1, f^2 \dots$ can be taken to be linearly independent (otherwise one could express ω as a sum with fewer terms).

The operation s decomposes into

$$s = -C^i\delta_i + s_c + s_1 + s_2\ , \tag{5.18}$$

where s_c acts only on spin and isospin ghosts C and preserves the number of translation ghosts and where s_1 and s_2 increase this number by 1 and 2,

$$s_c\,T = 0\ ,\qquad s_c\,c^a = 0\ ,\qquad s_c\,C^i = -\frac{1}{2}C^jC^k f_{jk}{}^i\ , \tag{5.19}$$

$$s_1\,T = -c^a D_a T\ ,\qquad s_1\,c^a = 0\ ,\qquad s_1\,C^i = 0\ , \tag{5.20}$$

$$s_2\,T = 0\ ,\qquad s_2\,c^a = 0\ ,\qquad s_2\,C^i = \frac{1}{2}c^a c^b F_{ab}{}^i\ . \tag{5.21}$$

Therefore, to lowest order in the translation ghosts c the conditions $s\,\omega = 0$ and $\delta\,\omega = 0$ lead to

$$0 = s_c\,\omega = s_c \sum_l \Theta_l(C) f^l(c, T) = \sum_l (s_c\,\Theta_l) f^l(c, T)\ , \tag{5.22}$$

which implies that each Θ_l is a solution to

$$s_c\,\Theta = 0\ , \Theta \bmod s_c\,\eta\ , \tag{5.23}$$

because the functions $f^1, f^2 \dots$ are linearly independent. If we change e.g. Θ_1 by $s_c\,\eta$, then ω is changed to lowest order in form degree by $s(\eta f^1) - (-)^{|\eta|}\eta\, s\, f^1$, i.e. up to a trivial term by a form $\hat\omega = -(-)^{|\eta|}\eta\, s\, f^1$. But the condition $s\,\omega = 0$ and $s(\omega + \hat\omega) = 0$, differs only by denomination and one obtains the same set of solutions, whether one uses Θ_1 or $\Theta_1 + s_c\,\eta$.

Therefore each Θ_i is an element of the Lie algebra cohomology of the spin and isospin ghosts C. In particular, it can be taken to be δ invariant, because $-\{s_c, \frac{\partial}{\partial C^i}\}$ generates the δ transformations of the ghosts C. Therefore all f^l are also δ invariant.

The algebra of the spin and isospin ghosts is a product algebra of the ghosts of the simple and abelian factors of the spin and isospin Lie algebra. Each factor of the algebra is left invariant by s_c. Therefore the space of invariant ghost polynomials can be determined separately for each factor of the Lie algebra. By Künneth's formula (theorem 3.6) the cohomology of the product algebra is the product of the cohomologies of the factors.

5.2 Lie Algebra Cohomology

The following results for simple Lie algebras can be found in the mathematical literature [8]: the cohomology of s_c has dimension 2^r where r is the rank of the Lie algebra. The cohomology is the algebra generated by r primitive polynomials $\theta_\alpha(C)$, $\alpha = 1,\dots,r$. These primitive polynomials cannot be written as a sum of products of other invariant polynomials. They have odd ghostnumber $\mathrm{gh}(\theta_\alpha(C)) = 2m(\alpha) - 1$ and therefore are fermionic. They can be obtained from traces of suitable matrices M_i which represent a basis of the Lie algebra and are given with a suitable normalization by

$$\theta_\alpha(C) = \frac{(-)^{m-1} m!(m-1)!}{(2m-1)!} \mathrm{tr}(C^i M_i)^{2m-1} \ , \ m = m(\alpha) \ , \ \alpha = 1,\dots,r \ . \tag{5.24}$$

The number $m(\alpha)$ is the degree of homogeneity of the corresponding Casimir invariant

$$I_\alpha(X) = \mathrm{tr}(X^i M_i)^{m(\alpha)} \ . \tag{5.25}$$

These Casimir invariants generate all invariant functions of a set of commuting variables X^i which transform as an irreducible multiplet under the adjoint representation.

The degrees $m(\alpha)$ for the classical simple Lie algebras are given by

$$\begin{array}{llll} SU(n+1) & A_n & m(\alpha) = \alpha + 1 & \alpha = 1,\dots,n\ , n \geq 1\ , \\ SO(2n+1) & B_n & m(\alpha) = 2\alpha & \alpha = 1,\dots,n\ , n \geq 2\ , \\ SP(2n) & C_n & m(\alpha) = 2\alpha & \alpha = 1,\dots,n\ , n \geq 3\ , \\ SO(2n) & D_n & m(\alpha) = 2\alpha & \alpha = 1,\dots,n-1,\ m(n) = n,\ n \geq 4\ . \end{array} \tag{5.26}$$

With the exception of the last primitive element of $SO(2n)$ the matrices M_i are the defining representation of the classical Lie algebras. The last primitive element θ_n and the last Casimir invariant I_n of $SO(2n)$ are constructed from the spin representation Γ_i. Up to normalization they are given by

$$\begin{aligned} \theta_n &\sim \epsilon_{a_1 b_1 \dots a_n b_n} (C^{a_1}{}_{c_1} C^{c_1 b_1}) \dots (C^{a_{n-1}}{}_{c_{n-1}} C^{c_{n-1} b_{n-1}})\, C^{a_n b_n} \\ I_n &\sim \epsilon_{a_1 b_1 \dots a_n b_n} X^{a_1 b_1} \dots X^{a_n b_n} \ . \end{aligned} \tag{5.27}$$

If n is even then the element θ_n of $SO(2n)$ is degenerate in ghostnumber with $\theta_{\frac{n}{2}}$.

The primitive elements for the exceptional simple Lie algebras G_2, F_4, E_6, E_7, E_8 can also be found in the literature [9]. Their explicit form is not important for our purpose. In each case the Casimir invariant with lowest degree m is quadratic ($m = 2$).

For a one dimensional abelian Lie algebra the ghost C is invariant under the adjoint transformation. It generates the invariant polynomials $\Theta(C) = a + b\,C$ which span a 2^r dimensional space where $r = 1$ is the rank of the abelian Lie algebra. The generator θ of this algebra of invariant polynomials has odd ghostnumber $\mathrm{gh}(C) = 2m - 1$ with $m = 1$.

$$\theta(C) = C \tag{5.28}$$

The Casimir invariant I of the one dimensional, trivial adjoint representation acting on a bosonic variable X is homogeneous of degree $m = 1$ in X and is simply given by X itself,

$$I(X) = X \ . \tag{5.29}$$

Polynomials of r anticommuting variables θ_α constitute a 2^r dimensional Grassmann algebra. The statement that the primitive elements $\theta = (\theta_1, \theta_2 \dots)$ generate the Lie algebra cohomology asserts that

$$s_c\,\Theta(C) = 0 \Leftrightarrow \Theta(C) = \Phi(\theta(C)) + s_c\,\eta \ . \tag{5.30}$$

Because the cohomology is 2^r dimensional there are no algebraic relations among the functions θ apart from the anticommutation relations which result from their odd ghostnumber,

$$\Theta(C) = \Phi(\theta(C)) = 0 \Leftrightarrow \Phi(\theta) = 0 \ . \tag{5.31}$$

The Casimir invariants $I = (I_1, I_2 \dots)$ generate the space of δ invariant polynomials in commuting variables X which transform under the adjoint representation

$$\delta_i P(X) = 0 \Rightarrow P(X) = f(I(X)) \ . \tag{5.32}$$

If there are no algebraic relations among the variables X apart from their commutation relations then there is no algebraic relation among the Casimir invariants $I(X)$ up to the fact that the I_α commute [8].

$$P(X) = f(I(X)) = 0 \Leftrightarrow f(I) = 0 \tag{5.33}$$

To sum up: if we expand the solution ω of $s\,\omega = 0$ into parts ω_n with definite degree n in translations ghosts c

$$\omega = \omega_{\underline{n}} + \sum_{n > \underline{n}} \omega_n \tag{5.34}$$

then, up to a trivial solution, $\omega_{\underline{n}}$ is a superfield Φ in the anticommuting primitive invariant polynomials $\theta = (\theta_1, \dots \theta_r)$

$$\omega_{\underline{n}}(C, c, T) = \Phi(\theta(C), c, T) \tag{5.35}$$

with coefficients, which are spin and isospin invariant forms of the translation ghosts c and tensors.

In next to lowest degree the equation $s\,\omega = 0$ imposes the restriction

$$s_1\,\omega_{\underline{n}} + (-C^i \delta_i + s_c)\omega_{\underline{n}+1} = 0 \ . \tag{5.36}$$

But s_1 (5.20) maps invariant functions of the translations ghosts and tensors to invariant functions and treats spin and isospin ghost as constants,

$$s_1 T = c^a D_a T \ , \ s_1 c^a = 0 \ , \ s_1 C^i = 0 \ . \tag{5.37}$$

Therefore $s_1\Phi$ is again a superfield in θ with coefficients $f(T)$, which are δ invariant forms of the translation ghosts and which depend on tensors. Such a superfield is not of the form $(-C^i\delta_i + s_c)\eta$, unless it vanishes, therefore (5.36) implies

$$s_1\,\Phi(\theta, c, T) = 0\ , \tag{5.38}$$

and because s_1 only acts on the coefficients f of the θ expansion of Φ, they have to satisfy

$$s_1\,f(c, T) = 0\ ,\ \ f \bmod s_1\,\eta\ . \tag{5.39}$$

Indeed, we can neglect a contribution of the form $(s_1\,\eta)\,g(\theta)$ because it can be written as $(s - s_2)(\eta\,g(\theta))$ because η and g are δ_I and s_c invariant. $s(\eta\,\Phi(\theta))$ changes $\omega = \omega_{\underline{n}} + \dots$ only by an irrelevant piece. $s_2(\eta\,g(\theta))$ can be absorbed in the parts $\dots$ with higher ghost degree. Therefore we can neglect contributions $(s_1\eta)\,g(\theta)$ to $\omega_{\underline{n}}$.

To determine the cohomology of s_1 on the algebra of undifferentiated translation ghosts and tensors, we split the ghost form $f(c, T)$ and $s_1\,f$ according to a number N of jet variables, where we do not count powers of the vierbein $e_m{}^a$ and its inverse $E_b{}^n$, but count the differentiated $e_m{}^a$, count the connections for isospin transformations and their derivatives with a weight 2 and count the ghosts and remaining variables with normal weight,

$$N = N_{\{\partial e\}} + 2N_{\{A\}} + N_{\{C\}} + N_{\{\phi\}}\ . \tag{5.40}$$

$$f = \sum_{n \geq \underline{n}} f_n\ ,\ \ N\,f_n = n\,f_n\ , \tag{5.41}$$

The operation s_1 decomposes into pieces $s_{1\,n}$, which increase the N number by n

$$s_1 = \sum_{n \geq 1} s_{1,n}\ ,\ \ [N, s_{1,n}] = n\ s_{1,n}\ . \tag{5.42}$$

Then the equation $s_1\,f = 0$ implies in lowest N order

$$s_{1,1}\,f_{\underline{n}} = 0\ ,\ \ f_{\underline{n}} \bmod s_{1,1}\,\eta\ . \tag{5.43}$$

We can neglect contributions $s_{1,1}\,\eta$ to $f_{\underline{n}}$ because up to terms of higher N number they are of the form $s_1\,\eta$ and therefore trivial.

The ghosts are invariant under s_1 (5.20), on tensors s_1 acts by

$$s_1\,T = -c^a D_a T = -c^m(\partial_m + A^i_m\delta_i)\,T\ . \tag{5.44}$$

Therefore $s_{1,1}$ acts on ghosts and tensors as

$$s_{1,1} = -c^m\partial_m \tag{5.45}$$

where the partial derivative ∂_m only acts on tensors, not on ghosts and not on $e_m{}^a$, because the differentiation of $e_m{}^a$ increases the N number.

The part $c^m A_m{}^i\delta_i$ increases the N number by at least 2, even if an isospin transformation decreases the number N of fields and transforms a field ϕ into $\delta_i\phi$ with a field independent part. If the field independent part $(\delta_i\phi)_0$ does not vanish, then the field ϕ is called a Goldstone field and the transformation is said to be nonlinear or to be a spontaneously broken symmetry.

Lorentz transformations are not spontaneously broken, they transform fields into linear combinations of fields. Isospin transformation may be spontaneously broken, but N counts their connection with a weight 2, so $c^m A_m{}^i\delta_i$ increases the N number by at least 2 and $c^m A_m{}^i\delta_i$ does not contribute to $s_{1,1}$.

To lowest N order, the partial derivative in $s_{1,1}$ does not differentiate ghosts and the vierbein, therefore the derivative $\partial_m c^n = \partial_m(c^a E^m_a)$ vanishes. This justifies to change the notation and denote c^m by dx^m. Then $s_{1,1}$ is the exterior derivative and, changing the name $f_{\underline{n}}$ to ω, it is a differential form, which to lowest order in the fields satisfies

$$d\,\omega(T, dx) = 0\ ,\ \ \omega \bmod d\eta(T, dx) \tag{5.46}$$

by (5.43) and depends on the lowest order parts of tensors T.

5.3 Covariant Poincaré Lemma

Because we consider only the terms with lowest N number, the covariant derivatives of tensors, which appear in ω, contribute only with their partial derivative. By the same reason, the field strength and curvature enter only as antisymmetrized derivative of the connection,

$$D_{m\ \text{linearized}} = \partial_m\ ,\ \ F_{mn\ \text{lin}} = \partial_m A_n - \partial_n A_m\ . \tag{5.47}$$

Collectively we call the linearized field strength and the remaining fields T (not the ghosts which we are going to introduce in (5.54)), on which by assumption the derivatives act freely, together with their (higher) derivatives linearized tensors.

Differential forms with coefficients, which are functions of linearized tensors are termed linearized tensor forms.

The cohomology of d acting on jet forms is given by the algebraic Poincaré lemma (theorem 3.3). This lemma, however, does not apply, if the differential forms are restricted to be linearized tensor forms, because the derivatives do not act freely, i.e. without restriction apart from the property that they commute, but subject to the Bianchi identities (and their derivatives) that the antisymmetrized derivatives of the field strength vanish,

$$\partial_k F_{lm\ \text{lin}} + \partial_l F_{mk\ \text{lin}} + \partial_m F_{kl\ \text{lin}} = 0\ . \tag{5.48}$$

The cohomology of d acting on linearized tensor forms $\omega(T, dx)$ has been derived for Yang Mills theories [11, 12, 13], Riemannian geometry [14] and gravitational Yang Mills theories [10]. We consider a slightly more general problem and analyse linearized tensor forms $\omega(T, x, dx)$ which may also depend on the coordinates x^m. On such forms we investigate the cohomology of the exterior derivative d_ϕ, which differentiates only the fields, and of $d = d_\phi + d_x$ (3.38) which also differentiates coordinates. The results apply, in particular, if background fields occur which are given functions of the coordinates. The solution of (5.46) is spelled out at the end of this subsection.

Theorem 5.2: *Linearized Covariant Poincaré Lemma for* $\mathrm{d} = \mathrm{d}_\phi + \mathrm{d}_x$
(i) Let ω *be a linearized tensor form which may depend on the* x^m*, then*

$$\begin{aligned} \mathrm{d}\,\omega(T,x,\mathrm{d}x) = 0 \Leftrightarrow \quad & \omega(T,x,\mathrm{d}x) = \mathcal{L}(T,x)\,\mathrm{d}^D x + P(\mathrm{d}A) + \mathrm{d}\eta(T,x,\mathrm{d}x), \\ \omega(T,x,\mathrm{d}x) = \mathrm{d}\,\chi \Leftrightarrow \quad & \omega(T,x,\mathrm{d}x) = P_1(\mathrm{d}A) + \mathrm{d}\eta(T,x,\mathrm{d}x)\,, \end{aligned} \tag{5.49}$$

where the Lagrangian form $\mathcal{L}\,\mathrm{d}^D x$ *and* η *are linearized tensor forms which may depend on the* x^m *and where* P *and* P_1 *are polynomials in the field strength two forms* $\mathrm{d}A^i = \frac{1}{2}\mathrm{d}x^m\mathrm{d}x^n(\partial_m A_n{}^i - \partial_n A_m{}^i)$ *and where* P_1 *has no constant (field independent) part,* $P_1(0) = 0$.
(ii) A polynomial P *in the field strength two forms cannot be written as exterior derivative* d *of a tensor form* η *which may depend on the* x^m*,*

$$P(\mathrm{d}A) + \mathrm{d}\eta(T,x,\mathrm{d}x) = 0 \;\Leftrightarrow\; P(\mathrm{d}A) = 0 = \mathrm{d}\eta(T,x,\mathrm{d}x)\ . \tag{5.50}$$

For the exterior derivative d_ϕ which differentiates only the fields one gets:

Theorem 5.3: *Linearized Covariant Poincaré Lemma for* d_ϕ
(i) Let ω *be a linearized tensor form which may depend on the* x^m*, then*

$$\begin{aligned} \mathrm{d}_\phi\,\omega(T,x,\mathrm{d}x) = 0 \Leftrightarrow \quad & \omega(T,x,\mathrm{d}x) = \mathcal{L}(T,x)\,\mathrm{d}^D x + P(\mathrm{d}A,x,\mathrm{d}x) \\ & \qquad + \mathrm{d}_\phi\,\eta(T,x,\mathrm{d}x)\ , \\ \omega(T,x,\mathrm{d}x) = \mathrm{d}_\phi\,\chi \Leftrightarrow \quad & \omega(T,x,\mathrm{d}x) = P_1(\mathrm{d}A,x,\mathrm{d}x) + \mathrm{d}_\phi\,\eta(T,x,\mathrm{d}x)\ , \end{aligned} \tag{5.51}$$

where the Lagrangian form $\mathcal{L}\,\mathrm{d}^D x$ *and* η *are linearized tensor forms which may depend on the* x^m *and where* P *and* P_1 *are polynomials in the field strength two forms* $\mathrm{d}A^i = \frac{1}{2}\mathrm{d}x^m\mathrm{d}x^n(\partial_m A_n{}^i - \partial_n A_m{}^i)$ *and the coordinate differentials* $\mathrm{d}x^m$ *which may depend on the* x^m *and where* P_1 *has no field independent part.*
(ii) A polynomial P *in the field strength two forms and the coordinate differentials* $\mathrm{d}x^m$ *which may depend on the* x^m *cannot be written as exterior derivative* d_ϕ *of a tensor form* η *which may depend on the coordinates,*

$$\begin{aligned} & P(\mathrm{d}A,x,\mathrm{d}x) + \mathrm{d}_\phi\,\eta(T,x,\mathrm{d}x) = 0 \Leftrightarrow \\ & P(\mathrm{d}A,x,\mathrm{d}x) = 0 = \mathrm{d}_\phi\,\eta(T,x,\mathrm{d}x) \end{aligned} \tag{5.52}$$

The part $\mathcal{L}(T,x)\,\mathrm{d}^D x$ is of the from $P(\mathrm{d}A)+\mathrm{d}\eta(T,x,\mathrm{d}x)$ or $P(\mathrm{d}A,x,\mathrm{d}x)+\mathrm{d}_\phi\,\eta(T,x,\mathrm{d}x)$ if and only if its Euler derivative (3.35) vanishes.

We prove the theorems by induction with respect to the form degree, starting with the proof of (5.49). (5.49) holds for 0-forms: By the algebraic Poincaré lemma (theorem 3.3), 0-forms are closed if and only if they are constant. A constant, however, is a polynomial $P(\mathrm{d}A)$.

Assume the theorems to hold for all forms with degree less than the form degree $p > 0$ of the form $\Omega(T,x,\mathrm{d}x)$ which solves $\mathrm{d}\,\Omega(T,x,\mathrm{d}x) = 0$ or which in case $p = D$ has vanishing Euler derivative. Then by the algebraic Poincaré lemma (theorem 3.3) Ω is of the form

$$\Omega = \mathrm{d}\,\omega_{p-1}\ , \tag{5.53}$$

where ω_{p-1} depends on the jetvariables x,$\mathrm{d}x$,A,∂A,$\partial\ldots\partial A$,T,∂T,$\partial\ldots\partial T$.

As Ω depends only on tensors, it is invariant under the transformation s (3.7), which annihilates $x, \mathrm{d}x$, the ghosts C and tensors and which anticommutes with d.

$$s\,\partial_{(m_1}\ldots\partial_{m_{k-1}}A^i_{m_k)} = \partial_{m_1}\ldots\partial_{m_k}C^i\ ,\ k = 1,2,\ldots \tag{5.54}$$

(The parentheses denote symmetrization of the enclosed indices).

From $s\,\Omega = s\,\mathrm{d}\,\omega_{p-1} = 0$ and $\{s,\mathrm{d}\} = 0$ one concludes $\mathrm{d}\,s\,\omega_{p-1} = 0$. So, by the algebraic Poincaré lemma, there is a jetform ω_{p-2} with ghostnumber 1 such that

$$s\,\omega_{p-1} + \mathrm{d}\,\omega_{p-2} = 0\ . \tag{5.55}$$

Applying s to this equation one concludes $s\,\mathrm{d}\,\omega_{p-2} = 0$ and derives iteratively the descent equations

$$s\,\omega_{p-k} + \mathrm{d}\,\omega_{p-k-1} = 0\ . \tag{5.56}$$

At some stage the iteration has to end

$$s\,\omega_{p-G} = 0\ , \tag{5.57}$$

because the form degree can not become negative.

In terms of $\tilde{\omega} = \sum_k \omega_{p-k}$ and $\tilde{s} = s + \mathrm{d}$ all equations are summed up in

$$\Omega = \tilde{s}\,\tilde{\omega}\ . \tag{5.58}$$

The operation $\tilde{s}$ transforms the variables

$$\tilde{C}^i = C^i + \mathrm{d}x^m A^i_m \tag{5.59}$$

into the field strength two form $\mathrm{d}A^i$ (making use of $s\,\mathrm{d}x^m A^i_m = -\,\mathrm{d}\,C^i$)

$$\tilde{s}\,\tilde{C}^i = \mathrm{d}A^i = \frac{1}{2}\mathrm{d}x^m\mathrm{d}x^n(\partial_m A^i_n - \partial_n A^i_m) \tag{5.60}$$

and defines variables $q^i_{m_1\ldots m_k}$ as transformed symmetrized derivatives of A^i

$$\tilde{s}\,\partial_{(m_1}\ldots\partial_{m_{k-1}}A^i_{m_k)} = q^i_{m_1\ldots m_k},\ q^i_{m_1\ldots m_k} = \partial_{m_1}\ldots\partial_{m_k}C^i + \ldots,\ k = 1,2\ldots \tag{5.61}$$

Up to nonlinear terms, they are the derivatives of the ghosts. Coordinates and tensors transform into

$$\tilde{s}\,x^m = \mathrm{d}x^m\ ,\ \tilde{s}\,T = \mathrm{d}_T\,T\ , \tag{5.62}$$

where the exterior derivative d_T is defined to differentiate only the tensors and to vanish on the ghosts, on $q^i_{m_1\ldots m_k}$ and on the symmetrized derivatives of A^i.

We perform a coordinate transformation in jet space and use the new variables $\tilde{C}^i$ and $q^i_{m_1\dots m_k}$ in place of the C^i and their partial derivatives as new coordinates. The transformation is invertible (4.49) because the linearized transformation is invertible.

On functions of these variables $\tilde{s}$ acts as the derivative

$$\tilde{s} = dx^m \frac{\partial}{\partial x^m} + dA^i \frac{\partial}{\partial \tilde{C}^i} + q^i_{m_1\dots m_k} \frac{\partial}{\partial(\partial_{(m_1}\dots A^i_{m_k)})} + d_T \ . \tag{5.63}$$

It commutes with the number operator $N = N_{\partial_{(A)}} + N_q$, which counts the vectorfields A^i, its symmetrized derivatives and the q-variables. The tensor form Ω does not depend on these variables, therefore $0 = N\Omega = \tilde{s}N\tilde{\omega}$ and only the part of $\tilde{\omega}$ with $N\tilde{\omega} = 0$ can contribute to Ω. Therefore we can restrict $\tilde{\omega}$ to a form, which depends on coordinates x, undifferentiated ghosts C^i and tensors,

$$\tilde{\omega} = \sum_{k=0}^{m} \frac{1}{k!} \tilde{C}^{i_1} \dots \tilde{C}^{i_k} \chi^{(k)}_{i_1\dots i_k}(T, x, dx). \tag{5.64}$$

The crux of the matter is to prove that $\tilde{\omega}$ actually can be taken to be at most linear in the $\tilde{C}^i$. To prove this, we examine the case that the highest order in the $\tilde{C}^i$ which occurs in $\tilde{\omega}$ is non-zero, i.e. $m > 0$. Since Ω does not depend on the $\tilde{C}^i$, we obtain from $\Omega = \tilde{s}\,\tilde{\omega}$ at order m in the $\tilde{C}^i$:

$$m > 0: \quad d\,\chi^{(m)}_{i_1\dots i_m}(T, x, dx) = 0. \tag{5.65}$$

By induction hypothesis (5.49) holds for all form degrees smaller than p. $\chi^{(m)}_{i_1\dots i_m}(T, x, dx)$ has form degree $p - m - 1$. Hence, we conclude

$$m > 0: \quad \chi^{(m)}_{i_1\dots i_m}(T, x, dx) = P_{i_1\dots i_m}(dA) + d\eta_{i_1\dots i_m}(T, x, dx). \tag{5.66}$$

With no loss of generality we can ignore the contribution $d\,\eta_{i_1\dots i_m}(T, x, dx)$ to $\chi^{(m)}_{i_1\dots i_m}$ because we can remove it by replacing $\tilde{\omega}$ with $\tilde{\omega}'$ given by

$$m > 0: \quad \tilde{\omega}' = \tilde{\omega} - \tilde{s}\Big(\frac{(-)^m}{m!} \tilde{C}^{i_1} \dots \tilde{C}^{i_m} \eta_{i_1\dots i_m}(T, x, dx)\Big). \tag{5.67}$$

Assuming that this redefinition has been performed and dropping the prime on $\tilde{\omega}'$, we obtain

$$\chi^{(m)}_{i_1\dots i_m}(T, x, dx) = P_{i_1\dots i_m}(dA). \tag{5.68}$$

At order $m - 1$ in the $\tilde{C}^i$ we now obtain from $\Omega = \tilde{s}\,\tilde{\omega}$:

$$\begin{aligned} m = 1: &\quad \Omega(T, x, dx) = dA^i\, P_i(dA) + d\,\chi^{(0)}(T, x, dx) \\ m > 1: &\quad 0 = \tilde{C}^{i_1} \dots \tilde{C}^{i_{m-1}} \Big(dA^{i_m} P_{i_1\dots i_m}(dA) + d\,\chi^{(m-1)}_{i_1\dots i_{m-1}}(T, x, dx)\Big) \end{aligned} \tag{5.69}$$

In the cases $m > 1$, this imposes

$$m > 1: \quad 0 = dA^{i_m} P_{i_1\dots i_m}(dA) + d\,\chi^{(m-1)}_{i_1\dots i_{m-1}}(T, x, dx). \tag{5.70}$$

$dA^{i_m} P_{i_1\dots i_m}(dA)$ has form degree $p - m + 1$ and thus, in the cases $m > 1$, $dA^{i_m} P_{i_1\dots i_m}(dA)$ has a form degree smaller than p. Hence, assuming that (5.50) holds for all form degrees smaller than p, we conclude from (5.70) by means of (5.50):

$$m > 1: \quad dA^{i_m} P_{i_1\dots i_m}(dA) = 0 = d\,\chi^{(m-1)}_{i_1\dots i_{m-1}}(T, x, dx) \ . \tag{5.71}$$

Equation (5.71) implies that $\tilde{C}^{i_1} \dots \tilde{C}^{i_m} P_{i_1\dots i_m}(dA)$ does not contribute to $\Omega = \tilde{s}\,\tilde{\omega}$ in the cases $m > 1$ because it is $\tilde{s}$-invariant, so the contribution $\tilde{C}^{i_1} \dots \tilde{C}^{i_m} \chi^{(m)}_{i_1\dots i_m}(T, x, dx)$ to $\tilde{\omega}$ can be assumed to vanish with no loss of generality whenever $m > 1$. In other words: with no loss of generality we can assume $m \le 1$.

Now, in the case $m = 1$, equation (5.69) yields $\Omega = d\,\eta(T, x, dx) + P(dA)$ with $\eta(T, x, dx) = \chi^{(0)}(T, x, dx)$ and $P(dA) = dA^i P_i(dA)$. If $m = 0$ then $\Omega = \tilde{s}\,\tilde{\omega}$ directly gives $\Omega = d\,\eta(T, x, dx)$ with $\eta(T, x, dx) = \chi^{(0)}(T, x, dx)$. This completes the inductive proof of (5.49).

Analogously one can prove (5.51) inductively, with d_ϕ in place of d and using the algebraic Poincaré lemma for d_ϕ (theorem 3.4).

The proof of (5.52) is direct because $P(dA, x, dx)$ does not contain a derivative of a linearized tensor while every monomial contained in $d_\phi\,\eta(T, x, dx)$ contains at least one such derivative.

The proof of (5.50) is technically somewhat more involved than the proof of (5.52) because d (in contrast to d_ϕ) contains the piece $d_x = dx^m \frac{\partial}{\partial x^m}$ which does not add derivatives of fields. To prove (5.50) we use induction with respect to the form degree of $P(dA)$. (5.50) is trivial in form degrees 0 and 1 because there is no $d\eta$ with form degree 0 and there is no $P(dA)$ with form degree 1. In the cases that $P(dA)$ has a form degree p with $p > 1$ we decompose η into pieces η_k with definite degree k in derivatives ∂. As the linearized tensors have definite degrees in derivatives, the pieces η_k are functions of x, dx and linearized tensors T,

$$\eta(T, x, dx) = \sum_{k=0}^{M} \eta_k(T, x, dx), \quad N_\partial\,\eta_k(T, x, dx) = k\,\eta_k(T, x, dx). \tag{5.72}$$

The decomposition terminates at some degree M in derivatives since η is a local form (by assumption). Each field monomial contained in $P(dA)$ has exactly $p/2 = r$ derivatives.

Assume now $M \ge r$. In these cases $d\,\eta(T, x, dx) + P(dA) = 0$ imposes $d_\phi\,\eta_M = 0$. The algebraic Poincaré lemma for d_ϕ (theorem 3.4) implies $\eta_M = d_\phi\,\chi_{M-1}$ for some χ_{M-1} with $M - 1$ derivatives ∂. (5.51) for form degree $p - 1$ now implies $\eta_M = d_\phi\,\chi'_{M-1}(T, x, dx)$ (a form $P'(dA, x, dx)$ cannot occur here because such a form would have form degree $p - 1$ and thus would only contain terms of degree $\le r - 1$ in the dA^i which do not contain more than $r - 1$ derivatives ∂, in contradiction to $M \ge r$). Now we consider $\eta' = \eta - d\chi'_{M-1}(T, x, dx)$ in place of η. η' fulfills $d\eta'(T, x, dx) + P(dA) = 0$ but contains only terms with less than M derivatives. In this way we successively remove from η all parts η_k with $k \ge r$.

Hence, we can assume $M < r$ with no loss of generality. In the case $M = r - 1$ we obtain $P(dA) = d_\phi\,\eta_{r-1}(T, x, dx)$ which by (5.52) implies $P(dA) = 0$. If $M < r - 1$ we directly obtain $P(dA) = 0$. So we conclude $P(dA) = 0 = d\eta(T, x, dx)$. This ends the proof of theorems 5.2 and 5.3.

Applied to tensor forms, which do not depend on the coordinates, theorem 5.3 states the solution of (5.46):

Theorem 5.4: *Linearized Covariant Poincaré Lemma for* d_ϕ *on tensor forms* $\omega(T, dx)$
(i) Let ω *be a linearized tensor form which does not depend on the* x^m*, then*

$$\begin{aligned} d_\phi\,\omega(T, dx) = 0 \;\Leftrightarrow\; & \omega(T, dx) = \mathcal{L}(T)\, d^D x + P(dA, dx) + d_\phi\,\eta(T, dx)\ , \\ \omega(T, dx) = d_\phi\,\chi \;\Leftrightarrow\; & \omega(T, dx) = P_1(dA, dx) + d_\phi\,\eta(T, dx)\ , \end{aligned} \tag{5.73}$$

where the Lagrangian form $\mathcal{L} d^D x$ *and* η *are linearized tensor forms which do not depend on the* x^m *and where* P *and* P_1 *are polynomials in the field strength two forms* $dA^i = \frac{1}{2} dx^m dx^n (\partial_m A_n{}^i - \partial_n A_m{}^i)$ *and the coordinate differentials* dx^m *and where* P_1 *has no field independent part,* $P_1(0, dx) = 0$*.*
(ii) A polynomial P *in the field strength two forms and the coordinate differentials* dx^m *cannot be written as exterior derivative* d_ϕ *of a tensor form* η *which does not depend on the* x^m*,*

$$P(dA, dx) + d_\phi\,\eta(T, dx) = 0 \;\Leftrightarrow\; P(dA, dx) = 0 = d_\phi\,\eta(T, dx)\ . \tag{5.74}$$

5.4 Chern Forms

For differential forms, which depend on tensors, this includes field strengths

$$F_{mn}{}^i = \partial_m A_n{}^i - \partial_n A_m{}^i - A_m{}^j A_n{}^k f_{jk}{}^i \tag{5.75}$$

and the covariant derivatives of tensors, and which are Lorentz invariant and isospin invariant, invariant for short, one immediately concludes:

Theorem 5.5: *Covariant Poincaré Lemma*
Let ω *be an invariant differential form which depends on tensors, then*

$$\begin{aligned} d\,\omega = 0 \;\Leftrightarrow\; & \omega = \mathcal{L}\, d^D x + P + d\eta\ , \\ \omega = d\,\chi \;\Leftrightarrow\; & \omega = P_1 + d\eta\ . \end{aligned} \tag{5.76}$$

The Lagrangian $\mathcal{L}$ *and the differential form* η *are invariant and depend on tensors,* P *and* P_1 *are invariant polynomials in the field strength two forms* $F^i = \frac{1}{2} dx^m dx^n F_{mn}{}^i$ *and* P_1 *has no constant,* F *independent part.*

The Lagrange density $\mathcal{L}\, d^D x$ cannot be written as $P(F) + d\eta$ if its Euler derivative does not vanish.

We call the invariant polynomials P Chern forms. They are polynomials in commuting variables, the field strength two forms $F = (F^1, F^2, \dots)$ which transform under the adjoint representation of the Lie algebra. These invariant polynomials are generated by the elementary Casimir invariants $I_\alpha(F)$.

The Chern forms comprise all topological densities which one can construct from connections for the following reason. If a functional is to contain only topological information its value must not change under continuous deformation of the fields. Therefore it has to be gauge invariant and invariant under general coordinate transformations. If it is a local functional it is the integral over a density which satisfies the descent equation and which can be obtained from a solution to $s\,\omega = 0$. If this density belongs to a functional which contains only topological information then the value of this functional must not change even under arbitrary differentiable variations of the fields, i.e. its Euler derivatives with respect to the fields must vanish. Therefore the integrand must be a total derivative in the space of jet variables. But it must not be a total derivative in the space of tensor variables because then it would be constant and contain no information at all. Therefore, by theorem 5.5, all topological densities which one can construct from connections are given by Chern polynomials in the field strength two form.

Theorem 5.5 describes also the cohomology of s_1 acting on invariant ghost forms because s_1 acts on invariant ghost forms exactly like the exterior derivative d acts on differential forms. We have to allow, however, for the additional variables $\theta(C)$ in $\omega_{\underline{n}}$. They generate a second, trivial algebra A_2 and can be taken into account by Künneth's theorem (theorem 3.6). If we neglect the trivial part $s_1\,\eta_{\text{inv}}$ then the solution to (5.36) is given by

$$\omega_{\underline{n}} = (-1)^D \mathcal{L}(\theta(C), T) c^1 c^2 \dots c^D + P(\theta(C), I(F)) \tag{5.77}$$

The δ_I invariant Lagrange ghost density satisfies already the complete equation $s\omega(C, T) = 0$ because it is a D ghost form. The solution to $\tilde{s}\,\tilde{\omega} = 0$ is given by $\tilde{\omega} = \omega(C + A, T)$ and the Lagrange density and the anomaly candidates are given by the part of $\tilde{\omega}$ with $d^D x$. The coordinate differentials come from $c^a - dx^m e_m{}^a$ (4.41) [2]. If one picks the D form part then one gets

$$dx^{m_1} \dots dx^{m_D} e_{m_1}{}^1 \dots e_{m_D}{}^D = \det(e_\cdot{}^\cdot)\, d^D x\ ,\ \det(e_\cdot{}^\cdot) =: \sqrt{g} \tag{5.78}$$

Therefore the solutions to the descent equations of Lagrange type are

$$\omega_D = \mathcal{L}(\theta(C), T)\, \sqrt{g}\, d^D x\ . \tag{5.79}$$

They are constructed in the well known manner from tensors T, including field strengths and covariant derivatives of tensors, which are combined to a Lorentz invariant and isospin invariant Lagrange function. This composite scalar field is multiplied by the density $\sqrt{g}$. Integrands of local gauge invariant actions are obtained from this formula by restricting ω_D to vanishing ghostnumber. Then the variables $\theta(C)$ do not occur. We indicate the ghostnumber by a superscript and have

$$\omega_D^0 = \mathcal{L}(T)\sqrt{g}\, d^D x\ . \tag{5.80}$$

Integrands of anomaly candidates are obtained by choosing D forms with ghostnumber 1. Only abelian factors of the Lie algebra allow for such anomaly candidates because the

[2] We can use the ghosts variables C or $\hat{C}$ (4.54). The expressions remain unchanged because they are multiplied by D translation ghosts.

primitive invariants θ_α for nonabelian factors have at least ghostnumber 3,

$$\omega^1_D = \sum_{i\,:\,\text{abelian}} C^i \mathcal{L}_i(T)\sqrt{g}\,d^D x\ . \tag{5.81}$$

The sum ranges over all abelian factors of the gauge group. Anomalies of this form actually occur as trace anomalies or β functions if the isospin algebra contains dilatations.

This completes the discussion of Lagrange densities and anomaly candidates coming from the first term in (5.77).

6 Chiral Anomalies

6.1 Chern Simons Forms

It remains to investigate solutions which correspond to

$$\omega_{\underline{n}} = P(\theta(C), I(F))\ . \tag{6.1}$$

Ghosts C^i for spin and isospin transformations and ghost forms F^i generate a subalgebra which is invariant under s and takes a particularly simple form if expressed in terms of matrices $C = C^i M_i$ and $F = F^i M_i$, where M_i represent a basis of the Lie algebra.

For nearly all algebraic operations it is irrelevant that F is a composite field. The transformation of C (5.4)

$$s\,C = C^2 + F \tag{6.2}$$

can be read as definition of an elementary (purely imaginary) variable F. The transformation of F follows from $s^2\,C = 0$ and turns out to be the adjoint transformation,

$$s\,F = -F\,C + C\,F\ . \tag{6.3}$$

Due to (6.2, 6.3) $s^2\,F$ vanishes identically.

If one changes the notation and replaces s by $d = dx^m\partial_m$ and C by $A = dx^m A_m{}^i M_i$ then the same equations are the definition of the field strengths in Yang Mills theories and their Bianchi identities,

$$F = d\,A - A^2\ ,\ \ d\,F - [A, F] = 0\ . \tag{6.4}$$

The equations are valid whether or not the anticommuting variables C and the nilpotent operation s are composite.[1]

The Chern polynomials I_α satisfy $s\,I_\alpha = 0$ because they are invariant under adjoint transformations. All I_α are trivial i.e. of the form $s\,q_\alpha$. To show this explicitly we define a one parameter deformation $F(t)$ of F ,

$$F(t) = tF - (t^2 - t)C^2 = t\ s\,C - t^2\,C^2\ ,\ \ F(0) = 0\ ,\ \ F(1) = F\ , \tag{6.5}$$

which allows to switch on F.

[1] This does not mean that there are no differences at all. For example the product of $D + 1$ matrix elements of the one form matrix A vanish.

All invariants I_α can be written as $\mathrm{tr}(F^{m(\alpha)})$ (if the representation matrices M_I are suitably chosen). We rewrite $\mathrm{tr}(F^m)$ in an artificially more complicated form

$$\mathrm{tr}(F^m) = \int_0^1 dt\,\frac{d}{dt}\,\mathrm{tr}(F(t)^m) = m\int_0^1 dt\ \mathrm{tr}\big((s\,C - 2t\,C^2)\,F(t)^{m-1}\big)\ . \tag{6.6}$$

The integrand coincides with

$$\begin{aligned} s\,\mathrm{tr}\,(CF(t)^{m-1}) &= \mathrm{tr}((s\,C)F(t)^{m-1} + t\,C\,[F(t)^{m-1}, C]) \\ &= \mathrm{tr}((s\,C)\,F(t)^{m-1} - 2t\,C^2\,F(t)^{m-1})\ . \end{aligned} \tag{6.7}$$

The Chern form I_α is the s transformation of the Chern Simons form q_α, these forms generate a subalgebra,

$$s\,q_\alpha = I_\alpha\ ,\ s\,I_\alpha = 0\ , \tag{6.8}$$

$$q_\alpha = m\int_0^1 dt\ \mathrm{tr}\Big(C\big(t\,F - (t^2 - t)\,C^2\big)^{m-1}\Big)\ ,\ m = m(\alpha)\ . \tag{6.9}$$

Using the binomial formula and

$$\int_0^1 dt\,t^k\,(1-t)^l = \frac{k!\,l!}{(k+l+1)!} \tag{6.10}$$

the t-integral can be evaluated. It gives the combinatorial coefficients of the Chern Simons form.

$$q_\alpha(C,F) = \sum_{l=0}^{m-1} \frac{(-)^l m!(m-1)!}{(m+l)!(m-l-1)!}\,\mathrm{tr_{sym}}\big(C(C^2)^l(F)^{m-l-1}\big) \tag{6.11}$$

It involves the traces of completely symmetrized products of the l factors C^2, the $m-l-1$ factors F and the factor C. The part with $l = m-1$ has form degree 0 and ghostnumber $2m-1$ and agrees with θ_α

$$q_\alpha(C,0) = \frac{(-)^{m-1}m!(m-1)!}{(2m-1)!}\,\mathrm{tr}\,C^{2m-1} = \theta_\alpha(C)\ ,\ m = m(\alpha)\ . \tag{6.12}$$

For notational simplicity we write θ for $\theta_1, \theta_2 \dots \theta_r$ and similarly q for $q_1, q_2 \dots q_r$ as well as I for $I_1, I_2 \dots I_r$.

Each polynomial $P(\theta, I)$ defines naturally a polynomial $P(q, I)$ and a form

$$\omega(C,F) = P(q(C,F), I(F)) \tag{6.13}$$

which coincides with $P(\theta(C), I(F))$ in lowest form degree,

$$\omega_{\underline{n}}(C,F) = P(\theta(C), I(F))\ . \tag{6.14}$$

On such polynomials $P(q, I)$ s acts simply as the operation

$$s = I_\alpha\frac{\partial}{\partial q_\alpha} \tag{6.15}$$

$$s\,\omega(C,F) = I_\alpha\frac{\partial}{\partial q_\alpha}P(q,I)_{|_{q(C,F),I(F)}} \tag{6.16}$$

The basic lemma (3.6) implies that among the polynomials $P(q, I)$ the only nontrivial solutions of $s\,P = 0$ are independent of q and I,

$$s\,P(q,I) = 0 \Leftrightarrow P(q,I) = P(0,0) + s\,Q(q,I)\ . \tag{6.17}$$

Though correct this result is misleading because we are not looking for solutions of $s\,P(q,I) = 0$, which depend on variables q and I, but solve $s\,\omega(q(C,F), I(F)) = 0$ in terms of functions of ghosts C and field strength two forms F. This equation has more solutions than (6.17) because $D+1$ forms vanish.

For $s\,P(q(C,F), I(F))$ to vanish it is necessary and sufficient that its lowest form degree is larger than D and for P not to vanish, its lowest form degree has to be D or smaller. The lowest form degree of Chern Simons forms $q_\alpha(C,0) = \theta_\alpha(C)$ vanishes, therefore the lowest form degree of polynomials $P(q, I)$ is counted by

$$N_{\mathrm{form}} = \sum_\alpha 2m(\alpha)\,I_\alpha\frac{\partial}{\partial I_\alpha}\ . \tag{6.18}$$

Because $I_\alpha = s\,q_\alpha$ has form degree $2m(\alpha)$, s increases the lowest form degree. However, the increase has different values.

6.2 Level Decomposition

To deal with this situation, we introduce the notation x_k for the variables q_α and I_α with a fixed $m(\alpha) = k$

$$\{x_k\} = \{q_\alpha, I_\alpha : m(\alpha) = k\} \tag{6.19}$$

and decompose the polynomial $P(q, I)$, which we consider as polynomial $P(x_1, x_2 \dots x_{\bar k})$, into pieces P_k which do not depend on the variables $x_1, \dots x_{k-1}$ and which vanish, if they do not depend on x_k [2],

$$P_1 = P(x_1, x_2, \dots, x_{\bar k}) - P(0, x_2, \dots, x_{\bar k})\ , \tag{6.20}$$

$$P_k = P(0, \dots, 0, x_k, x_{k+1}, \dots, x_{\bar k}) - P(0, \dots, 0, 0, x_{k+1}, \dots, x_{\bar k})\ , \tag{6.21}$$

$$P = \sum_k P_k\ . \tag{6.22}$$

We decompose the polynomial P_k à la Hodge (3.14) with s and

$$r_k = \sum_{m(\alpha)=k} q_\alpha\frac{\partial}{\partial I_\alpha}\ ,\ \{s, r_k\} = N_{x_k}\ , \tag{6.23}$$

[2] We assume without loss of generality $P(0,0\dots0)$ to vanish. It is not affected by the shift $C \mapsto C + A$ and does not contribute to a D-form which satisfies the descent equations.

as a sum of an s exact piece and an r_k exact piece

$$P_k = s\,\rho + r_k\,\sigma \tag{6.24}$$

No constant occurs in this Hodge decomposition of P_k, because P_k vanishes by construction if as function of x_k it is constant. Hodge decomposing σ shows that it can be taken to be s exact without loss of generality. Because the decomposition of P_k is unique, $r_k\,\sigma$ is nontrivial and can be taken to represent the nontrivial contribution to ω

$$P_k = r_k\,\sigma\ ,\ \ \sigma = s\,\eta\ . \tag{6.25}$$

It corresponds to a nontrivial solution $\omega(C,T) = P_k(q(C,F), I(F))$, if its lowest form degree is not larger than D, and if the lowest form degree of $s\,P_k$ is larger than D. Because r_k lowers the lowest form degree by 2k and s increases the lowest form degree of P_k by 2k, this means, that the lowest form degree D' of σ has to satisfy

$$D' - 2k \leq D < D'\ . \tag{6.26}$$

If we want to obtain a solution ω with a definite ghostnumber then we have to choose σ as eigenfunction of the ghost counting operator

$$N_C = \sum_\alpha \big(2m(\alpha) I_\alpha \frac{\partial}{\partial I_\alpha} + (2m(\alpha) - 1) q_\alpha \frac{\partial}{\partial q_\alpha}\big) \tag{6.27}$$

which counts the number of translation ghosts, spin and isospin ghosts. The total ghostnumber of $\omega = r_k\,\sigma$ is G if the total ghostnumber of σ is $G+1$, because r_k lowers the total ghostnumber by 1.

We obtain the long sought solutions ω^g_D of the relative cohomology (2.66) which for ghostnumber $g = 0$ gives Lagrange densities of invariant actions (2.52) and which for $g = 1$ gives anomaly candidates (2.65) if we substitute in ω the ghosts C by ghosts plus connection one forms $C + A$ and if we pick the part with D differentials. Therefore the total ghostnumber G of σ has to be chosen to be $G = g + D + 1$ to obtain a solution ω which contributes to ω^g_D. If the ghost variables $\hat{C}$ (4.54) are used to express ω then ω^g_D is simply obtained if all translation ghosts c^m are replaced by $-dx^m$ and the part with the volume element $d^D x$ is taken.

$$\begin{gathered}\omega(C,F) = \big(r_k\,\sigma\big)_{|q(C,F),I(F)}\ ,\ \ N_C\,\sigma = (g + D + 1)\,\sigma\ ,\ \ N_{\text{form}}\,\sigma = D'\sigma\ ,\\ \omega^g_D = \omega(\hat{C}, \frac{1}{2}\,dx^m dx^n\,F_{mn})_{|D\text{ form part}}\end{gathered} \tag{6.28}$$

These formulas end our general discussion of the BRST cohomology of gravitational Yang Mills theories. The general solution of the consistency equations is a linear combination of the Lagrangian solutions and the chiral solutions.

6.3 Anomaly Candidates

Let us conclude by spelling out the general formula for $g = 0$ and $g = 1$. If $g = 0$ then σ can contain no factors q_α because the complete ghostnumber $G \geq D'$ is not smaller than the ghostnumber D' of translation ghosts. D' has to be larger than D (6.26) and not larger than $G = g + D + 1 = D + 1$ which leaves $D' = D + 1$ as only possibility. D' is even (6.18), therefore chiral contributions to Lagrange densities occur only in odd dimensions.

If, for example $D = 3$, then σ is an invariant 4 form.

For $k = 1$ such a form is given by $\sigma = F_i F_j a^{ij}$ with $a^{ij} = a^{ji} \in \mathbb{R}$ if the isospin group contains abelian factors with the corresponding abelian field strength F_i and i and j enumerate the abelian factors. The form $\omega = r_1\,\sigma = 2 q_i F_j a^{ij}$ yields the gauge invariant abelian Chern Simons action in 3 dimensions which is remarkable because it cannot be constructed from tensor variables alone and because it does not contain the metric.

To construct ω^0_3 one has to express $q(C) = C$ by $C = \hat{C} + C^m A_m$. Then one has to replace all translation ghosts by differentials dx^m and to pick the volume form. One obtains

$$\omega^0_{3\text{abelian}} = dx^m A_{m\,i} dx^k dx^l F_{kl\,j} a^{ij} = \epsilon^{klm} A_{m\,i} F_{kl\,j} a^{ij} d^3x\ . \tag{6.29}$$

For $k = 2$ the form $\sigma = \text{tr}\,F^2$ of each nonabelian factor contributes to the nonabelian Chern Simons form. One has $I_1 = \text{tr}\,F^2 = s\,q_1$ and $\omega = r_2 I_1$ is given by the Chern Simons form q_1(6.11)

$$\omega = \text{tr}(C\,F - \frac{1}{3} C^3)\ . \tag{6.30}$$

The corresponding Lagrange density is

$$\omega^0_{3\text{nonabelian}} = \text{tr}(A\,F - \frac{1}{3} A^3) = \frac{1}{2}(A_m{}^i F_{rs}{}^i - \frac{1}{3} A_m{}^i A_r{}^j A_s{}^k f_{jk}{}^i)\,\epsilon^{mrs}\,d^3x \tag{6.31}$$

The integrands of chiral anomalies ω^1_D have ghostnumber $g = 1$. This fixes the total ghostnumber of σ to be $G = D + 2$ and because G is not less than $D' > D$ we have to consider the cases $D' = D + 1$ and $D' = D + 2$.

The first case can occur in odd dimensions only, because D' is even, and only if the level k, the smallest value of $m(\alpha)$ of the variables occurring in σ, is 1 because the missing total ghostnumber $D + 2 - D'$, which is not carried by $I_\alpha(F)$, has to be contributed by a Chern Simons polynomial q_α with $2m(\alpha) - 1 = 1$, i.e. with $m(\alpha) = 1$. Moreover $\sigma = s\,\eta$ has the form

$$\sigma = \sum_{ij\text{ abelian}} a^{ij}(I_\alpha)\,q_i\,I_j\ ,\ \ a^{ij} = -a^{ji}\ , \tag{6.32}$$

where the sum runs over the abelian factors and the form degrees contained in the antisymmetric a^{ij} and in the abelian $I_j = F_j$ have to add up to $D + 1$. In particular this anomaly can occur only if the gauge group contains at least two abelian factors because a^{ij} is antisymmetric. In $D = 3$ dimensions a^{ij} is linear in abelian field strengths and one has

$$\sigma = \sum_{ijk\,\text{abelian}} a^{ijk}\,q_i\,I_j\,I_k\ ,\ \ a^{ijk} = a^{ikj}\ ,\ \ \oint_{ijk} a^{ijk} = 0\ . \tag{6.33}$$

This leads to

$$\omega = r_1\,\sigma = \sum_{ijk\,\text{abelian}} b^{ijk}\, q_i\, q_j\, I_k = \sum_{ijk\,\text{abelian}} b^{ijk} C_i C_j F_k \ , \ b^{ijk} = -a^{ijk} + a^{jik} \tag{6.34}$$

and the anomaly candidate is

$$\omega^1_3 = 2 \sum_{ijk\,\text{abelian}} b^{ijk}\,\hat{C}_i\,A_j\,F_k = \sum_{ijk\,\text{abelian}} b^{ijk}\,\hat{C}_i\,A_{m\,j}\,F_{rs\,k}\,\epsilon^{mrs}\,d^3x\ . \tag{6.35}$$

If one considers $g = 1$ and $D = 4$ then $D' = 6$ because it is bounded by $G = D + 1 + g = D + 2$, larger than D and even. This leaves $D' = G$ as only possibility, so the total ghostnumber is carried by the translation ghosts contained in $\sigma = \sigma(I_\alpha)$ which is a cubic polynomial in the field strength two forms F. Abelian two forms can occur in the combination

$$\sigma = \sum_{ijk\,\text{abelian}} d^{ijk}\,F_i\,F_j\,F_k \tag{6.36}$$

with completely symmetric coefficients d^{ijk}. These polynomials are s exact. They lead to the abelian anomaly

$$\begin{aligned}\omega^1_{4\text{abelian}} &= \frac{3}{4} \sum_{ijk\,\text{abelian}} d^{ijk}\,\hat{C}_i\,F_{mn\ j}\,F_{rs\ k}\,\epsilon^{mnrs}\,d^4x \\ &= 3 \sum_{ijk\,\text{abelian}} d^{ijk}\,\hat{C}_i\,dA_j\,dA_k\end{aligned} \tag{6.37}$$

Abelian two forms F_i can also occur in σ multiplied with $\text{tr}(F_k)^2$ where i enumerates abelian factors and k nonabelian ones. The mixed anomaly which corresponds to

$$\sigma = \sum_{ik} c^{ik} F_i\,\text{tr}_k(F^2) \tag{6.38}$$

is similar in form to the abelian anomaly

$$\omega^1_{4\text{mixed}} = -\frac{1}{4} \sum_{ik} c^{ik}\,\hat{C}_i\,\Big(\sum_I F_{mn}{}^I F_{rs}{}^I\Big)_k\,\epsilon^{mnrs}\,d^4x\ . \tag{6.39}$$

The sum, however extends now over abelian factors enumerated by i and nonabelian factors enumerated by k. Moreover we assumed that the basis, enumerated by I, of the simple Lie algebras is chosen such that $\text{tr}\,M_I M_J = -\delta_{IJ}$ holds for all k. Phrased in terms of dA the mixed anomaly differs from the abelian one because the nonabelian field strength contains also A^2 terms [3].

$$\omega^1_{4\text{mixed}} = \sum_{ik} c^{ik}\hat{C}_i\,\text{tr}_k\,d\Big(A\,dA + \frac{2}{3}A^3\Big) \tag{6.40}$$

[3]The trace over an even power of one form matrices A vanishes.

The last possibility to construct a polynomial σ with form degree $D' = 6$ is given by the Chern form $\text{tr}(F)^3$ itself. Such a Chern polynomial with $m = 3$ exists for classical algebras only for the algebras $SU(n)$ for $n \geq 3$ (5.26) [4]. In particular the Lorentz symmetry in $D = 4$ dimensions is not anomalous. The form ω which corresponds to the Chern form is the Chern Simons form

$$\omega(C, F) = \text{tr}\Big(C\,F^2 - \frac{1}{2}C^3\,F + \frac{1}{10}C^5\Big)\ . \tag{6.41}$$

The nonabelian anomaly follows after the substitution $C \to C + A$ and after taking the volume form

$$\begin{aligned}\omega^1_{4\text{nonabelian}} &= \text{tr}\Big(\hat{C}F^2 - \frac{1}{2}(\hat{C}\,A^2F + A\hat{C}\,A\,F + A^2\hat{C}\,F) + \frac{1}{2}\hat{C}\,A^4\Big) \\ &= \text{tr}\Big(\hat{C}\,d\big(A\,dA + \frac{1}{2}A^3\big)\Big)\ .\end{aligned} \tag{6.42}$$

[4]The Lie algebra $SO(6)$ is isomorphic to $SU(4)$.

7 Inclusion of Antifields

7.1 BRST-Antifield Formalism

The BRST-antifield formalism (or field-antifield formalism, or BV formalism) originated in the context of the renormalization of Yang-Mills theories where external sources for the BRST transformations of the fields and ghost fields were introduced [15]. Later it was realized that on the field antifield algebra one could extend the BRST methods to gauge theories with open algebras, i.e. with commutator algebras which close only on-shell [16, 17, 18]. With fields and antifields one can treat the equations of motions, Noether identities and further reducibility identities as objects which occur in a cohomological problem [19, 20]. Here, we restrict ourselves to discuss some selected features of this cohomology. For the general structure of the formalism we refer to the literature [4, 21].

The formalism comprises "fields" and "antifields". The set of fields contains the fields of the classical theory which we denote by φ^i and ghost fields denoted by $\hat{C}^\alpha$ which correspond to the gauge symmetries (α enumerates a generating set of gauge symmetries [20, 4]). In addition the set of fields may contain further fields, such as "ghosts of ghosts" when the gauge transformations are reducible, or antighost fields $\bar{C}$ (which must not be confused with the antifields of the ghosts) used for gauge fixing but this is not relevant to the matters to be discussed later on. To distinguish fields from antifields we mark the latter by a superscript $\star$ (which must not be confused with the symbol $*$ used for complex conjugation). There is one antifield $\varphi^\star$ for each equation of motion of the classical theory and one antifield $C^\star$ for each nontrivial identity of these equations of motion. In a Lagrangian field theory with Lagrangian $\mathcal{L}(x,\{\varphi\})$ there is one equation of motion for each field φ^i given by $\frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\varphi^i} = 0$ which sets to zero the Euler derivative (3.35) of $\mathcal{L}$ with respect to φ^i. Furthermore, the gauge symmetries of a Lagrangian correspond one-to-one to the nontrivial (Noether) identities relating the equations of motion (by Noethers second theorem). Therefore, in a Lagrangian field theory the fields φ^i and $\hat{C}^\alpha$ correspond one-to-one to antifields $\varphi^\star_i$ and $C^\star_\alpha$.

We denote the infinitesimal gauge transformations of the fields φ^i by $\delta_\epsilon\varphi^i = R^i_\alpha\epsilon^\alpha$ where ϵ^α are the parameters of gauge transformations and R^i_α are (in general field dependent) differential operators acting on ϵ^α according to

$$\delta_\epsilon\varphi^i = R^i_\alpha\,\epsilon^\alpha, \quad R^i_\alpha\,\epsilon^\alpha = \sum_k (\partial_{m_1}\dots\partial_{m_k}\epsilon^\alpha)\, r_\alpha^{i m_1\dots m_k}(x,\{\varphi\}). \tag{7.1}$$

By assumption these gauge transformations generate symmetries of a Lagrangian $\mathcal{L}$, i.e. the Lagrangian $\mathcal{L}$ transforms under δ_ϵ into a total derivative,

$$\delta_\epsilon\,\mathcal{L} = \partial_m K^m. \tag{7.2}$$

The Euler derivative of (7.2) with respect to ϵ^α gives the Noether identity for the αth gauge symmetry. This Noether identity reads

$$R_\alpha^{i\dagger}\,\frac{\hat\partial \mathcal{L}}{\hat\partial \varphi^i} = 0 \tag{7.3}$$

where $R_\alpha^{i\dagger}$ is an operation transposed to R_α^i,

$$R_\alpha^{i\dagger}\chi = \sum_k (-)^k \partial_{m_1}\dots\partial_{m_k}(r_\alpha^{im_1\dots m_k}(x,\{\varphi\})\,\chi). \tag{7.4}$$

By assumption the set of gauge transformations is closed under commutation up to trivial gauge transformations, i.e. the commutator of any two gauge transformations is again a gauge transformation at least on-shell,

$$[\delta_\epsilon, \delta_{\epsilon'}] \approx \delta_f\,,\ \ f^\alpha = f^\alpha(x,\{\epsilon,\epsilon',\varphi\}) \tag{7.5}$$

where $f^\alpha(x,\{\epsilon,\epsilon',\varphi\})$ are local structure functions of the parameters ϵ^α, ϵ'^α, the fields φ^i and derivatives thereof, and $\approx$ denotes "weak equality" defined according to

$$F(x,\{\varphi\}) \approx 0 \quad \Leftrightarrow \quad F(x,\{\varphi\}) = \sum_k g^{im_1\dots m_k}(x,\{\varphi\})\partial_{m_1}\dots\partial_{m_k}\frac{\hat\partial\mathcal{L}}{\hat\partial\varphi^i}\,. \tag{7.6}$$

Hence, two functions are weakly equal iff they differ only by terms which are at least linear in the Euler derivative $\frac{\hat\partial\mathcal{L}}{\hat\partial\varphi^i}$ or its derivatives. As the Euler derivative vanishes on-shell (i.e., for all solutions of the equations of motion), weak equality is "equality on-shell".

The BRST transformations of the $\varphi, \hat C, \varphi^\star, C^\star$ take the form[1]

$$\begin{aligned}
s\,\varphi^i &= -R_\alpha^i(x,\{\varphi\},\partial)\hat C^\alpha + \dots\ ,\\
s\,\hat C^\alpha &= \frac{1}{2}\, f^\alpha(x,\{\epsilon,\epsilon',\varphi\})\Big|_{\epsilon^\alpha=\hat C^\alpha,\,\epsilon'^\alpha=(-)^{|\hat C^\alpha|}\hat C^\alpha} + \dots\ ,\\
s\,\varphi_i^\star &= (-)^{|\varphi^i|}\,\frac{\hat\partial\mathcal{L}(x,\{\varphi\})}{\hat\partial\varphi^i} + \dots\ ,\\
s\,C_\alpha^\star &= (-)^{|\hat C^\alpha|}R_\alpha^{i\dagger}(x,\{\varphi\},\partial)\varphi_i^\star + \dots
\end{aligned} \tag{7.7}$$

with ellipsis indicating antifield dependent contributions. The grading $|\hat C^\alpha|$ of the ghost $\hat C^\alpha$ is opposite to the grading of the corresponding parameter ϵ^α. Furthermore the grading of an antifield is always opposite to the grading of the corresponding field:

$$|\hat C^\alpha| = |\epsilon^\alpha| + 1 \bmod 2, \quad |\varphi_i^\star| = |\varphi^i| + 1 \bmod 2, \quad |C_\alpha^\star| = |\hat C^\alpha| + 1 \bmod 2. \tag{7.8}$$

The ghostnumbers of a field and the corresponding antifield add up to minus one:

$$\mathrm{gh}(\varphi^i) = 0,\ \mathrm{gh}(\varphi_i^\star) = -1,\ \mathrm{gh}(\hat C^\alpha) = 1,\ \mathrm{gh}(C_\alpha^\star) = -2. \tag{7.9}$$

[1] We use conventions such that $s : \chi \mapsto (S,\chi)$ where $S = \int d^Dx(\mathcal{L} + (R_\alpha^i\hat C^\alpha)\varphi_i^\star + \dots)$ solves the master equation $(S,S) = 0$ with the standard antibracket [18, 4, 21].

A very useful concept for discussing various aspects of the BRST-antifield formalism is the decomposition of the ghostnumber into a pure ghostnumber (pgh) and an antifield number (af) according to

$$\begin{aligned}
&\mathrm{gh} = \mathrm{pgh} - \mathrm{af},\\
&\mathrm{af}(\varphi^i) = 0,\ \mathrm{af}(\hat C^\alpha) = 0,\ \mathrm{af}(\varphi_i^\star) = 1,\ \mathrm{af}(C_\alpha^\star) = 2,\\
&\mathrm{pgh}(\varphi^i) = 0,\ \mathrm{pgh}(\hat C^\alpha) = 1,\ \mathrm{pgh}(\varphi_i^\star) = 0,\ \mathrm{pgh}(C_\alpha^\star) = 0.
\end{aligned} \tag{7.10}$$

s decomposes into parts of various antifield numbers ≥ -1,

$$s = \delta + \gamma + \dots,\quad \mathrm{af}(\delta) = -1,\quad \mathrm{af}(\gamma) = 0 \tag{7.11}$$

where the ellipsis indicates parts with antifield numbers ≥ 1. The parts δ and γ are the two crucial ingredients of s. In particular they determine the structure of the BRST cohomology. The part δ is often called the Koszul-Tate differential. It is the part of s which lowers the antifield number and therefore vanishes on the fields

$$\delta\varphi^i = 0,\ \delta\hat C^\alpha = 0\ ,\ \delta\varphi_i^\star = (-)^{|\varphi^i|}\frac{\hat\partial\mathcal{L}}{\hat\partial\varphi^i}\ ,\ \delta C_\alpha^\star = (-)^{|\hat C^\alpha|}R_\alpha^{i\dagger}\varphi_i^\star\,. \tag{7.12}$$

In particular the Koszul-Tate differential is the part of s which implements the equations of motion and the Noether identities in cohomology by the δ-transformations of the $\varphi^\star$ and $C^\star$. It is nilpotent by itself. For instance, owing to (7.3) one gets

$$\delta^2 C_\alpha^\star = (-)^{|\hat C^\alpha|+|\epsilon^\alpha|+|\varphi^i|}R_\alpha^{i\dagger}\delta\varphi_i^\star = -R_\alpha^{i\dagger}\frac{\hat\partial\mathcal{L}}{\hat\partial\varphi^i} = 0. \tag{7.13}$$

In Yang Mills theories, Einstein gravity and gravitational Yang Mills theories with the standard gauge transformations the decomposition (7.11) of s terminates with γ, i.e. in these cases one simply has $s = \delta + \gamma$, because the commutator algebra of the gauge transformations closes even off-shell (i.e. (7.5) holds with $=$ in place of $\approx$) and because the structure functions f^α do not depend on fields φ. Hence, on the fields $\varphi, \hat C$ one has in these cases $s\,\varphi = \gamma\varphi$ and $s\,\hat C = \gamma\hat C$. With respect to γ the antifields $\varphi^\star, C^\star$ are tensors or, in the gravitational case, tensor densities with weight one. For instance, in pure Yang Mills theories in flat space-time, with semisimple isospin Lie algebra and Lagrangian $\mathcal{L} = -\frac{1}{4}d_{ij}F_{mn}{}^iF^{mnj}$ (where $F_{mn}{}^i = \partial_m A_n{}^i - \partial_n A_m{}^i - A_m{}^jA_n{}^kf_{jk}{}^i$ and d_{ij} is the Cartan-Killing metric of the isospin Lie algebra), the φ are the components $A_m{}^i$ of the gauge fields and the $\hat C$ are the Yang Mills ghosts $\hat C^i$.[2] Denoting the corresponding antifields $A^{\star m}{}_i$ and $C^\star{}_i$ one obtains

$$\begin{aligned}
\delta A_m{}^i &= 0\,, & \gamma A_m{}^i &= \partial_m\hat C^i + \hat C^jA_m{}^kf_{jk}{}^i\,,\\
\delta\hat C^i &= 0\,, & \gamma\hat C^i &= \frac{1}{2}\,\hat C^j\hat C^kf_{jk}{}^i\,,\\
\delta A^{\star m}{}_i &= d_{ij}D_nF^{nmj}\,, & \gamma A^{\star m}{}_i &= -\hat C^jf_{ji}{}^kA^{\star m}{}_k\,,\\
\delta C^\star{}_i &= D_mA^{\star m}{}_i\,, & \gamma C^\star{}_i &= -\hat C^jf_{ji}{}^kC^\star{}_k
\end{aligned} \tag{7.14}$$

[2] Depending on the context, i numbers all fields when we refer to a general gauge theory, whereas in Yang Mills theories it enumerates a basis of the Lie algebra.

where $D_m = \partial_m + A_m{}^i\delta_i$ denotes the covariant derivative. Notice that one has $\gamma A^{\star m}{}_i = -\hat{C}^j\delta_j A^{\star m}{}_i$ and $\gamma C^\star{}_i = -\hat{C}^j\delta_j C^\star{}_i$, i.e. the antifields are indeed treated as tensors by γ.

7.2 The Antifield Dependent BRST Cohomology

We briefly indicate how one can compute the BRST cohomology in gravitational Yang Mills theories in presence of antifields along the lines of sections 3 to 6 and discuss two different strategies to adapt the analysis in order to deal with the antifields. The first strategy eliminates the antifields by a suitable change of variables [22, 23]. The second strategy keeps the antifields throughout the analysis [25, 26, 27, 24]. Even though both strategies appear to be rather different, they are closely related and, of course, they lead to the same results.

7.2.1 First Strategy

Equations (7.7) indicate that the antifields and all their derivatives might be removed from the cohomological analysis for $\tilde{s}$ (and analogously for s) by the arguments given in section 4.3 because each antifield variable $(\varphi^\star, C^\star, \partial\varphi^\star, \dots)$ might be taken as a variable u or be replaced by a variable $\tilde{s}\,u$, respectively. Indeed, for a standard Lagrangian $\mathcal{L}$, $\tilde{s}\,\varphi^\star_i$ contains a piece linear in the fields φ and their derivatives given by the linearized Euler derivative $\frac{\hat{\partial}\mathcal{L}}{\partial\varphi^i}$. Analogously, $\tilde{s}\,C^\star_\alpha$ contains a piece that is linear in the $\varphi^\star$ and their derivatives given by the linearization of $(-)^{|\hat{C}^\alpha|}R^{i\dagger}_\alpha\varphi^\star_i$.

As a consequence, one can eliminate the antifields and all their derivatives from the cohomological analysis for $\tilde{s}$ provided one can construct a new set of variables replacing all the field and antifield variables and consisting of variables u and $\tilde{s}\,u$ and complementary variables $\tilde{w} = (\tilde{C}, \tilde{T})$ such that $\tilde{s}\,\tilde{w} = F(\tilde{w})$ with a set of "generalized tensors" $\tilde{T}$.[3] It can be shown quite generally that such a set of variables exists [23]. In particular it exists for standard Yang Mills theories, Einstein gravity and gravitational Yang Mills theories. However, two important consequences of this strategy have to be pointed out.

Firstly, the set of generalized tensors $\tilde{T}$ contains *fewer* variables than the corresponding set of tensors T in the antifield independent cohomology because along with the elimination of the antifields one also eliminates tensors T that correspond to the Euler derivatives $\frac{\hat{\partial}\mathcal{L}}{\partial\varphi^i}$ and their derivatives. For instance, (7.14) shows that in pure Yang Mills theories the set of generalized tensors $\tilde{T}$ does not contain elements corresponding to the tensors $D_n F^{nmi}$ as these are eliminated along with the antifields $A^{\star m}{}_i$ and an analogous statement applies to all covariant derivatives of the $D_n F^{nmi}$. Hence, there are (combinations of) tensors T which have no counterpart in the set of generalized tensors $\tilde{T}$ because they are set to zero by the equations of motion and their derivatives (and, in fact, there are infinitely many such tensors).

[3]It turns out that in standard Yang Mills theories, Einstein gravity and gravitational Yang Mills theories one can use the same variables $\tilde{C}$ in the antifield dependent case as in the antifield independent case. Therefore we do not change the notation concerning these variables.

Secondly, each generalized tensor $\tilde{T}$ has an antifield independent part $\tilde{T}_0 = \tilde{T}|_{\varphi^\star=0=C^\star}$ and the set of $\tilde{T}_0$ may be taken as a subset of the set of tensors T. However, some of the $\tilde{T}$ also contain antifield dependent contributions [23]. As a consequence, even though the cohomology can be computed completely in terms of the variables $\tilde{C}, \tilde{T}$, some of the nontrivial representatives of the cohomology may depend on antifields through the dependence of variables $\tilde{T}$ on antifields. This is analogous to the way in which the undifferentiated gauge fields $A_m{}^i$ enter nontrivial representatives of the antifield independent cohomology. Namely, the $A_m{}^i$ are variables u but are also used within the construction of the variables $\tilde{C}$ and T through field strengths, covariant derivatives of tensors and $\tilde{C} = C + A$. Hence, even though the undifferentiated gauge fields $A_m{}^i$ are eliminated from the cohomological analysis as variables u, they nevertheless enter representatives of the cohomology through their occurrence within the variables $\tilde{C}$ and T.

Let us now assume that the antifields have been eliminated. In that case one is left with the computation of the cohomology in a space of functions $f(\tilde{C}, \tilde{T})$. On the variables $\tilde{C}, \tilde{T}$ one has $\tilde{s}\,\tilde{T} = -\tilde{C}^N\Delta_N\tilde{T}$ and $\tilde{s}\,\tilde{C}^N = -\frac{1}{2}\tilde{C}^K\tilde{C}^L\tilde{F}_{LK}{}^N$, analogously to the antifield independent case. As a consequence, in order to analyse the cohomology of $\tilde{s}$ in the space of functions $f(\tilde{C}, \tilde{T})$ one can proceed exactly as in the antifield independent case until one arrives at the counterpart of equation (5.46) which we write as

$$d_{\tilde{T}}\,\omega(\tilde{T}, dx) = 0\ ,\ \ \omega \text{ mod } d_{\tilde{T}}\eta(\tilde{T}, dx) \tag{7.15}$$

where $\omega(\tilde{T}, dx)$ and $\eta(\tilde{T}, dx)$ are spin and isospin invariant forms which depend on the generalized tensors $\tilde{T}$. Here we use the notation $d_{\tilde{T}}$ to stress that this operation is the exterior derivative on forms $\omega(\tilde{T}, dx)$ of the generalized tensors $\tilde{T}$. $d_{\tilde{T}}$ acts on forms $\omega(\tilde{T}, dx)$ substantially different from d acting on forms $\omega(T, dx)$ because the ideal of tensors T, which contain an Euler derivative or derivatives of Euler derivatives, have been eliminated together with the antifields from the tensor algebra.

To make this point clear, let us compare $d\,\tilde{T}_0$ to the antifield independent part $(d_{\tilde{T}}\,\tilde{T})_0$ of $d_{\tilde{T}}\,\tilde{T}$. $d\,\tilde{T}_0$ in general contains tensors T which are eliminated along with the antifields whereas $(d_{\tilde{T}}\,\tilde{T})_0$ does not contain any of these tensors T. Now recall that the tensors T which are eliminated along with the antifields are just those that are set to zero by the equations of motion (or by the linearized equations of motion when one uses linearized tensors) and derivatives thereof. Hence, in general $d\,\tilde{T}_0$ is only weakly equal to $(d_{\tilde{T}}\,\tilde{T})_0$! Therefore, the problem posed by equation (7.15) is equivalent to the *weak cohomology* of d (i.e., the cohomology of d on-shell) on invariant tensor forms $\omega(\tilde{T}_0, dx)$, with the cocycle condition

$$d\,\omega(\tilde{T}_0, dx) \approx 0 \tag{7.16}$$

and coboundaries fulfilling $\omega(\tilde{T}_0, dx) \approx d\eta(\tilde{T}_0, dx)$. The solution of this cohomological problem is the analog of the covariant Poincaré lemma in the antifield independent cohomology and may therefore be termed "weak covariant Poincaré lemma" [24] and will be briefly discussed below. By means of the weak covariant Poincaré lemma one can finish the computation of the BRST cohomology in presence of the antifields along the lines applied in the antifield independent case. We shall briefly sketch the results below.

7.2.2 Second Strategy

The second strategy treats the antifields as additional tensors $T^\star$ ("antitensors"). This is possible because, as we have pointed out above, the antifields $\varphi^\star, C^\star$ transform as tensors or (in the gravitational case) as tensor densities under the part γ of s. Hence, the undifferentiated antifields $\varphi^\star, C^\star$ or (in the gravitational case) $\varphi^\star/e, C^\star/e$ (with $e = \det e_m{}^a$) can be viewed as tensors. Standard covariant derivatives of these antifield variables transform again as tensors under γ and are used as antitensors $T^\star$ that substitute for the derivatives of the $\varphi^\star, C^\star$. The difference of the $T^\star$ and the T is that the Koszul-Tate part δ of s acts nontrivially on the $T^\star$. However δ maps each antitensor $T^\star$ to (a combination of) tensors T or antitensors $T^\star$. Therefore the space of functions $f(C, T, T^\star)$ is invariant under s, just as the space of functions $f(C, T)$.

This allows one to extent the methods used in the antifield independent case straightforwardly to the antifield dependent case, with $T, T^\star$ in place of T and with $s = \delta + \gamma$, until one arrives at equation (5.18). In the latter equation one now gets $\delta + s_c$ in place of s_c. In place of equation (5.22) one therefore gets $(\delta + s_c)\omega = 0$ for the part of lowest order in the translation ghosts. As δ acts nontrivially only on the antitensors $T^\star$ and does not affect the dependence on the C at all, and as s_c acts nontrivially only on the C and does not affect the dependence on tensors or antitensors at all, one gets (using Künneth's Theorem) $\omega = \sum_l \Theta_l(C) f^l(c, T, T^\star)$ with $s_c\, \Theta_l(C) = 0$ and $\delta f^l(c, T, T^\star) = 0$. $\delta f^l(c, T, T^\star) = 0$ implies that one may take $f^l = f^l(c, T_0)$ where the T_0 form a subset of the tensors T which corresponds to the set of generalized tensors $\tilde{T}$ of the first strategy. The reason is the following: just as one can eliminate the antifields from the cohomology of $\tilde{s}$ along with a subset of weakly vanishing tensors T, one can also eliminate the antifields from the cohomology of δ along with the same subset of weakly vanishing tensors T. The remaining sets of generalized tensors $\tilde{T}$ and tensors T_0 correspond to each other, and one may actually take the set of T_0 identical to the set of $\tilde{T}_0$. The conditions $s_c\, \Theta_l(C) = 0$ are treated exactly as in the antifield independent case. As a consequence one gets $\omega_{\underline{n}} = \Phi(\theta(C), c, T_0)$ in place of equation (5.35).

The next change is in equation (5.36) where again $\delta + s_c$ replaces s_c. Consequently one gets a δ-exact term $\delta(\dots)$ in the subsequent equations (5.38), (5.39) and (5.43). In particular, in place of equation (5.43) one gets $s_{1,1}\, f_{\underline{n}}(c, T_0) + \delta f_{\underline{n}+1}(c, T, T^\star) = 0$ (with $f_{\underline{n}} \bmod s_{1,1}\, \eta_{\underline{n}-1} + \delta\eta_{\underline{n}}$). Eventually equation (5.46) is replaced by

$$
\begin{aligned}
&d\,\omega_p(T_0, dx) + \delta\omega_{p+1}(T, T^\star, dx) = 0\,,\\
&\omega_p(T_0, dx) \mod d\eta_{p-1}(T, dx) + \delta\eta_p(T, T^\star, dx)\,.
\end{aligned}
\tag{7.17}
$$

Now, the problem posed by (7.17) is exactly the same as the problem associated with equation (7.16). Namely, in (7.17) $\omega_{p+1}(T, T^\star, dx)$ has antifield number one because $\omega_p(T_0, dx)$ has antifield number zero (since $\omega_p(T_0, dx)$ does not depend on antifields at all). Hence, $\omega_{p+1}(T, T^\star, dx)$ is linear in those $T^\star$ that correspond to the $\varphi^\star_i$ and their derivatives. As the δ-transformations of these antifields vanish weakly, one has $\delta\omega_{p+1} \approx 0$ and thus $d\,\omega_p(T_0, dx) \approx 0$ which reproduces precisely (7.16) because, as we mentioned above, the set of tensors T_0 corresponds to the set of tensors $\tilde{T}_0$. Analogous statements apply to the coboundary conditions.

7.3 Characteristic Cohomology and Weak Covariant Poincaré Lemma

The cohomological problem posed by (7.16) and (7.17) correlates the BRST cohomology to the weak cohomology of d on forms $\omega(\{\varphi\}, x, dx)$ (without restricting these forms to tensor forms or invariant tensor forms). This cohomology has been termed characteristic cohomology (of the equations of motion) [28] and is interesting on its own because it generalizes the concept of conserved currents. To explain this, we write a p-form $\omega(\{\varphi\}, x, dx)$ as

$$
\omega_p = \frac{1}{p!(D-p)!}\, dx^{m_1} \dots dx^{m_p}\, \varepsilon_{m_1\dots m_D}\, j^{m_{p+1}\dots m_D}(\{\varphi\}, x) \tag{7.18}
$$

where the ε-symbol is completely antisymmetric and $\varepsilon_{0\dots(D-1)} = 1$. The condition $d\,\omega_p \approx 0$ of the characteristic cohomology in form degree $p < D$ is equivalent to

$$
\partial_{m_1} j^{m_1\dots m_{D-p}}(\{\varphi\}, x) \approx 0\,. \tag{7.19}
$$

For $p = D-1$ this gives $\partial_m j^m \approx 0$ which determines conserved currents. The representatives of the characteristic cohomology with $p < D$ are thus conserved differential p-forms of the fields which for $p = D-1$ provide the conserved currents. Coboundaries of the characteristic cohomology are weakly d-exact forms $\omega_p \approx d\,\eta_{p-1}$ which are equivalent to $j^{m_1\dots m_{D-p}}(\{\varphi\}, x) \approx \partial_{m_0} k^{m_0\dots m_{D-p}}(\{\varphi\}, x)$ where $k^{m_0\dots m_{D-p}} = k^{[m_0\dots m_{D-p}]}$ is completely antisymmetric.

A remarkable feature of the characteristic cohomology is that the reducibility order r of a theory gives a bound on the formdegrees p below which the characteristic cohomology is trivial, provided the theory is a "normal theory" [29]. In this context one assigns reducibility order $r = -1$ to a theory which has no nontrivial gauge symmetries, $r = 0$ to a gauge theory with irreducible gauge transformations (such as standard Yang Mills theories or Einstein gravity), $r = 1$ to a gauge theory with gauge transformations which are reducible of first order etc. It was proved for "normal theories" and $D > r + 2$ that the characteristic cohomology is trivial in all form degrees smaller than $D - r - 2$ [29]:

$$
\begin{aligned}
0 < p < D - r - 2: &\quad d\,\omega_p \approx 0 \quad \Leftrightarrow \quad \omega_p \approx d\,\eta_{p-1}\,,\\
p = 0: &\quad d\,\omega_0 \approx 0 \quad \Leftrightarrow \quad \omega_0 \approx \text{constant}\,.
\end{aligned}
\tag{7.20}
$$

For standard Yang Mills theories, Einstein gravity and gravitational Yang Mills theories this result implies that the characteristic cohomology is trivial in formdegrees $p < D-2$ if $D > 2$. Furthermore for these theories the characteristic cohomology in formdegree $p = D-2$ is represented by $(D-2)$-forms corresponding one-to-one to "free Abelian gauge symmetries" which act nontrivially only on the corresponding Abelian gauge fields and leave invariant all other fields [29]. In formdegree $p = D-1$ the characteristic cohomology is represented by "Noether forms" J involving the nontrivial Noether currents j^m,

$$
J = \frac{1}{(D-1)!}\, dx^{m_1} \dots dx^{m_{D-1}}\, \varepsilon_{m_1\dots m_D}\, j^{m_D}\,, \quad \partial_m\, j^m \approx 0\,. \tag{7.21}
$$

We shall now sketch a derivation of the result (7.20) for irreducible gauge theories ($r = 0$). In these theories there are only antifields $\varphi^\star$ with antifield number 1 and antifields $C^\star$ with antifield number 2 (but no antifields with antifield numbers $k > 2$). The derivation relates the characteristic cohomology to the cohomology of δ modulo d on forms $\omega(\{\varphi, \varphi^\star, C^\star\}, x, dx)$ of fields and antifields via descent equations for δ and d and analyses these descent equations. In order to simplify the notation we shall assume that all fields φ and antifields $C^\star$ are bosonic.

As discussed above the cocycle condition $d\,\omega^0_p(\{\varphi\}, x, dx) \approx 0$ can be written as $d\,\omega^0_p + \delta\,\omega^1_{p+1} = 0$ where the subscript of a form ω^k_p denotes the formdegree and the superscript denotes the antifield number. Applying d to $d\,\omega^0_p + \delta\,\omega^1_{p+1} = 0$ one obtains $\delta\,(d\,\omega^1_{p+1}) = 0$. As $d\,\omega^1_{p+1}$ has antifield number 1 and the cohomology of δ is trivial in positive antifield numbers this implies $d\,\omega^1_{p+1} + \delta\,\omega^2_{p+2} = 0$ for some form ω^2_{p+2} with antifield number 2. Repeating this reasoning one obtains descent equations for δ and d related to $d\,\omega^0_p \approx 0$,

$$\begin{aligned} 0 &= d\,\omega^0_p + \delta\,\omega^1_{p+1}\,, \\ 0 &= d\,\omega^1_{p+1} + \delta\,\omega^2_{p+2}\,, \\ &\ \ \vdots \\ 0 &= d\,\omega^{D-p-1}_{D-1} + \delta\,\omega^{D-p}_{D}\,. \end{aligned} \tag{7.22}$$

We now analyse the last equation in (7.22), i.e. the equation at formdegree D, using $\omega^{D-p}_D = d^D x\, a^{D-p}$ where a^{D-p} has antifield number $D - p$. This gives

$$\delta\, a^{D-p} = \partial_m a^{m,\,D-p-1} \tag{7.23}$$

with $a^{m,\,D-p-1}$ arising from ω^{D-p-1}_{D-1}. We analyse (7.23) by considering the linearized Koszul-Tate differential δ_0 acting as

$$\delta_0\,\varphi^i = 0\,,\ \ \delta_0\,\varphi^\star_i = D_{ij}\,\varphi^j\,,\ \ \delta_0\,C^\star_\alpha = U^{i\dagger}_\alpha\,\varphi^\star_i \tag{7.24}$$

where $D_{ij}\,\varphi^j$ are the linearized Euler derivatives $\frac{\hat\partial \mathcal{L}}{\hat\partial \varphi^i}$ of the Lagrangian and $U^{i\dagger}_\alpha\,\varphi^\star_i$ is the linearized δ-transformation of $C^\star_\alpha$ (with $D_{ij} = \sum_k d^{m_1\dots m_k}_{ij}\partial_{m_1}\dots\partial_{m_k}$ etc.).

At lowest order in the fields and antifields, (7.23) imposes

$$\delta_0\,\underline{a} = \partial_m b^m \tag{7.25}$$

where $\underline{a}$ is the part of a^{D-p} with lowest degree of homogeneity in the fields and antifields, and b^m is the corresponding part of $a^{m,\,D-p-1}$.

Taking Euler derivatives of (7.25) with respect to the $C^\star$, $\varphi^\star$ and φ we obtain (with $D^\dagger_{ji} = \sum_r (-)^k d^{m_1\dots m_k}_{ji}\partial_{m_1}\dots\partial_{m_k}$ etc.)

$$\delta_0\,\frac{\hat\partial \underline{a}}{\hat\partial C^\star_\alpha} = 0,\quad \delta_0\,\frac{\hat\partial \underline{a}}{\hat\partial \varphi^\star_i} = U^i_\alpha\,\frac{\hat\partial \underline{a}}{\hat\partial C^\star_\alpha}\,,\quad \delta_0\,\frac{\hat\partial \underline{a}}{\hat\partial \varphi^i} = -D^\dagger_{ji}\,\frac{\hat\partial \underline{a}}{\hat\partial \varphi^\star_j}\,. \tag{7.26}$$

Assume now that $p < D-2$. In this case a^{D-p} and $\underline{a}$ have antifield number $D-p > 2$. Hence, all Euler derivatives in equations (7.26) have positive antifield numbers. Since

the cohomology of δ_0 vanishes for positive antifield numbers, the first equation (7.26) implies

$$\frac{\hat\partial \underline{a}}{\hat\partial C^\star_\alpha} = \delta_0 f^\alpha \tag{7.27}$$

for some f^α with antifield number $D-p-1$. Using (7.27) in the second equation (7.26) one gets

$$\delta_0\left(\frac{\hat\partial \underline{a}}{\hat\partial \varphi^\star_i} - U^i_\alpha f^\alpha\right) = 0 \tag{7.28}$$

where the expression in parentheses has antifield number $D - p - 1 > 1$. We conclude that this expression is the δ_0-transformation of some f^i with antifield number $D - p$ which yields

$$\frac{\hat\partial \underline{a}}{\hat\partial \varphi^\star_i} = U^i_\alpha f^\alpha + \delta_0 f^i. \tag{7.29}$$

Using (7.29) in the third equation (7.26) we obtain, owing to the operator identity $U^{i\dagger}_\alpha D_{ij} = 0$ which follows from (7.3):

$$\delta_0\left(\frac{\hat\partial \underline{a}}{\hat\partial \varphi^i} + D^\dagger_{ji}\, f^j\right) = 0 \tag{7.30}$$

where the expression in parentheses has positive antifield number $D - p > 2$. We conclude that this expression is the δ_0-transformation of some f_i with antifield number $D - p + 1$ which yields

$$\frac{\hat\partial \underline{a}}{\hat\partial \varphi^i} = -D^\dagger_{ji}\, f^j + \delta_0 f_i\,. \tag{7.31}$$

Analogously to equation (3.36) one can reconstruct $d^D x\,\underline{a}$ from the Euler derivatives of $\underline{a}$ with respect to the $C^\star$, $\varphi^\star$ and φ up to a d-exact form,

$$d^D x\,\underline{a} = \frac{1}{N}\,d^D x\left(C^\star_\alpha\,\frac{\hat\partial \underline{a}}{\hat\partial C^\star_\alpha} + \varphi^\star_i\,\frac{\hat\partial \underline{a}}{\hat\partial \varphi^\star_i} + \varphi^i\,\frac{\hat\partial \underline{a}}{\hat\partial \varphi^i}\right) + d(\dots) \tag{7.32}$$

where N is the degree of homogeneity of $\underline{a}$ in the fields and antifields. Using now equations (7.27), (7.29) and (7.31) in (7.32) one obatins

$$\begin{aligned} d^D x\,\underline{a} &= \delta_0\, k^{D-p+1}_D + d\, k^{D-p}_{D-1}\,, \\ k^{D-p+1}_D &= \frac{(-)^D}{N}\,d^D x\,(C^\star_\alpha f^\alpha - \varphi^\star_i f^i + \varphi^i f_i). \end{aligned} \tag{7.33}$$

One now considers $\omega'^{D-p}_D = \omega^{D-p}_D - \delta\, k^{D-p+1}_D - d\, k^{D-p}_{D-1}$. If ω'^{D-p}_D vanishes one gets $\omega^{D-p}_D = \delta\, k^{D-p+1}_D + d\, k^{D-p}_{D-1}$. Otherwise ω'^{D-p}_D is treated as ω^{D-p}_D before and the procedure is iterated. For "normal theories" the linearized theory contains the maximum number of derivatives and the iteration can be shown to terminate [29] resulting in

$$\omega_D^{D-p} = \delta\,\eta_D^{D-p+1} + d\,\eta_{D-1}^{D-p}\ . \tag{7.34}$$

Using (7.34) in the last equation (7.22) the latter gives

$$0 = d(\omega_{D-1}^{D-p-1} - \delta\,\eta_{D-1}^{D-p}). \tag{7.35}$$

Using the algebraic Poincaré lemma (theorem 3.3) one concludes that the form in parentheses is equal to $d\,\eta_{D-2}^{D-p-1}$ for some $(D-2)$-form η_{D-2}^{D-p-1} with antifield number $D-p-1$. This gives

$$\omega_{D-1}^{D-p-1} = \delta\,\eta_{D-1}^{D-p} + d\,\eta_{D-2}^{D-p-1}\ . \tag{7.36}$$

In the same way one derives that all the forms ω_{p+k}^k with $k > 0$ in (7.22) are δ-exact modulo d in the case $p < D-2$:

$$p < D-2,\ k > 0:\ \omega_{p+k}^k = \delta\,\eta_{p+k}^{k+1} + d\,\eta_{p+k-1}^k\ . \tag{7.37}$$

Using this result for $k = 1$ in the first equation (7.22) one eventually gets

$$p < D-2:\ 0 = d(\omega_p^0 - \delta\,\eta_p^1)\ . \tag{7.38}$$

The algebraic Poincaré lemma now implies that the form in parentheses is constant if $p = 0$ and d-exact if $p > 0$,

$$p < D-2:\ \omega_p^0 = \begin{cases} \delta\,\eta_p^1 + d\,\eta_{p-1}^0 & \text{if } p > 0, \\ \delta\,\eta_0^1 + \text{constant} & \text{if } p = 0. \end{cases} \tag{7.39}$$

Owing to $\delta\,\eta_p^1 \approx 0$ this yields (7.20) for $r = 0$.

In the case $p = D-2$ the form ω_D^2 can be taken as

$$\omega_D^2 = d^Dx\,(C_\alpha^\star g^\alpha(\{\varphi\},x) + h(\{\varphi,\varphi^\star\},x)) \tag{7.40}$$

where $h(\{\varphi,\varphi^\star\},x)$ is quadratic in the antifields $\varphi^\star$ and their derivatives. The last equation in (7.22) now gives

$$0 = d\,\omega_{D-1}^1 + (-)^D d^Dx\,\big((-R_\alpha^{i\dagger}\varphi_i^\star)g^\alpha(\{\varphi\},x) + \delta\,h(\{\varphi,\varphi^\star\},x)\big). \tag{7.41}$$

The Euler derivative of this equation with respect to $\varphi_i^\star$ yields $R_\alpha^i g^\alpha(\{\varphi\},x) \approx 0$ which imposes that the functions $g^\alpha(\{\varphi\},x)$ are (possibly field dependent) parameters of weakly vanishing gauge transformations. Without going into further detail we note that in Yang Mills theories, Einstein gravity and gravitational Yang Mills theories and dimensions $D > 2$ this implies $g^\alpha = 0$ for all α except for $g^\alpha =$ constant when α labels a "free Abelian gauge symmetry", and that the characteristic cohomology for $p = D-2$ is in these theories represented by forms ω_{D-2}^0 related via the descent equations (7.22) to volume forms given simply by $d^Dx\,C_{i'}^\star$ where i' enumerates the "free Abelian gauge symmetries" [29].

For $p = D-1$ the descent equations (7.22) reduce to the single equation $0 = d\,\omega_{D-1}^0 + \delta\,\omega_D^1$ which directly provides (7.21) with $\omega_{D-1}^0 \equiv J$.

Using (7.20) one can derive the linearized weak covariant Poincaré lemma for $p < D-2$ along the same lines as the linearized covariant Poincaré lemma in section 5.3 by replacing there equalities (=) by weak equalities ($\approx$) in various places using that the characteristic cohomology is trivial on forms $\omega_p^0(\{\varphi,\hat{C}\},x,dx)$ with positive pure ghostnumber,

$$d\,\omega_p^0 \approx 0,\ \text{pgh}(\omega_p^0) = g > 0,\ p < D\ \Rightarrow\ \omega_p^0 \approx \begin{cases} d\,\eta_{p-1}^0 & \text{if } p > 0, \\ \text{constant} & \text{if } p = 0. \end{cases} \tag{7.42}$$

To prove this we use again the descent equations (7.22) related to $d\,\omega_p^0 \approx 0$. In these equations all forms ω_{p+k}^k can be assumed to have the same pure ghostnumber g because d and δ do not alter the pure ghostnumber. Using $\omega_D^{D-p} = d^Dx\,a$, the Euler derivative of the last equation (7.22) with respect to the ghost $\hat{C}^\alpha$ gives $\delta\frac{\hat\partial a}{\partial C^\alpha} = 0$ which implies $\frac{\hat\partial a}{\partial C^\alpha} = \delta f_\alpha$ if $g > 0$. Analogously to (7.32)–(7.39) this implies $\omega_D^{D-p} = \delta\,\eta_D^{D-p+1} + d\,\eta_{D-1}^{D-p}$ with $\eta_D^{D-p+1} = g^{-1}(-)^{D+1}d^Dx\,\hat{C}^\alpha f_\alpha$ and consequently (7.42).

Using (7.42), the reasoning in section 5.3 can be carried over to the weak cohomology of d on linearized tensor forms starting from (5.53) which now reads $\Omega \approx d\,\omega_{p-1}$ and implies $d\,s\,\omega_{p-1} \approx 0$. In the latter equation $s\,\omega_{p-1}$ has pure ghostnumber 1 and by (7.42) this implies $s\,\omega_{p-1} + d\,\omega_{p-2} \approx 0$ in place of (5.55) etc. Eventually this yields that in ordinary and gravitational Yang Mills theories the invariant characteristic cohomology, i.e. the characteristic cohomology on spin and isospin invariant tensor forms $\omega_p(T,dx) = dx^{a_1}\ldots dx^{a_p}f_{a_1\ldots a_p}(T)$, is represented in formdegrees $p < D-2$ solely by Chern forms $P(F)$. This result extends to formdegree $p = D-2$ if there are no "free Abelian gauge symmetries" and if $D > 2$ and provides the weak covariant Poincaré lemma for these theories [24]:

$$d\,\omega(T,dx) \approx 0\ \Leftrightarrow\ \omega(T,dx) \approx P(F) + J_{\text{inv}} + d^Dx\,e\,f_{\text{inv}}(T) + d\,\eta(T,dx) \tag{7.43}$$

where $P(F)$ can contain a constant, J_{inv} is of the form (7.21) with $j^m = e\,E_a^m\,j^a(T)$ and $\eta(T,dx)$ is invariant.

7.4 Antifield Dependent Representatives of the BRST Cohomology

Each Noether form J_{inv} in (7.43) gives rise to a cocycle of the cohomology of $\tilde{s}$. We denote these cocycles by $\tilde{J}_{\text{inv}}$. Explicitly one may use

$$\begin{aligned} \tilde{J}_{\text{inv}} = {} & \frac{1}{(D-1)!}\,\tilde{c}^{m_1}\ldots\tilde{c}^{m_{D-1}}\varepsilon_{m_1\ldots m_D}j^{m_D} \\ & + \frac{1}{D!}\,\tilde{c}^{m_1}\ldots\tilde{c}^{m_D}\varepsilon_{m_1\ldots m_D}\,e\,G(T,T^\star) \end{aligned} \tag{7.44}$$

where j^m are the components of the Noether current occurring in $\tilde{J}_{\text{inv}}$, $G(T, T^\star)$ is a spin and isospin invariant function of the tensors T and antitensors $T^\star$ which has antifield number 1 and fulfills $\partial_m j^m = \delta(e\, G)$, and $\tilde{c}^m = \hat{c}^m + dx^m$ is the sum of the translation ghost $\hat{c}^m$ and the coordinate differential dx^m.

Owing to $\tilde{s}\,\tilde{J}_{\text{inv}} = 0$ one can finish the investigation of the antifield dependent BRST cohomology along the lines of section 6 by considering the $\tilde{J}_{\text{inv}}$ as Chern forms which have no Chern-Simons forms. Each $\tilde{J}_{\text{inv}}$ provides a representative ω_D^{-1} with ghostnumber -1 of the cohomology of s modulo d given by

$$\omega_D^{-1} = d^D x \; e\, G(T, T^\star) \; . \tag{7.45}$$

Further antifield dependent representatives ω_D^g of the cohomology of s modulo d arise from products of $\tilde{J}_{\text{inv}}$ and Chern Simons forms q_α (with the latter written in terms of the $\tilde{C}$). Among others this provides representatives ω_D^g with ghostnumbers $g = 0$ and $g = 1$ given by

$$\omega_D^0 = d^D x \left(e\, \hat{C}^i G(T, T^\star) - A_m{}^i j^m \right) , \tag{7.46}$$

$$\omega_D^1 = d^D x \left(e\, \hat{C}^i \hat{C}^j G(T, T^\star) + (A_m{}^i \hat{C}^j - A_m{}^j \hat{C}^i)\, j^m \right) \tag{7.47}$$

where $\hat{C}^i, \hat{C}^j$ are Abelian ghosts and $A_m{}^i, A_m{}^j$ are the corresponding Abelian gauge fields.

Finally we comment on the case that “free Abelian gauge symmetries” are present. As remarked in subsection 7.3, each of these symmetries gives rise to a nontrivial cohomology class of the characteristic cohomology in form degree $D - 2$. Accordingly it induces corresponding modifications of the BRST cohomological results. These modifications depend on the Lagrangian of the respective theory under consideration. We shall not discuss them in general here but restrict our comments to the cases that the gauge fields $A_m{}^{i'}$ of “free Abelian gauge symmetries” occur in the Lagrangian solely via terms $-\frac{1}{4}\, e\, F_{mn}{}^{i'} F^{mn\, j'} \delta_{i'j'}$ where i' numbers the “free Abelian gauge symmetries” and $F_{mn}{}^{i'} = \partial_m A_n{}^{i'} - \partial_n A_m{}^{i'}$ denote the corresponding field strengths. In this case the representatives of the characteristic cohomology in form degree $D - 2$ are the Poincaré duals ${}^*F_{i'}$ of the field strength 2-forms $dA^{i'}$ of the $A^{i'}_m$,

$$ {}^*F_{i'} = \frac{1}{2!(D-2)!}\, dx^{m_1} \dots dx^{m_{D-2}} \varepsilon_{m_1 \dots m_D}\, e\, F^{m_{D-1} m_D\, j'} \delta_{i'j'} \; . \tag{7.48}$$

Accordingly the weak covariant Poincaré lemma for $D > 2$ (7.43) gets additional contributions $\lambda^{i'}\,{}^*F_{i'}$ which are linear combinations of the dual $(D-2)$-forms ${}^*F_{i'}$ with numerical coefficients $\lambda^{i'}$. These $(D-2)$-forms give rise to cocycles of $\tilde{s}$ given by

$$\begin{aligned} {}^*\tilde{F}_{i'} = & \frac{1}{2!(D-2)!}\, \tilde{c}^{m_1} \dots \tilde{c}^{m_{D-2}} \varepsilon_{m_1 \dots m_D}\, e\, F^{m_{D-1} m_D j'} \delta_{i'j'} \\ & + \frac{1}{(D-1)!}\, \tilde{c}^{m_1} \dots \tilde{c}^{m_{D-1}} \varepsilon_{m_1 \dots m_D} A^{\star m_D}{}_{i'} \\ & + \frac{1}{D!}\, \tilde{c}^{m_1} \dots \tilde{c}^{m_D} \varepsilon_{m_1 \dots m_D} C^\star{}_{i'} \; . \end{aligned} \tag{7.49}$$

Each of these $\tilde{s}$-cocycles ${}^*\tilde{F}_{i'}$ contains a representative ω_D^{-2} with ghostnumber -2 of the cohomology of s modulo d given by

$$\omega_D^{-2} = d^D x\, C^\star{}_{i'} \; . \tag{7.50}$$

Antifield dependent representatives ω_D^g with ghostnumbers $g > -2$ of the cohomology of s modulo d arise from products of ${}^*F_{i'}$ and Chern Simons forms q_α written in terms of the $\tilde{C}$.

A Appendix

A.1 Massive Vectorfield

The Lagrangian $\mathcal{L}$ of the vectorfield $A = (A_0, A_1, A_2, A_3)$ ($F_{mn} = \partial_m A_n - \partial_n A_m$)

$$\mathcal{L}(x, A, \partial A) = -\frac{1}{4} F_{mn} F^{mn} + \frac{m^2}{2} A_n A^n - \frac{\lambda}{2} (\partial_n A^n)^2 \tag{A.1}$$

corresponds to the coupled system of equations of motion

$$\frac{\hat{\partial} \mathcal{L}}{\hat{\partial} A_m} = (\Box + m^2) A^m + (\lambda - 1) \partial^m \partial_n A^n = 0 \ . \tag{A.2}$$

Differentiating with ∂_m implies

$$(\lambda \Box + m^2) \partial_m A^m = 0 \tag{A.3}$$

which, upon differentiation of (A.2) with $(\lambda \Box + m^2)$, leads to

$$(\lambda \Box + m^2)(\Box + m^2) A^m = 0 \ . \tag{A.4}$$

So, if $m^2 > 0$ and $\lambda > 0$, the Fourier transformation of A^m is restricted to the mass shells $k^2 = m^2$ and $k^2 = m^2/\lambda$ and the solutions A_m are superpositions of plane waves with suitable polarization vectors ε and amplitudes a, b, $a^\dagger$ and $b^\dagger$,

$$\begin{aligned} A_m(x) = &\sum_\tau \int \tilde{d}k_{|_{k^0 = \sqrt{m^2 + \vec{k}^2}}} \big(\varepsilon_m^{*\tau}(k)\, a_\tau^\dagger(k)\, e^{ik\,x} + \text{h.c.} \big) + \\ &+ \sum_\tau \int \tilde{d}k_{|_{k^0 = \sqrt{m^2/\lambda + \vec{k}^2}}} \big(\varepsilon_m^{*\tau}(k)\, b_\tau^\dagger(k)\, e^{ik\,x} + \text{h.c.} \big) \ . \end{aligned} \tag{A.5}$$

They solve the equations of motion (A.2) if on the mass shell $k^2 = m^2$ the polarisation vectors are orthogonal to k: $k\,\varepsilon^i = 0$, $i = 1, 2, 3$. We take the three linearly independent (spacelike) solutions as mutually orthogonal and normalized, $\varepsilon^{*i}\,\varepsilon^j = -\delta^{ij}$. On the mass shell $k^2 = m^2/\lambda$ (A.2) implies that ϵ is a multiple of k. There, we choose $\epsilon = k/m$.

$$\begin{aligned} A_m(x) = &\sum_{i=1}^{3} \int \tilde{d}k_{|_{k^0 = \sqrt{m^2 + \vec{k}^2}}} \big(\varepsilon_m^{*i}(k)\, a_i^\dagger(k)\, e^{ik\,x} + \text{h.c.} \big) + \\ &+ \int \tilde{d}k_{|_{k^0 = \sqrt{m^2/\lambda + \vec{k}^2}}} \big(\frac{k_m}{m}\, b^\dagger(k)\, e^{ik\,x} + \text{h.c.} \big) \end{aligned} \tag{A.6}$$

The propagator (1.11) is the Green function of the differential operator in (A.2)

$$\langle \mathrm{T} A^m(x) A_n(0)\rangle = -i \lim_{\epsilon\to 0+}\int \frac{dp}{(2\pi)^4}\, e^{ipx}\Big(\frac{\delta^m{}_n - \frac{p^m p_n}{m^2}}{p^2 - m^2 + i\epsilon} + \frac{\frac{p^m p_n}{m^2}}{p^2 - \frac{m^2}{\lambda} + i\epsilon}\Big)\ . \qquad (A.7)$$

The right hand side conincides with the vacuum expectation value of the time ordered product of the fields (A.29) on the left, if the amplitudes a, b, $a^\dagger$ and $b^\dagger$ are creation and annihilation operators with nonvanishing commutators given by

$$\begin{aligned}[a_i(\vec{k}), a_j^\dagger(\vec{k}')] &= \delta_{ij}(2\pi)^3\, 2\sqrt{m^2 + \vec{k}^2}\, \delta^3(\vec{k} - \vec{k}')\ ,\\ [b(\vec{k}), b^\dagger(\vec{k}')] &= -(2\pi)^3\, 2\sqrt{m^2/\lambda + \vec{k}^2}\, \delta^3(\vec{k} - \vec{k}')\ .\end{aligned} \qquad (A.8)$$

The negative sign in the last commutation relation implies that the one-particle states $b^\dagger(\vec{k})\Omega$ have negative norm.

If λ or m^2 were negative, then the field A_m would contain tachyons and could not be decomposed into a creation part and an annihilation part such that Lorentz transformations act differentiably. If $\lambda = 0$, then the last term in (A.7) vanishes. This is the so called unitary gauge. But then the propagator does not fall off asymptotically with $1/p^2$ and the power counting of the divergencies of the integration over loop momenta is spoiled. Therefore, renormalizable models are consistent only for $\lambda > 0$.

A.2 Stueckelberg Lagrangian

If one enlarges the field content of an abelian gauge field A_m, its ghost C, its antighost $\bar{C}$ and the real auxiliary field B by a real scalar boson ϕ with vanishing ghost number with the BRST transformation

$$\begin{aligned} sA_n = \partial_n C\ ,\quad s\phi = mC\ ,\quad sC = 0\ ,\\ s\bar{C} = iB\ ,\quad sB = 0\ ,\end{aligned} \qquad (A.9)$$

then each BRST invariant Lagrangian can depend arbitrarily on the gauge invariant vector field $\hat{A}_n = D_n\phi$, which can also be considered as the covariant derivative of ϕ,

$$D_n\phi = \partial_n\phi - mA_n\ . \qquad (A.10)$$

The fields C, ϕ, $\bar{C}$ and B are trivial pairs and contribute to the Lagrangian only via gauge fixing terms $-isX$.

One is then tempted to conclude that models which realize the BRST transformations with this field content are equivalent to models without the fields $\bar{C}$, B, ϕ and C and a vector field without any BRST transformation. The conclusion is true if the gauge fixing term does not contain A, e.g.

$$\begin{aligned} X &= \bar{C}\big(\frac{1}{2}B + \frac{1}{m}\Box\phi + m\,\phi\big)\ ,\\ -isX &= \frac{1}{2}\big(B + \frac{1}{m}\Box\phi + m\,\phi\big)^2 - \frac{1}{2m^2}\big((\Box + m^2)\phi\big)^2 + i\,\bar{C}(\Box + m^2)C\ .\end{aligned} \qquad (A.11)$$

The corresponding equation of motion $(\Box + m^2)^2\phi = 0$ contains higher derivatives and ϕ generates, after quantization, two scalars with mass m, one of which with negative norm (1.57). Together with the fermionic ghosts and antighosts, they realize an unphysical BRST quartet of states at mass m, which are either not invariant or trivial.

If higher derivatives are excluded, then the quadratic Lagrangian of the gauge invariant vector field $\hat{A}$ is given by (A.1), where $\hat{A}$ replaces A. But then, if $\lambda = 0$, the propagator of $\hat{A} \approx (\eta^{mn} - p^m p^n/m^2)/p^2$ does not fall off like $1/p^2$ in all directions and the model is not renormalizable. Or, if $\lambda > 0$, then $\hat{A}$ creates a BRST invariant particle with negative norm and mass $m/\sqrt{\lambda}$.

To avoid the negative norm state of ϕ in the BRST quartet and to replace it by the negative norm state of $\partial_m A^m$, one has to replace $\Box\phi$ in X by ∂A (2.45, 2.46),

$$\begin{aligned} X &= \bar{C}\big(\frac{1}{2\lambda}B + \partial_n A^n + \frac{m}{\lambda}\phi\big)\ ,\\ -isX &= \frac{1}{2\lambda}\big(B + \lambda(\partial_n A^n + \frac{m}{\lambda}\phi)\big)^2 - \frac{\lambda}{2}\big(\partial_n A^n + \frac{m}{\lambda}\phi\big)^2 + i\,\bar{C}(\Box + \frac{m^2}{\lambda})C\ .\end{aligned} \qquad (A.12)$$

The gauge invariant Lagrangian is an arbitrary function of (derivatives of) the field strength F_{mn} and the covariant derivative $D_n\phi$. If one excludes higher derivatives and chooses a convenient normalization, the quadratic part is given by

$$\mathcal{L}_{\mathrm{inv}}(\phi, A, \partial\phi, \partial A) = -\frac{1}{4}F_{mn}F^{mn} + \frac{1}{2}(\partial_n\phi - mA_n)(\partial^n\phi - mA^n)\ . \qquad (A.13)$$

The product $-m\,A^n\,\partial_n\phi$ mixes the vector and the scalar field. However, together with the mixing terms in the gauge fixing $-isX$, both mixing terms combine to a complete derivative $-m\partial_n\big(A^n\phi\big)$ and can be dropped. This convenient gauge fixing, which cancels the vector scalar mixing, goes back to 't Hooft [1] and is called the R_ξ gauge.

After the cancellation of the mixing terms, one is left with the Stueckelberg Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{inv}} - isX \approx &-\frac{1}{4}F_{mn}F^{mn} + \frac{1}{2}\,m^2 A^2 - \frac{\lambda}{2}(\partial A)^2 +\\ &+\frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2\lambda}\phi^2 + \frac{1}{2\lambda}\big(B + \lambda\partial A + m\phi\big)^2 +\\ &+ i\,\bar{C}\big(\Box + \frac{m^2}{\lambda}\big)C\ .\end{aligned} \qquad (A.14)$$

It contains the Lagrangian (A.1) for a massive vectorfield A with a gauge fixing term. Its propagator falls off like $1/p^2$ in all directions in momentum space as do the propagators of ϕ and $\bar{C}$ and C. The field B is auxiliary.

The field A creates and annihilates real bosonic spin-1 particles with mass m and a real, bosonic scalar with mass $m/\sqrt{\lambda}$, the states of which have wrong sign norm. Moreover, at mass $m/\sqrt{\lambda}$ there is the real bosonic scalar, which is created by ϕ and has positive norm, and the two fermionic scalars from $\bar{C}$ and C. These scalars form a BRST

[1] In the name 't Hooft the apostrophe precedes the t.

quartet of states, which are either not invariant or which are trivial. Only the vector particle with mass m is physical.

BRST invariant, renormalizable interactions with charged matter fields are obtained by the completion of their partial derivatives ∂_n to covariant derivatives $D_n = \partial_n + A_n\delta$. Couplings of the covariant derivative $D_n\phi$ are excluded in renormalizable models.

So the Stueckelberg model (A.14) yields a massive vector field which can interact with charged matter. The ghosts $\bar{C}$ and C and the scalars ϕ and $\partial_m A^m$ remain free and unobservable.

A.3 The Higgs Effect

For vector fields A^a, where $a = 1, \dots d$, enumerates a basis of the Lie algebra of gauge transformations, and for real scalar fields ϕ^i, $i = 1, \dots N$, on which these transformations act, we consider the Lagrangian[2]

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^a_{mn}F^{a\,mn} + \frac{1}{2}D_m\phi^i\,D^m\phi^i - V(\phi) + \\ &- \mathrm{is}\,\bar{C}^a\big(\frac{1}{2\lambda}B^a + \partial_m A^{a\,m} + \frac{g}{\lambda}\phi^T T_a v\big)\ .\end{aligned} \tag{A.15}$$

The gauge coupling g, which can be chosen freely for each factor of the Lie algebra in front of the F^2-term, is chosen to be absorbed in redefined vector fields and ghost fields, $A' = A/g$ and $C' = C/g$, such that all kinetic terms F^aF^a are equally normalized. The gauge couplings then appear as factors gA' and gC' in the nonlinear parts of the field strength and in the covariant derivatives $D_m = \partial_m + gA'^a_m\delta_a$ (4.24). For readability, we have dropped the primes in (A.15) and in the following.

The scalar fields ϕ transform under the isospin transformations

$$\delta_a\phi = -T_a\phi + k_a \tag{A.16}$$

linearly with a possible inhomogeneity k_a. Otherwise, if higher powers of the fields occured, then, in four dimensional space time, the gauge invariant $D_m\phi^i D^m\phi^i$ would already contain non-renormalizable vertices. The matrices T_a are real representations of the Lie algebra

$$[\delta_a, \delta_b] = f_{ab}{}^c\,\delta_c\ ,\ [T_a, T_b] = f_{ab}{}^c\,T_c\ , \tag{A.17}$$

which have to be antisymmetric

$$T_a^T = -T_a \tag{A.18}$$

in order to leave the quadratic form $D\phi \cdot D\phi$ invariant.

The constants k_a are vectors in the space on which T_a act. They have to satisfy

$$T_a k_b - T_b k_a = f_{ab}{}^c k_c\ . \tag{A.19}$$

[2] If one deals with complex scalar fields and complex representation matrices, one has to decompose them into their real and imaginary parts in order to make our considerations applicable.

These conditions are obviously fulfilled by vectors of the form $T_a w$, as they arise by the shift $\phi' = \phi - w$ of the scalar field. Assume this shift chosen such that the summed length squared of the inhomogeneities k_a is as small as possible. Then $\sum_a(k_a + T_a w)^2$ is minimal at $w = 0$, its linear term $2k_a \cdot T_a w$ vanishes for all w and k_a satisfies

$$T_a k_a = 0\ . \tag{A.20}$$

Using this when acting with T_a on (A.19), we can replace $T_aT_bk_a$ by the comutator $[T_a, T_b]k_a$ and obtain

$$T_aT_ak_b = f_{ab}{}^c\big(T_ak_c + T_ck_a\big)\ . \tag{A.21}$$

But the right hand side vanishes: the structure constants $f_{ab}{}^c$ are the matrix elements of the adjoint representation of the Lie algebra. These adjoint transformations have to preserve the length square F^aF^a, i.e. they are infinitesimal rotations which means that the structure constants $f_{ab}{}^c$ are antisymmetric under the interchange of a and c,[3]

$$T_aT_ak_b = 0\ . \tag{A.22}$$

The operator T_aT_a is negative in all irreducible subspaces where not all T_a vanish. There all k_b have to vanish. They can be different from zero only in the subspace, where all T_a vanish. There (A.19) implies

$$f_{ab}{}^c k_c = 0\ , \tag{A.23}$$

i.e. only the vectors k_a which correspond to abelian factors of the gauge group can be different from zero. So the covariant derivatives of the scalars either have the form

$$D_m\phi^i = \partial_m\phi^i - gA^a_m T_a{}^i{}_j\phi^j\ ,\ i = 1, \dots M\ , \tag{A.24}$$

or, in the subspace where all T_a vanish, they are as in the Stueckelberg model (A.10)

$$D_m\phi^i = \partial_m\phi^i - A^a_m k^i_a\ ,\ i = M+1, \dots N \tag{A.25}$$

and these $N - M$ scalar fields transform only under the abelian isospin transformations.

The vector v in the last term of (A.15) is the vacuum expectation value of ϕ, which is to say that the potential $V(\phi)$ becomes minimal at v

$$V(v) = V_{\min}\ ,\ v = \langle\phi(x)\rangle \tag{A.26}$$

and $\phi(x) = v$ is a solution to the equations of motion which is invariant under translations and Lorentz transformations. If $v \neq 0$ is not a fixpoint of the isospin transformations (A.16) then one calls them spontaneously broken. That, as we shall see, a nonvanishing vacuum expectation value of the scalar fields results in masses of the vector particles is the so called Higgs effect.

[3] This is why the isospin transformations form a compact or an abelian group.

Because the energy density is bounded from below by V_{min}, $\phi(x) = v$ is a lowest energy solution of the equations of motion. Therefore small deviations φ from v

$$\phi(x) = v + \varphi(x) \tag{A.27}$$

are expected to remain small, because they do not contain enough energy for larger deviations, and to satisfy approximately the linearized field equations (linear in φ) which derive from the quadratic part of the Lagrangian.

To determine this quadratic part, we observe that the linear part of the covariant derivative has the Stueckelberg form (A.25)

$$D_m\phi = \partial_m\varphi - gA_m^a T_a v + \dots = \partial_m\varphi - A_m^a k_a + \dots \ , \text{ where } k_a^i = g T_a{}^i{}_j v^j \ , \tag{A.28}$$

with the only difference, that in the last equation $i = 1, \dots M$, enumerates components in the irreducible subspaces, where not all T_a vanish. Consider k_a to be the vectors with N components given by (A.25, A.28).

Then $D\phi \cdot D\phi$ contribute the quadratic terms

$$\begin{aligned} \frac{1}{2}D_m\phi \cdot D^m\phi &\approx \frac{1}{2}(\partial_m\varphi - A_m^a k_a)\cdot(\partial^m\varphi - A^{b\,m}k_b) \\ &\approx \frac{1}{2}\partial_m\varphi \cdot \partial^m\varphi - (\partial_m\varphi \cdot k_a)A^{a\,m} + \frac{1}{2}M_{ab}^2 A_m^a A^{b\,m} \ , \\ M_{ab}^2 &= k_a \cdot k_b \ . \end{aligned} \tag{A.29}$$

The mass squared M_{ab}^2 are the matrix elements of a real quadratic form in the space of vector fields. By an orthogonal change of the basis it can be diagonalized, preserving the quadratic form $F^a F^a$. We assume to work with a basis which diagonalizes M^2 and denote their diagonal elements by m_a^2,

$$k_a \cdot k_b = \begin{cases} 0 & \text{if} \quad a \neq b \\ m_a^2 & \text{if} \quad a = b \end{cases} . \tag{A.30}$$

But this means, that in the space of the scalar fields the vectors $k_a = m_a e_a$ are multiples of orthonormal vectors e_a, which we can suitably complete to an orthonormal basis. Assuming, that we work in this basis already (and no longer in a basis adapted to the irreducible subspaces), then the quadratic form $D\phi \cdot D\phi$ remains unchanged and $\varphi \cdot k_a$ is simply

$$\varphi \cdot k_a = m_a \varphi^a \ , \text{ no sum over } a \ . \tag{A.31}$$

Altogether, the covariant derivatives of the scalar fields contribute

$$\frac{1}{2}D_m\phi \cdot D^m\phi \approx \frac{1}{2}\partial_m\varphi \cdot \partial^m\varphi + \frac{1}{2}\sum_a m_a^2 A_m^a A^{a\,m} - \sum_a m_a \partial_m \varphi^a A^{a\,m} \tag{A.32}$$

to the quadratic Lagrangian. The gauge fixing fermion X is

$$X = \sum_a \bar{C}^a\big(\frac{1}{2\lambda}B^a + \partial_m A^{a\,m} + \frac{m_a}{\lambda}\varphi^a\big) \ . \tag{A.33}$$

So, apart from the contributions from the potential $V(v+\varphi)$, the quadratic Lagrangian is just a sum of Stueckelberg Lagrangians (A.14) with masses m_a.

The potential V is invariant under isospin transformations

$$g\delta_a V = (g\delta_a\phi)^i \frac{\partial V}{\partial \phi^i} = 0 \ . \tag{A.34}$$

Differentiating again and evaluating at $\phi = v$, where the first derivative of V vanishes (V is minimal there), we obtain

$$k_a^i \frac{\partial^2 V}{\partial\phi^i \partial\phi^j}_{|\phi=v} = 0 \ . \tag{A.35}$$

In other words, the potential does not contribute masses to the scalar fields in the space spanned by the vectors k_a. They would be massless, were it not for the gauge fixing term $-is\big(\bar{C}\, m_a/\lambda\, \phi^a\big)$ which brakes the isospin symmetry explicitly and makes Goldstone's theorem inapplicable that massless bosons occur if an isospin symmetry breaks down spontaneously.

In the directions which are orthogonal to all k_a, the expansion of V provides masses for the remaining and observable scalar fields, the Higgs fields.

The observable particles are n vector bosons with masses m_a, $a = 1, 2, \dots n$, where n is the rank of the mass matrix $k_a \cdot k_b$. The n Goldstone bosons combine together with ghost and antighost particles and the negativ norm scalars, contained in the vector field to n BRST quartets at the masses $m_a/\sqrt{\lambda}$. These scalars are either not invariant or trivial and do not belong to the physical sector.

Each massive vector particle has three possible polarisation states as compared to two transverse polarisation states of the vector particle and one scalar particle which become observable if the mass vanishes. So the number of states (per given momentum) does not change if one switches on the mass parameter. In this case the scalar state becomes unobservable and the vector massive because, as the cannibalistic interpretation has it, it has eaten the scalar.

The massless vector fields create and annihilate two transverse, physical particles. The other states with polarisation in direction of the momentum k or in direction $\bar{k}$ combine together with ghosts and antighosts to unphysical, massless BRST quartetts.

$N-n$ scalar fields remain observable. In particular, in the standard model, one of the four real scalar fields is orthogonal to the Goldstone fields and creates and annihilates a physical particle, the Higgs.

Note that by a Stueckelberg contribution to the mass matrix, the photon can have a mass without an additional observable scalar and without violating a gauge symmetry. That the photon is massless is an observational input into the standard model and not a theoretical deduction.

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